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Nonlinear state feedback $\mathcal{H}_\infty$-control of mechanical systems under unilateral constraints

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Abstract: The work focuses on the state feedback synthesis of hybrid mechanical systems under unilateral constraints. The problem of robust control of mechanical systems is addressed under unilateral constraints by designing a nonlinear state feedback $\mathcal{H}_\infty$-controller developed in the hybrid setting, covering impact phenomena. Performance issues of the developed nonlinear $\mathcal{H}_\infty$-tracking controller are illustrated with numerical tests on a seven-link biped robot.

Keywords: hybrid systems, robust control, tracking, mechanical systems, biped, walking gait

1. INTRODUCTION

The study of hybrid dynamical systems has recently attracted a significant research interest, basically, due to the wide variety of applications and the complexity that arises from the analysis of this type of systems. See, e.g., the relevant works by Hamed and Grizzle [2013], Goebel et al. [2009], and references quoted therein. Particularly, the disturbance attenuation problem for hybrid dynamical systems has been addressed by Haddad et al. [2005], Nešić et al. [2008], where impulsive control inputs were admitted to counteract/compensate disturbances/uncertainties at time instants of instantaneous changes of the underlying state. It should be noted, however, that even in the state feedback design, a pair of independent Riccati equations, separately coming from continuous and discrete dynamics, was required to possess a solution that satisfies both equations, thus yielding a restrictive condition on the feasibility of the proposed synthesis. Moreover, the physical implementation of impulsive control inputs was impossible in many practical situations, e.g., while controlling walking biped robots.

Thus motivated, the present investigation introduces a new control strategy, which is feasible under certain conditions and which avoids using impulsive control inputs while ensuring the asymptotic stability of the undisturbed hybrid system of interest and possessing the $\mathcal{L}_2$-gain of its disturbed version to be less than an appropriate disturbance attenuation level. The work focuses on impulse hybrid systems, which are recognized as dynamical systems under unilateral constraints (Brogliato [1999]). The $\mathcal{H}_\infty$ approach, that has recently been developed by Orlov and Aguilar [2014] towards nonsmooth mechanical applications, is now generalized in the presence of unilateral constraints. An essential feature, adding the value to the present investigation, is that in contrast to the existing literature where the perfect knowledge of the restitution rule at the collision time instants is assumed, not only standard external disturbances but also their discrete-time counterparts are attenuated.

The paper is outlined as follows. Section 2 presents the hybrid model of interest subject to an unilateral constraint and the $\mathcal{H}_\infty$-control problem is then stated. Section 3 derives sufficient conditions for a global/local solution of the problem in question to exist, and a state feedback controller is synthesized and developed for n-DOF mechanical manipulators. Capabilities of the developed state feedback synthesis are illustrated in Sect. 4 in a numerical study of the orbital stabilization of a seven-link biped robot with feet required to track a walking gait composed of single support phases separated by impacts. Finally, conclusions of this work are presented in Sect. 5.

2. PROBLEM STATEMENT

Given a scalar unilateral constraint $\mathbf{F}(\mathbf{x_1}) \geq \mathbf{0}$, consider a nonlinear system, evolving within the above constraint, which is governed by continuous dynamics of the form

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{x}_2 \\ \dot{\mathbf{x}}_2 &= \mathbf{\Phi}(\mathbf{x}_1, \mathbf{x}_2, t) + \mathbf{\Psi}_1(\mathbf{x}_1, \mathbf{x}_2, t)\mathbf{w} + \mathbf{\Psi}_2(\mathbf{x}_1, \mathbf{x}_2, t)\mathbf{u} \end{aligned} \tag{1}$$

$$\mathbf{z} = \mathbf{h_1}(\mathbf{x}_1, \mathbf{x}_2, t) + \mathbf{k_{12}}(\mathbf{x}_1, \mathbf{x}_2, t)\mathbf{u} \tag{2}$$

beyond the surface $\mathbf{F}(\mathbf{x_1}) = \mathbf{0}$ when the constraint is inactive, and by the algebraic relations

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$$\mathbf{x}_1(t_i^+) = \mathbf{x}_1(t_i^-)$$
$$\mathbf{x}_2(t_i^+) = \mu_0(\mathbf{x}_1(t_i), \mathbf{x}_2(t_i^-), t_i) + \omega(\mathbf{x}_1(t_i), \mathbf{x}_2(t_i^-), t_i)\mathbf{w_d^i} \tag{3}$$

