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Solution of Static Reduced Decoupling Problem for Linear Systems

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Abstract: We propose a structural solution by non-regular static state feedback to the diagonal, or row by row decoupling problem for linear systems. Without being completely general, this solution concerns the Reduced Morgan's Problem, that is we do not want it increases the essential orders of the original system. The solution that we propose relies on properties between, on one hand, some partial infinite structures extracted from well chosen interactors and on the other hand, the controllability indices of a specific shifted system. To my knowledge, there was at this date only very partial solutions to this problem deemed structurally hard.

Keywords: linear systems, non regular control, state feedback, diagonal decoupling, structural properties.

1. INTRODUCTION

The diagonal decoupling of linear systems by static state feedbacks $u = Fx + Gv$, or Morgan's Problem, is a very difficult control problem when G should be not invertible. Yet such non regular static solutions can exist even when there is none regular. Such non regular feedbacks can transform all the structures underlying the regular solutions to control problems. The challenge is to find new structures that will allow us to solve the decoupling Problem. The non-regular decoupling by dynamic state feedback was solved by Dion and Commault [1988] due to new invariants, the essential orders (E.O.). Now, non regular decoupling always comes down to a problem of increasing structures. The relative simplicity of the dynamic case comes from the facts that the E.O. are the minimal infinite structure to achieve for decoupling and that it is always possible to get it if there is a solution. To make this increase, dynamic feedbacks use integrators that are external to the system. This is no more the case for static feedbacks that must only use internal dynamics. We consider here the Static Reduced Morgan's Problem (SRMP), say the static decoupling without increasing the E.O.: it provides insight into the complex mechanisms of structural changes by non-regular controls and, from the practical point of view, this restricted problem is not without interest: indeed, if a static solution exists with an increasing of the E.O., there always exists a dynamic one without this increasing od E.O.. There are so far only very partial results for the SRMP: when it is sufficient to increase only one element of the infinite structure (Herrera H and Lafay [1993]), or for trivial internal structures, (Zagalak et al. [1998]),(Lafay [2013]). The specific difficulties of SRMP are of two kinds: firstly, the increases of infinite structure for solving SRMP depend on the order of the outputs of the system, while the sum of the sizes of these increases does not depend of this order. This lock has been lift in Lafay [2013], where it is proved that there is a unique "minimal list of decoupling indices" such that if SRMP has a static solution, one solution exists with this list. Secondly we must take into account internal couplings which are unobservable from the outputs to be decoupled. For that, we develop a non trivial general formulation of SRMP inspired by (Herrera H and Lafay [1993]). The general solution of SRMP relies on the controllability of a well defined shifted system. We propose here only the structural aspects of decoupling, without considerations on the internal stability.

2. NOTATIONS AND BACKGROUND

2.1 Notations

Let $\Sigma(C, A, B)$, denoted shortly Σ, a linear system whose state is supposed to be measured or reconstructible:

$$\Sigma \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t). \end{cases} \tag{1}$$

$x \in \mathcal{X} \subset \mathbb{R}_n, u \in \mathcal{U} \subset \mathbb{R}_m$ and $y \in \mathcal{Y} \subset \mathbb{R}_p$. B is monic and C epic. Σ is controllable and right invertible. $\mathcal{V}^*$ and $\mathcal{R}^*$ are the supremal (A, B)invariant and the supremal controllability subspaces in the kernel of C. As the structure of $\mathcal{V}^*/\mathcal{R}^*$ does not affect the decouplability by static state feedbacks (cf the Morse's canonical form Morse [1973]) we can assume without any loss of generality that $\mathcal{V}^*{=}\mathcal{R}^*$, say that Σ has no finite zeros (they concern only internal stability of the state, which is not addressed here). A set of p elements is noted $\{\bullet\}_p$. Polynomial and rational functions in variable s are respectively noted $\bullet[s]$ and $\bullet(s)$. We note $\partial p[s]$ the degree of $p[s]$ and $\partial_{c_i} M[s]$ the highest degree of the ith column of matrix M[s]. $f^{(n)}(t)$ is the derivative of order n of $f(t)$. $\mathbb{I}_n$ is the identity matrix of order n and $diag\{a_i\}_p$ the $p \times p$ matrix which diagonal terms are a_i, for $i \leq p$.

Definition 1. The interactor of Σ,(Wolovich and Falb [1976]), is the unique $p \times p$ triangular and non-singular polynomial matrix $\Phi[s] = [\varphi_{i,j}[s]]$ such that there exists a biproper $m \times m$ (non unique) matrix $B(s)$ satisfying : $T(s) = C(sI_n - A)^{-1}B = \left[\Phi^{-1}0\right] B(s)$ where:

- $\varphi_{i,i} = s^{d_i}$, $i = 1, ..., p$, d_i being positive integers
- $\varphi_{i,j}$ is zero or $\varphi_{i,j}/s^{d_j}$ is divisible by s, $\forall i > j$.