$$\mathbf{z_i^d} = \mathbf{x}_2(t_i^+) \tag{4}$$

at *a priori* unknown collision time instants $t = t_i$, $i = 1, 2, \ldots$, when the system trajectory hits the surface $\mathbf{F}(\mathbf{x_1}) = \mathbf{0}$. In the above relations, $\mathbf{x}^\top = [\mathbf{x}_1^\top, \mathbf{x}_2^\top] \in \mathbb{R}^{2n}$ represents the state vector with components $\mathbf{x}_1 \in \mathbb{R}^n$ and $\mathbf{x}_2 \in \mathbb{R}^n$; $\mathbf{u} \in \mathbb{R}^n$ is the control input of dimension n; $\mathbf{w} \in \mathbb{R}^l$ and $\mathbf{w_d^i} \in \mathbb{R}^q$ collect exogenous signals affecting the motion of the system (external forces, including impulsive ones, as well as model imperfections). The variable $\mathbf{z} \in \mathbb{R}^s$ represents a continuous time component of the system output to be controlled whereas the post-impact value of the only state component $\mathbf{x}_2(t)$ subjected to the instantaneous change is pre-specified as a discrete component $\mathbf{z_i^d}$ of the to-be-controlled output. The overall system in the closed-loop should be dissipative with respect to the output thus specified. Throughout, the functions $\mathbf{\Phi}$, $\mathbf{\Psi_1}$, $\mathbf{\Psi}_2$, $\mathbf{h_1}$, $\mathbf{k_{12}}$, $\mathbf{F}$, μ_0, and ω are of appropriate dimensions, which are continuously differentiable in their arguments and uniformly bounded in t. The origin is assumed to be an equilibrium of the unforced system (1)-(4), which is located beyond the unilateral constraint, i.e., $F(0) \neq 0$, $\mathbf{\Phi}(0, 0, t) = 0$, $\mathbf{h_1}(0, 0, t) = 0$, for all t and $\mu_0(0, 0, 0) = 0$.

If interpreted in terms of mechanical systems, equation (1) describes the continuous dynamics before the underlying system hits the reset surface $\mathbf{F}(\mathbf{x}_1) = 0$, depending on the position only, whilst the restitution law, given by equation (3), is a physical law for the instantaneous change of the velocity when the resetting surface is hit. Thus, the position is not instantaneously changed at the collision time instants whereas the post-impact velocity $\mathbf{x}_2(t^+)$ is a function of both the pre-impact state $(\mathbf{x}_1(t), \mathbf{x}_2(t^-))$ and a discrete perturbation $\mathbf{w_d}$ accounting for inadequacies of the restitution law. In order to deal with systems dissipating the energy during the collision, only motions of the finite sort (Mabrouk [1998]) are admitted throughout with the restitution function μ meeting the condition

$$\|\mu_\mathbf{0}(\mathbf{x}_1, \mathbf{x}_2)\| \leq \|\mathbf{x}_2\| \tag{5}$$

for all $(\mathbf{x}_1, \mathbf{x}_2) \in \mathbb{R}^{2n}$. The inequality

$$\|\mathbf{x}_2(t_i^+)\| \leq \|\mathbf{x}_2(t_i^-)\| \tag{6}$$

is thus ensured for all collision time instants t_i, $i = 1, 2, \ldots$ for the undisturbed system with $\mathbf{w_d} = \mathbf{0}$. For later use, the notion of an admissible controller is specified for the underlying system. Consider a causal feedback controller

$$\mathbf{u} = \kappa(\eta) \tag{7}$$

with the function $\kappa(\eta)$ of class C^1 such that $\kappa(0) = 0$. Such a controller is said to be a locally (globally) *admissible controller* iff the undisturbed $(\mathbf{w}, \mathbf{w_d^i} = \mathbf{0})$ closed-loop system (1)–(4) is uniformly (globally) asymptotically stable.

The $\mathcal{H}_\infty$-control problem of interest consists in finding an admissible global controller (if any) such that the $\mathcal{L}_2$-gain of the disturbed system (1)–(4) is less than a certain attenuation level $\gamma > 0$, that is the inequality

$$\int_{t_0}^{T} \|\mathbf{z}\|^2 \mathrm{d}t + \sum_{i=1}^{N_T} \|\mathbf{z_i^d}\|^2 \leq \gamma^2 \left[\int_{t_0}^{T} \|\mathbf{w}\|^2 \mathrm{d}t + \sum_{i=1}^{N_T} \|\mathbf{w_d^i}\|^2 \right] + \sum_{j=0}^{N} \beta_j(\mathbf{x}(t_j^-), t_j) \tag{8}$$

locally holds for some positive definite functions $\beta_j(\mathbf{x}, t)$, $j = 0, \ldots, N_T$, for all segments $[t_0, T]$ and a natural N_T such that $t_{N_T} \leq T < t_{N_T+1}$, and for all piecewise continuous disturbances $\mathbf{w}(t)$ and discrete ones $\mathbf{w_d^i}$, $i = 1, 2, \ldots$. In turn, a locally admissible controller (7) is said to be a local solution of the $\mathcal{H}_\infty$-control problem if there exists a neighborhood $\mathcal{U} \in \mathbb{R}^{2n}$ of the origin, validating inequality (8) for some positive definite functions $\beta_j(\mathbf{x}, t)$, $j = 0, \ldots, N_T$, for all segments $[t_0, T]$ and a natural N_T such that $t_{N_T} \leq T < t_{N_T+1}$, for all piecewise continuous disturbances $\mathbf{w}(t)$ and discrete ones $\mathbf{w_d^i}$, $i = 1, 2, \ldots$, for which the state trajectory of the closed-loop system starting from an initial point $(\mathbf{x}(t_0) = \mathbf{x_0}) \in \mathcal{U}$ remains in $\mathcal{U}$ for all $t \in [t_0, T]$.

It is worth noticing that the above $\mathcal{L}_2$-gain definition is consistent with the notion of dissipativity introduced by Willems [1972] and Hill and Moylan [1980], and it represents a natural extension to hybrid systems (see, e.g. the works by Nešić et al. [2008], Yuliar et al. [1998] and Lin and Byrnes [1996]).

3. NONLINEAR $\mathcal{H}_\infty$-CONTROL SYNTHESIS

For later use, the continuous dynamics (1) are rewritten in the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) + \mathbf{g_1}(\mathbf{x}, t)\mathbf{w} + \mathbf{g_2}(\mathbf{x}, t)\mathbf{u} \tag{9}$$

whereas the restitution rule is represented as follows

$$\mathbf{x}(t_i^+) = \mu(\mathbf{x}(t_i^-), t_i) + \Omega(\mathbf{x}(t_i^-), t_i)\mathbf{w_d^i}, \ i = 1, 2, \ldots \tag{10}$$

with $\mathbf{x}^\top = [\mathbf{x}_1^\top, \mathbf{x}_2^\top]$, $\mathbf{f}^\top(\mathbf{x}, t) = [\mathbf{x}_2^\top, \mathbf{\Phi}^\top(\mathbf{x}, t)]$, $\mathbf{g_1}^\top(\mathbf{x}, t) = [\mathbf{0}, \mathbf{\Psi}_1^\top(\mathbf{x}, t)]$, $\mathbf{g}_2^\top(\mathbf{x}, t) = [\mathbf{0}, \mathbf{\Psi}_2^\top(\mathbf{x}, t)]$, $\mu^\top(\mathbf{x}, t) = [\mathbf{x}_1^\top, \mu_0^\top(\mathbf{x}, t)]$, and $\Omega^\top(\mathbf{x}, t) = [\mathbf{0}, \omega(\mathbf{x}, \mathbf{t})]$. In order to simplify the synthesis to be developed and to provide reasonable expressions for the controller design, the following assumptions

$$\mathbf{h_1}^\top \mathbf{k_{12}} = \mathbf{0}, \ \mathbf{k_{12}}^\top \mathbf{k_{12}} = \mathbf{I} \tag{11}$$

which are standard in the literature (see, e.g., Orlov [2009]) are made. Relaxing these assumptions is indeed possible, but it would substantially complicate the formulas to be worked out.

3.1 Global state-space solution

Below we list the hypotheses under which a solution to the problem in question is derived. Given $\gamma > 0$, in a domain $\mathbf{x} \in B_\delta^{2n}, t \in \mathbb{R}$, where $B_\delta^{2n} \in \mathbb{R}^{2n}$ is a ball of radius $\delta > 0$, centered around the origin,

H1) The norm of the matrix function ω is upper bounded by $\frac{\sqrt{2}}{2}\gamma$, i.e.,

$$\|\omega(x, t)\| \leq \frac{\sqrt{2}}{2}\gamma. \tag{12}$$

H2) there exist a smooth, positive definite, decrescent function $V(\mathbf{x}, t)$ and a positive definite function $R(x)$ such that the Hamilton–Jacobi–Isaacs inequality

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \mathbf{x}}(\mathbf{f}(\mathbf{x},t) + \mathbf{g_1}(\mathbf{x},t)\alpha_1 + \mathbf{g_2}(\mathbf{x},t)\alpha_2) + \mathbf{h_1}^\top \mathbf{h_1} + \alpha_2{}^\top \alpha_2 - \gamma^2 \alpha_1{}^\top \alpha_1 \le -R(\mathbf{x}) \quad (13)$$

holds with

$$\alpha_1 = \frac{1}{2\gamma^2}\mathbf{g}_1^\top(\mathbf{x},t)\left(\frac{\partial V}{\partial \mathbf{x}}\right)^\top,\ \alpha_2 = -\frac{1}{2}\mathbf{g}_2^\top(\mathbf{x},t)\left(\frac{\partial V}{\partial \mathbf{x}}\right)^\top$$

H3) Hypotheses H1 is satisfied with the function $V(\mathbf{x},t)$ which decreases along the direction μ in the sense that the inequality

$$V(\mathbf{x},t) \ge V(\mu(\mathbf{x}),t) \quad (14)$$

holds in the domain of V.

The main result of the present work is as follows.