$\Phi[s]$ is invariant under the action of the group (T, F, G), T and G being changing of bases on $\mathcal{X}$ and $\mathcal{U}$, and F is a state feedback. Without any loss of generality, we can suppose that Φ is row reduced (r.r.), (Lafay et al. [1992]). Four lists of integers characterize partially the structure of Σ, (see Morse [1973],Commault et al. [1986]):
- $\{c_i\}_m$: controllability indices of (A, B) ,
- $\{n'_i\}_p$: I_4 Morse's list of the orders of the zeroes at infinity. If the interactor of Σ is r.r., $d_i = n'_i, i \leq p$.
- $\{\sigma_j\}_{m-p}$: I_2 Morse's list providing the structure of $\mathcal{R}^*$,
- $\{n_{ie}\}_p$: the p E.O. of the outputs $y_i(t)$: $n_{ie} = \partial_{c_i}\Phi[s]$, $i \leq p$, .

These lists are invariant under the action of the group (T, F, G, Π), where Π is a permutation of the outputs of Σ and G is invertible (Herrera H et al. [1997]).
An extended system $\Sigma^e(C_e, A, B)$ is defined in Herrera H et al. [1997] by adding $m - p$ fictitious outputs, to reflect the structure of $\mathcal{R}^*$.

Proposition 1. The $m \times m$ interactor of Σ^e, called "extended interactor of Σ", has the structure:

$$\Phi^e[s] = \begin{bmatrix} \Phi_1^e[s] & (0) \\ \Phi_2^e[s] & \Phi_3^e[s] \end{bmatrix} = \left[\varphi^e{}_{i,j}[s]\right], where \qquad (2)$$

- $\Phi_1^e[s] = \Phi[s]$, then if Φ is r.r., $\varphi_{i,i}^e = s^{n'_i}, i = \leq p$,
- $\Phi_3^e[s] = diag\{s^{\sigma^i}\}_{m-p}$,
- polynomials $\left[\varphi^e{}_{2i,j}[s]\right] = \sum_{r=l}^{h} \alpha_r s^r$ of $\Phi_2^e[s]$ are zero or verify: $\partial\varphi_{1j,j}^e + 1 \leq l$ and $h \leq \sigma_i - 1$.

Note $\Phi[s] = W(s)diag\{s^{n_{ie}}\}_p$. $W(s)$ is a proper matrix, of rank at infinity k, called "proper part of $\Phi[s]$", and:

Proposition 2. (Dion and Commault [1988]) There exists output's permutation(s) Π such that the $p \times p$ interactor Φ_Π of $\Sigma_\Pi(\Pi C, A, B)$ has the structure:

$$\Phi_\Pi = \begin{bmatrix} \Phi_{\Pi,1}[s] & (0) \\ \Phi_{\Pi,2}[s] & \Phi_{\Pi,3}[s] \end{bmatrix} = \left[\varphi_{\Pi i,j}[s]\right], where \qquad (3)$$

- $\Phi_{\Pi,3}[s] = diag\{s^{n_{je}}\}_k$, n_{je} being the E.O. of the k last outputs of Σ_Π,
- Nonzero infinite zeros of $W(s)$ are given by $\{\delta_i\}_{p-k} = \{n_{ie} - f_i\}_{p-k}$, where $f_i = \partial\varphi_{\Pi i,i}[s]$, $i \leq p - k$.
- $\{\delta_i\}_{p-k}$ is called: list of decoupling indices of Σ_π.
- There are always permutations Π_m such that $\{\delta_i\}_{p-k}$ is the unique *minimal list* $\{\Delta_i\}_{p-k}$ of decoupling indices of Σ.

$\{\Delta_i\}_{p-k}$ is defined in (Lafay [2013]) using the notion of "minor" list:

Definition 2. Let two lists of integers $\{\delta_i\}_{k1}$ and $\{\gamma_i\}_{k2}$ such that $\sum_{i=1}^{k1} \delta_i = \sum_{i=1}^{k2} \gamma_i$. Note $\{\hat{\delta}_i\}_{sup\delta_i}$ and $\{\hat{\gamma}_i\}_{sup\gamma_i}$ their dual lists. $\{\delta_i\}_{k1}$ "minor" the list $\{\gamma_i\}_{k2}$ if, for $i = 1, .., \text{sup}(\text{sup}_j(\delta_j), \text{sup}_j(\gamma_j))$, we have:

$$\sum_{j=1}^{i} \hat{\delta}_j \leq \sum_{j=1}^{i} \hat{\gamma}_j. \qquad (4)$$

2.2 Impact of a permutation Π on the extended interactor

Assume $\Phi^e[s]$ as in (2).