Theorem 3.1. Consider system (1)-(4) subject to (5). Given $\gamma > 0$, suppose Hypotheses H1) and H2) are satisfied in a domain $\{\mathbf{x} \in B_\delta^{2n}, t \in \mathbb{R}\}$. Then, the closed-loop system (1)-(4), driven by the controller

$$\mathbf{u} = \alpha_2(\mathbf{x},t), \quad (15)$$

locally possesses a $\mathcal{L}_2$-gain less than γ. Moreover, the disturbance-free closed-loop system (1)-(4), (15) is uniformly asymptotically stable provided that Hypothesis H3) is satisfied as well. If in addition, Hypotheses H1)–H3) remain in force globally with $V(\mathbf{x},t)$ radially unbounded, then the results hold true globally.

Proof. Since the proof follows the same line of reasoning as that in the book by Orlov [2009] for the impact-free case here we provide only a sketch. Similar to the proof of [Orlov, 2009, Theorem 7.1], let us consider the function $V(\mathbf{x},t)$ whose time derivative, computed along the disturbed closed-loop system (1)-(4) between collision time instants $t \in (t_k, t_{k+1}),\ \ k = 0,1,\dots,$ is estimated as follows [Orlov, 2009, p.138]:

$$\frac{\mathrm{d}V}{\mathrm{d}t} \le -\|\mathbf{z}\|^2 + \gamma^2\|\mathbf{w}\|^2 - R(\mathbf{x}). \quad (16)$$

Then integrating (16) from t_k to $t_{k+1},\ k = 0,1,\dots,$ yields

$$\int_{t_k}^{t_{k+1}} [\gamma^2\|\mathbf{w}\|^2 - \|\mathbf{z}\|^2]\mathrm{d}t \ge \int_{t_k}^{t_{k+1}} R(\mathbf{x}(t))\mathrm{d}t + \int_{t_k}^{t_{k+1}} \frac{\mathrm{d}V(\mathbf{x}(t),t)}{\mathrm{d}t}\mathrm{d}t > 0. \quad (17)$$

Skipping positive terms in the right-hand side of (17), it follows that

$$\int_{t_0}^{T} (\gamma^2\|\mathbf{w}\|^2 - \|\mathbf{z}\|^2)\mathrm{d}t \ge V(\mathbf{x}(T),T) + \sum_{i=1}^{N_T}[V(\mathbf{x}(t_i^-),t_i) - V(\mathbf{x}(t_i^+),t_i)] - V(\mathbf{x}(t_0),t_0). \quad (18)$$

Since the function V is smooth by Hypothesis H2), the following relation

$$|V(\mathbf{x}(t_i^-),t_i) - V(\mathbf{x}(t_i^+),t_i)| \le L_i^V|\mathbf{x}(t_i^-) - \mathbf{x}(t_i^+)| \quad (19)$$

holds true with $L_i^V > 0$ being a local Lipschitz constant of V, in the ball of radius $\|\mathbf{x}(t_i^+)\|$, centered around $\mathbf{x}(t_i^-)$. Relations (18) and (19), coupled together, result in

$$\int_{t_0}^{T} (\gamma^2\|\mathbf{w}\|^2 - \|\mathbf{z}\|^2)\mathrm{d}t \ge -\sum_{i=1}^{N_T}[2(L_i^V)\|\mathbf{x}(t_i^-)\| - V(\mathbf{x}(t_0),t_0) \quad (20)$$

Apart from this, inequality

$$\sum_{i=1}^{N_T}\|\mathbf{z_i^d}\|^2 = \sum_{i=1}^{N_T}\|\mathbf{x_2}(t_i^+)\|^2 \le \sum_{i=1}^{N_T}[2\|\mu_0\|^2] + 2\sum_{i=1}^{N_T}[\|\omega\mathbf{w_d^i}\|^2] \le \gamma^2\sum_{i=1}^{N_T}\|\mathbf{w_d^i}\|^2 + \sum_{i=1}^{N_T}[2\|\mu_0\|^2] \quad (21)$$

is ensured by H1. Thus, combining (20)-(21), one derives

$$\int_{t_0}^{T}\|\mathbf{z}\|^2\mathrm{d}t + \sum_{i=1}^{N_T}\|\mathbf{z_i^d}\|^2 \le V(\mathbf{x}(t_0),t_0) + \sum_{i=1}^{N_T}[2\|\mu_0\|^2] + \gamma^2\left[\int_{t_0}^{T}\|\mathbf{w}\|^2\mathrm{d}t + \sum_{i=1}^{N_T}\|\mathbf{w_d^i}\|^2\right] + \sum_{i=1}^{N_T}[(2L_i^V)\|\mathbf{x}(t_i^-)\|, \quad (22)$$

i.e., the disturbance attenuation inequality (8) is established with

$$\beta_0(\mathbf{x}(t_0),t_0) = V(\mathbf{x}(t_0),t_0), \quad \beta_i(\mathbf{x}(t_i),t_i) = (2L_i^V)\|\mathbf{x}(t_i^-)\| + 2\|\mu_0(\mathbf{x}(t_i^-),t_i)\|^2 \quad (23)$$

with $i = 1,\dots,N$. To complete the proof it remains to establish the asymptotic stability of the undisturbed ($\mathbf{w} = \mathbf{0},\ \ \mathbf{w_d^i} = \mathbf{0},\ \ i = 1,2,\dots$) version of the closed-loop system (1)-(4),(15). In order to do that, we can use [Haddad et al., 2006, Theorem 2.4] specified to the present case with $x_1 = x$ and $x_2 = t$. Indeed, according to this result, Hypothesis H1 and the negative definiteness (16) of the time derivative of the Lyapunov function $V(\mathbf{x},t)$ between the collision time instants ensure that the system is uniformly asymptotically stable. If in addition, Hypotheses H1–H3 hold globally with the radially unbounded function $V(x,t)$ then the results of the theorem hold globally. ■

3.2 Local state-space solution

To present a local solution to the problem in question the underlying system is linearized to

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{B_1}(t)\mathbf{w} + \mathbf{B_2}(t)\mathbf{u}, \quad (24)$$

$$\mathbf{z} = \mathbf{C_1}(t)\mathbf{x} + \mathbf{D_{12}}(t)\mathbf{u}, \quad (25)$$

within impact-free time intervals (t_{i-1},t_i) where t_0 is the initial time instant and $t_i,\ i = 1,2,\dots$ are the collision time instants, whereas $\mathbf{A}(t) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}}$, $\mathbf{B_1}(t) = \mathbf{g_1}(0,t)$, $\mathbf{B_2}(t) = \mathbf{g_2}(0,t)$, $\mathbf{C}(t) = \left.\frac{\partial \mathbf{h}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}}$, $\mathbf{D_{12}}(t) = \mathbf{k_{12}}(0,t)$.

By the time-varying strict bounded real lemma [Orlov and Aguilar, 2014, p.46], the following condition is necessary and sufficient for the linear $\mathcal{H}_\infty$ control problem (24)-(25) to possess a solution: given $\gamma > 0$,

C) there exists a positive constant ε_0 such that the differential Riccati equation

$$-\dot{\mathbf{P}}_\varepsilon(t) = \mathbf{P}_\varepsilon(t)\mathbf{A}(t) + \mathbf{A}^\top(t)\mathbf{P}_\varepsilon(t) + \mathbf{C_1}^\top(t)\mathbf{C_1}(t) + \mathbf{P}_\varepsilon(t)[\frac{1}{\gamma^2}\mathbf{B_1}\mathbf{B_1}^\top - \mathbf{B_2}\mathbf{B_2}^\top](t)\mathbf{P}_\varepsilon(t) + \varepsilon\mathbf{I} \quad (26)$$

has a uniformly bounded symmetric positive definite solution $\mathbf{P}_\varepsilon(t)$ for each $\varepsilon \in (0,\varepsilon_0)$;

As shown below, this condition, if coupled to Hypothesis H1 and a certain monotonicity condition, is also sufficient for a local solution to the nonlinear $\mathcal{H}_\infty$ control problem to exist under unilateral constraints.

Theorem 3.2. Let condition C be satisfied with some $\gamma > 0$. Then Hypothesis H2 hold locally around the equilibrium ($\mathbf{x} = 0$) of the nonlinear system (1)-(4) with

$$V(\mathbf{x}, t) = \mathbf{x}^\top \mathbf{P}_\varepsilon(t)\mathbf{x}, \quad R(\mathbf{x}) = \frac{\varepsilon}{2}\|\mathbf{x}\|^2 \tag{27}$$

and the closed-loop system driven by the state feedback

$$\mathbf{u} = -\mathbf{g_2}(\mathbf{x}, t)^\top \mathbf{P}_\varepsilon(t)\mathbf{x} \tag{28}$$

locally possesses a $\mathcal{L}_2$-gain less than γ provided that Hypothesis H1 holds as well. If in addition, Hypothesis H3 is satisfied with the quadratic function $V(\mathbf{x}, t)$, given in (27), then the disturbance-free closed-loop system (1)-(4), (28) is uniformly asymptotically stable.

Proof. Due to [Orlov and Aguilar, 2014, Theorem 24], Hypothesis H2 locally holds with (27). Then by applying Theorem 3.1, the validity of Theorem 3.2 is concluded.

4. A CASE STUDY: ORBITAL STABILIZATION OF A BIPED ROBOT

4.1 Model of a biped with feet

The bipedal robot considered in this section is walking on a rigid and horizontal surface. It is modeled as a planar biped, which consists of a torso, hips, two legs with knees and feet (see Fig. 1). The walking gait takes place on the sagittal plane, and is composed of single support phases separated by impacts. The complete model of the biped robot consists of two parts: the differential equations describing the dynamics of the robot during the swing phase, and an impulse model of the contact event (the impact between the swing leg and the ground is modeled as a contact between two rigid bodies like that of Chevallereau et al. [2003]).