Property 1. Let Π any permutation of the outputs of Σ. The extended interactor of Σ_Π is given by

$$\Phi_\Pi^e[s] = \begin{bmatrix} \Phi_{\Pi,1}^e[s] & (0) \\ \Phi_{\Pi,2}^e[s] & \Phi_{\Pi,3}^e[s] \end{bmatrix}, where: \qquad (5)$$

- $\Phi_{\Pi,1}^e[s] = \Phi_\Pi[s] = B_1(s)\Phi[s]\Pi^{-1}$, where $\Phi_\Pi[s]$ is the interactor of Σ_Π and $B_1(s)$ is a biproper $p \times p$ matrix,
- $\Phi_{\Pi,2}^e[s] = B_2(s)\Phi_{2e}[s]\Pi^{-1}$, where $B_2(s)$ is a proper $(m - p) \times p$ matrix, which can add polynomials to whose of $\Phi^e{}_2[s]$ but, in each column j, these polynomials are all of degree less than or equal to n_{je}.
- $\Phi_{\Pi,3}^e[s] = \Phi_3^e[s]$.

The proof is not given here for sake of shortness.

Remark 1. Let us consider (2) and (5). If, for $j \leq p - k$, one keeps in the jth columns of $\Phi_2^e[s]$ and $\Phi_{\Pi,2}^e[s]$ only monomials of degree higher than n_{je}, then the "truncated" matrices $\Phi_2^{e,t}[s]$ and $\Phi_{\Pi,2}^{e,t}[s]$ satisfy $\Phi_{\Pi,2}^{e,t}[s] = \Phi_2^{e,t}[s]\Pi^{-1}$.

2.3 On the model of the system

All the changes made so far on the transfer matrix of Σ correspond to left biproper operations which can be globally realized by a regular static state feedback (F_m, G_m), (Hautus and Heymann [1978]): they did not affect the static decouplability of Σ and we can suppose that the transfer matrix $T_m(s)$ of Σ_m is $\Phi_m{}^{-1}(s)$. We note $\Sigma_m(C_m, A_m, B_m)$, where $A_m = A + BF_m$, $B_m = BG_m$ and $C_m = \Pi_m C$.

In the following it will be assumed, unless explicitly mentioned, that $\Sigma = \Sigma_m$, (Proposition 2). Then,

$$\Phi_m^e[s] = \left[\varphi_{mi,j}^e[s]\right] = \left[\begin{array}{c|c|c} \Phi_{m1,1}^e & (0) & (0) \\ \hline \Phi_{m1,2}^e & \Phi_{m1,3}^e & (0) \\ \hline \Phi_{m2,1}^e & \Phi_{m2,2}^e & \Phi_{m2,3}^e \end{array}\right] \qquad (6)$$

where the $p \times p$ interactor of Σ_m is:

$$\Phi_m[s] = \left[\begin{array}{c|c} \Phi_{m1,1}^e & (0) \\ \hline \Phi_{m1,2}^e & \Phi_{m1,3}^e \end{array}\right] \qquad (7)$$

- $\Phi_{m1,3}^e = diag\{n_{ie}\}_k$.
- $\Phi_{m1,2}^e$is a $p - k \times k$ matrix with $\partial_{c_i}\Phi_{m1,2}^e = n_{ie}$,
- $\Phi_{m1,1}^e$ is a $p - k \times p - k$ matrix with: $\varphi_{mi,i}^e = s^{f_i}$, $\varphi_{mi,j}^e = 0$ for $j > i$ and, for $i < j$, the non null polynomials $\varphi_{mi,j}^e$ are such that $f_j + 1 \leq \partial_{min}\, \varphi_{m\,i,j}^e$ and $\partial_{max}\, \varphi_{m\,i,j}^e \leq n_{je} - 1$,
- the list $\{n_{ie} - f_i\}_{p-k} = \{\Delta_i\}_{p-k}$,
- $\Phi_{m2,3}^e = diag\{s^{\sigma^i}\}_{m-p}$,
- for $j \leq p$, polynomials of $\Phi_{m2}^e = \left[\,\Phi_{m2,1}^e \middle| \Phi_{m2,2}^e\,\right]$, which are non null, are such that $f_j + 1 \leq \partial_{min}\varphi_{mi,j}^e$ and $\partial_{max}\, \varphi_{m\,i,j}^e < \sigma_i$ but it can be greater than n_{je}.