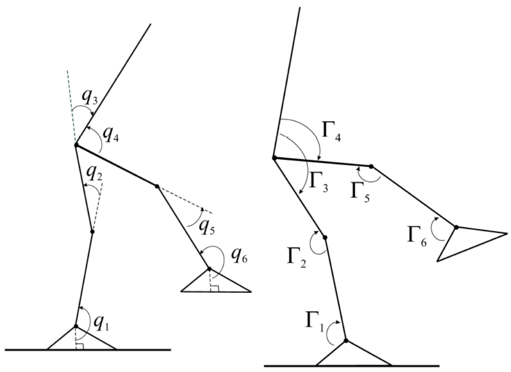

Fig. 1. Seven-link bipedal robot

Dynamic model in a single support In the single support phase, considering a flat foot contact of the stance foot with the ground (i.e. there is no take off, no rotation, and no sliding during this phase), the dynamic model of the biped can be written as follows:

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{H}(\mathbf{q}, \dot{\mathbf{q}}) = \mathbf{D_\Gamma}\mathbf{\Gamma} + \mathbf{w_1} \tag{29}$$

with $\mathbf{q} = (q_1, q_2, q_3, q_4, q_5, q_6)^\top$ the 6×1 vector of generalized coordinates, $\mathbf{D}$ is the symmetric, positive definite 6×6 inertia matrix, $\mathbf{D_\Gamma}$ is a 6×6 constant and nonsingular matrix; $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6)^\top$ is the 6×1 vector of joint torques (see Fig. 1); the term $\mathbf{H}(\mathbf{q}, \dot{\mathbf{q}})$ is the 6×1 vector of the centrifugal, coriolis and gravity forces; and $\mathbf{w_1}$ is the 6×1 vector of external disturbances.

Impact model Now, assuming a flat foot contact, the double support phase is instantaneous and it can be modeled through passive impact equations, *i.e.* impulsive torques are applied in the interlink joints (Formalskii [2009]). An impact appears at a time $t = T_I$ when the swing leg touches the ground. We shall assume that the impact is passive, absolutely inelastic, and that the legs do not slip (Tlalolini et al. [2010]). Given these conditions, the ground reactions can be viewed as impulsive forces. The algebraic equations, allowing one to compute the jumps of the velocities, can be obtained through integration of the dynamic equations of the motion, taking into account the ground reactions during an infinitesimal time interval from T_I^- to T_I^+ around an instantaneous impact. The torques supplied by the actuators at the joints, the centrifugal, Coriolis and gravity forces have finite values, thus not influencing an impact.

The impact is assumed to be with complete surface of the foot sole touching the ground. This means that the velocity of the swing foot impacting the ground is zero after impact. After an impact, the right foot (previous stance foot) takes off the ground, so the vertical component of the velocity of the taking-off foot just after an impact must be directed upwards and the impulsive ground reaction in this foot equals zeros. Thus, the impact dynamic model can be represented in the form (Haq et al. [2012]):

$$\dot{\mathbf{q}}^+ = \phi(\mathbf{q})\dot{\mathbf{q}}^- + \mathbf{w_d} \tag{30}$$

where $\dot{\mathbf{q}}^-$ is the velocity of the robot before the impact and $\dot{\mathbf{q}}^+$ is the velocity after the impact; $\phi(\mathbf{q})$ represents a restitution law that determines the relations between the velocities before and after the impacts; $\mathbf{q}$ is the position at the impact. The additive term $\mathbf{w_d}$ is introduced to account for inadequacies in this restitution law.

The unilateral constraint can be defined as $\mathbf{F}(\mathbf{q})$, which represents the height of swing foot, as a function of the generalized coordinates of the implicit-contact model (29). In the next section, a specific trajectory invoked to generate a cyclic motion of the undisturbed model (29)-(30), is designed so it can be used in our tracking problem as a reference trajectory.

4.2 Motion Planning

The walking gait, which is composed of single support phases and impacts, has been defined by $\mathbf{q_d}(t)$ and $\dot{\mathbf{q}}_\mathbf{d}(t)$ satisfying the conditions of contact using an off-line optimization (Haq et al. [2012]).

The control task is in driving the biped in such a manner that each joint angle follows its own reference trajectory. The reference walking minimizes the integral of the norm of the torque vector for a given distance. The walking velocity is selected to be $0.5\ m/s$. The duration of one step is $0.53\ s$. Since the impact is instantaneous and passive, the control law is defined only during the single support phase. The restitution law during the impact phase is given by:

$$\dot{\mathbf{q}}_\mathbf{d}(t_k^+) = \phi(\mathbf{q_d}(t_k))\dot{\mathbf{q}}_\mathbf{d}(t_k^-), \quad k = 1, 2, \dots . \tag{31}$$

4.3 Pre-feedback desing

Our objective is to design a pre-feedback controller of the form

$$\mathbf{\Gamma} = \mathbf{D_\Gamma}^{-1}[\mathbf{D}(\ddot{\mathbf{q}}_\mathbf{d} + \mathbf{u}) + \mathbf{H}] \tag{32}$$

that imposes on the undisturbed biped motion desired stability properties around $\mathbf{q_d}$ while also locally attenuating the effect of the disturbances. Thus, the controller to be constructed consists of the feedback linearizing terms of (32) subject to $u = 0$, which are responsible for the trajectory compensation, and a disturbance attenuator $\mathbf{u}$, internally stabilizing the closed-loop system around the desired trajectory. In what follows, we confine our research to the trajectory tracking control problem where the output to be controlled is given by

$$\mathbf{z} = \begin{bmatrix} \mathbf{0} \\ \rho_p(\mathbf{q_d} - \mathbf{q}) \\ \rho_v(\dot{\mathbf{q}}_\mathbf{d} - \dot{\mathbf{q}}) \end{bmatrix} + \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \mathbf{u} \tag{33}$$

$$\mathbf{z_d} = \mathbf{q_d}(t_k^+) - \mathbf{q}(t_k^+) \tag{34}$$

with positive weight coefficients ρ_p, ρ_v.

Now, let us introduce the state deviation vector $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)^\top$, where $\mathbf{x_1}(t) = \mathbf{q_d}(t) - \mathbf{q}(t)$ is the position deviation from the desired trajectory, and $\mathbf{x_2}(t) = \dot{\mathbf{q}}_d(t) - \dot{\mathbf{q}}(t)$ is the velocity deviation from the desired velocity.

Then, rewriting the state equations (29)-(34) in terms of the errors $\mathbf{x_1}$ and $\mathbf{x_2}$, we obtain an error system in the form (1)-(4), being specified with

$$\mathbf{f}(\mathbf{x}, t) = \begin{bmatrix} \mathbf{x}_2 \\ \mathbf{0} \end{bmatrix}, \mathbf{g_1}(\mathbf{x}, t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{D^{-1}}(\mathbf{q_d} - \mathbf{x_1}) \end{bmatrix}, \tag{35}$$

$$\mathbf{g_2}(\mathbf{x}, t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{1} \end{bmatrix}, \mathbf{h}(\mathbf{x}) = \begin{bmatrix} \mathbf{0} \\ \rho_p \mathbf{x}_1 \\ \rho_v \mathbf{x}_2 \end{bmatrix}, \mathbf{k_{12}}(\mathbf{x}) = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{36}$$

$$\mu(\mathbf{x}, t) = \begin{bmatrix} \mathbf{x_1} \\ \phi(\mathbf{q_d})\dot{\mathbf{q}}_\mathbf{d} - \phi(\mathbf{q_d} - \mathbf{x_1})(\dot{\mathbf{q}}_\mathbf{d} - \mathbf{x_2}) \end{bmatrix}, \tag{37}$$

$$\omega(\mathbf{x}, t) = -\mathbf{I} \tag{38}$$

where as a matter of fact, the zero symbols and unit ones, among the matrix entries, stand for zero and identity matrices of appropriate dimensions.

4.4 $\mathcal{H}_\infty$-Control Synthesis

In this section, we involve the results of Sect. 3.2 to robustly track the time-reference trajectory $\mathbf{q_d}(t)$, constructed in Sect 4.2 for the error system given in terms of the angular deviations of the biped from the reference.

The parameters in (33), required to design the Riccati equation (26), were selected by specifying certain values of ρ_p and ρ_v. Then by iterating on γ in (26), a minimum value γ_{min} was found, such that (26) had a periodic, symmetric and positive definite solution. The value of γ to subsequently be used in simulations was chosen slightly bigger than the minimum one for avoiding a high gain controller design. The controller parameters, solving (26) were thus selected as $\epsilon = 0.01$, $\rho_p = 500$ and $\rho_v = 1$, and $\gamma = 1.45$. That value of γ was straightforwardly verified to meet Hypothesis H1 with ω being an identity matrix.

In order to find a uniformly bounded solution of the differential Riccati equation (26), variables t and $\dot{\mathbf{P}}_\varepsilon$ were set to zero, and the resulting algebraic Riccati equation was solved to specify an initial condition $\mathbf{P}_\varepsilon^0$ of (26) under which we were able to numerically arrive at a periodic positive definite symmetric solution of (26). Hypothesis H3 was also numerically verified in simulations.

Thus, Theorem 3.2 proved to be applicable to the error system (1)-(4), specified with (35)-(38). By applying Theorem 3.2, the control law (15) was carried out to render a local solution to the robust tracking problem for the biped with the desired trajectory $\mathbf{q_d}(t)$ to follow.

4.5 Numerical study

The model (29)-(30) is used in this section to show numerical simulations of a stable walking gait by achieving robust tracking via the $\mathcal{H}_\infty$-controller designed in Sect. 4. The biped parameters are taken from (Haq et al. [2012]). The contact constraints presented in section 4.1, are verified online in order to confirm the validity of (29)-(30). The robustness of the tracking control (15) is verified by introducing a resultant disturbance force $F_{xw} = 80\ N$ in the horizontal plane, applied to the hip of the robot. Such a force is used for the duration of $0.07\ s$ to simulate a disturbance effect. The effect of F_{xw} represents a disturbance in the continuous phase of the dynamics (29) as it starts from $0.8\ s$ in the first cycle of the biped which belongs to the continuous phase of the trajectory.

For the impact phase, instead of using equation (30), the contact with the ground is stated as a linear complementary constraint problem (Rengifo et al. [2011]). This approach belongs to the family of time-stepping approaches. Let the vector $\mathbf{R} \in \mathbb{R}^4$ be the reaction force vector, which is obtained by stacking the reaction force vectors of the two edges of each foot. Vector $\mathbf{R}^k$ at $t = t_k$ is expressed at each sampling period as a function of the augmented generalized position vector $\mathbf{q}^k(9 \times 1)$ composed of the variable orientation of each link and the Cartesian coordinates x, y of the hips, the associate velocity vector $\dot{\mathbf{q}}^k(9 \times 1)$ for the biped and $\mathbf{\Gamma}^k$ with an algebraic equation $G(\mathbf{R}^k, \mathbf{q}^k, \dot{\mathbf{q}}^k), \mathbf{\Gamma}^k) = 0$. Let vector v^{k+1} be the Cartesian velocities of the corners in contact with the ground at $t = t_k$. The normal components must be non negative to avoid interpenetration. The identity $v^{k+1} = 0$ means that the contact remains and the inequality $v^{k+1} > 0$ means that the contact vanishes. The normal components $r^k > 0$ of $\mathbf{R}^k$, when contact occurs, are also subject to non negative constraints. These components can avoid interpenetration but they cannot avoid the stance foot take-off. It is clear that the variables v^{k+1} and r^k are complementarity quantities ($v^{k+1} \geq 0 \perp r^k \geq 0$). Furthermore, the variables v^{k+1} and r^k are subject to constraints imposed by friction which leads to a linear complementarity condition. The valid cases of contact for each edge can be determined using constrained optimization (Rengifo et al. [2011]). The difference between these two methods for solving the contact (the above mentioned complementarity approach and equation (30)) represents a discrete disturbance in our simulation runs.

Figure 2 shows the heights of the feet for six consecutive steps. The corresponding velocities of the feet in vertical direction are depicted in Fig. 3. Legends "P1" and "P3" represent the "toe" of the right and left foot, respectively.

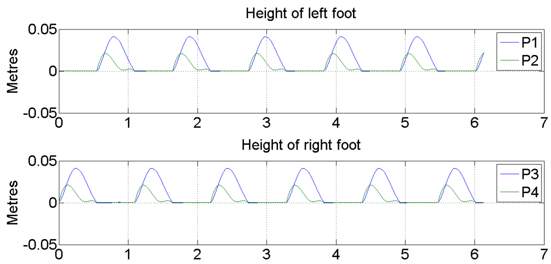

Fig. 2. Feet height in the walking gait

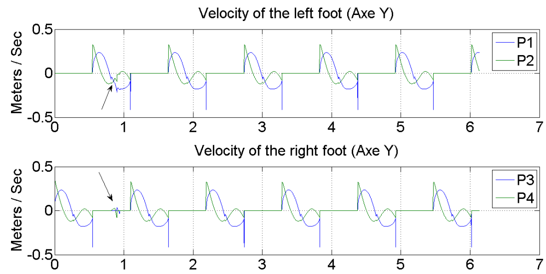

Fig. 3. Feet velocities in vertical direction in the walking gait

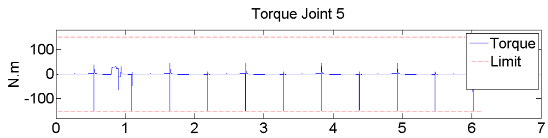

Fig. 4. Torque appearing in joint 5, where the effect of the disturbance is more evident

Similarly, "P2" and "P4" represent the "heel" of the right and left foot, respectively. The disturbance effect and its attenuation can readily be concluded from Fig. 3 where impact instants are pointed out by arrows, as well as from the torque 5, presented in Fig. 4, where one can see that the torque remains between the actuator limitations ($+/-100\ Nm$). Clearly, the biped returns to its desired gait after the discrete disturbance disappears.

5. CONCLUSION

In this paper, the state feedback $\mathcal{H}_\infty$-control problem for orbital stabilization of n-DOF, fully actuated mechanical systems subject to unilateral constraints is solved. Sufficient conditions for the existence of a global (local) solution of the tracking problem in question are carried out in terms of two coupled inequalities: a standard Hamilton-Jacobi-Isaacs inequality (perturbed differential Riccati equation) for the continuous dynamics, and a novel inequality, imposed on the corresponding solutions of the Hamilton-Jacobi-Isaacs inequality (perturbed differential Riccati equation) at the impact time instants. The proposed robust synthesis constitutes the contribution the paper makes to the existing literature. Effectiveness of the resulting design procedure is supported by numerical tests on a seven-link biped, exhibiting the desired disturbance attenuation in the presence of disturbances in the single support phase and uncertainty in the impact phase.

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