3. MORGAN'S PROBLEM

3.1 Formulation of the problem and consequences

The static diagonal decoupling *without stability* of Σ, or Static Morgan's Problem, can be stated as follows: Given Σ described by (1), does it exists a static state feedback $u = Fx + Gv = Fx + G_1v_1 + ... + G_pv_p, v_i \in \mathcal{U}$ and $G_i \in \mathbb{R}^{m\times 1}$ such that, for any $i \in 1, 2, ..., p$, v_i controls the scalar output y_i without affecting the $p-1$ other outputs y_j? If such a feedback exists, the transfer matrix $T_{FG}(s)$ of $\Sigma(C, A+BF, BG)$ is $T_{FG}(s) = diag\{s^{-r_1}, ..., s^{-r_p}\}, r_i \in \mathbb{R}$. If G is regular,this problem has a solution if and only if the ordered lists $\{n'_i\}_p$ and $\{n_{ie}\}_p$ are the same. So, the interactor of Σ is diagonal (Commault et al. [1986]). If $\{n'_i\}_p \neq \{n_{ie}\}_p$, Σ will be decouplable if and only if it is possible to increase the structure at infinity so that it matches to (new) E.O.. This can only be achieved by a non regular feedback, say with a loss of inputs. With such controls, the list of E.O. of Σ is not always the minimal infinite structure to reach for the static decoupling.
Static Restricted Morgan's Problem, (SRMP), is the particular case of Static Morgan's Problem where the E.O. of Σ *should be* the infinite structure of the decoupled system. Finally, as any non regular decoupling reduces to increase infinite structure, the solution will be based on a Theorem of Loiseau [1988] that we remind now:

Theorem 1. Let a linear system, $\{n'_i\}$ it's infinite structure and $\{\sigma_i\}$ it's I_2 Morse's list. Let $\{p_i\}$ a list of integers. Note $\{v_i\}$, $\{\alpha_i\}$ and $\{\pi_i\}$ the dual lists of, respectively, $\{n'_i\}$, $\{\sigma_i\}$ and $\{p_i\}$. Let $\{\Gamma_i\}$ obtained by arranging the differences $(\pi_i - v_i)$ in a non increasing order. Then, there exists a static state feedback such that the structure at infinity of the closed loop system is the list $\{p_i\}$ if and only if:

$$v_1 - v_i \geq \pi_1 - \pi_i, \forall i \geq 1, \tag{8}$$

$$\sum_{i=1}^{j} \alpha_i \geq \sum_{i=1}^{j} \Gamma_i, \forall j \geq 1. \tag{9}$$

Let us consider now the dynamic solution of decoupling:

3.2 The Dynamic Morgan's Problem (DMP)

Proposition 3. (Dion and Commault [1988]). DMP is solvable if and only if Σ is right invertible and $m - p \geq p - k$, k being the rank at infinity of $W(s)$. E.O. can always be unmodified and such minimal solutions need $\sum_{i=1}^{p} n_{ie} - \sum_{i=1}^{p} n'_i$ external integrators.

The dynamic solution puts Σ in the form Σ_Π (3), but $\{\delta_i\}_{p-k} = \{n_{ie} - f_i\}_{p-k}$ is not necessarily the list of *minimal* decoupling indices. DMP amounts to annihilating $\{\delta_i\}_{p-k}$ by the following iterative procedure for $i \leq p-k$:
- for $i = 1$, u_1 is replaced by an *external* chain of δ_1 integrators controlled by an entry of $\mathcal{R}^*$, for instance $v_1 = u_{p+1}$. This chain is independent of the $(m-p)$ chains of σ_i integrators of $\mathcal{R}^*$. This substitution amounts to multiply the first row of (3) by s^{δ_1}. So $\partial\varphi_{\Pi 1,1}$ becomes n_{1e} and it is possible, by a left biproper operation, to eliminate all the other polynomials of the first column of Φ_Π .
- we made the same operation for the other rows of $\Phi_{\Pi,1}$, taking at each step a new entry v_i of $\mathcal{R}^*$, which is possible as $m - p \geq p - k$. The final interactor is $diag\{s^{n_{ie}}\}_p$ and the system with entries $\{v_1, v_2, ..., v_{p-k}, u_{p-k+1}, ..., u_p\}$ is regularly decouplable. So we need $p - k$ independent chains of integrators coming from a dynamic extension of dimension $\sum_{i=1}^{p} n_{ie} - \sum_{i=1}^{p} n'_i$.
Note that the right invertibility or Σ expresses the independence of it's outputs. So this property will be also necessary for the SRPM.

4. THE SOLUTION OF SRMP

Let Σ_m as in Subsection 2.3. We note $B_m = \left[\, B_b \middle| B_s \middle| B_r \,\right]$ and $u(t) = \left[\, u_b(t) \middle| u_S(t) \middle| u_r(t) \,\right]^T$, where $B_b = [b_1, ..., b_{p-k}]$, $B_s = [b_{p-k+1}, ..., b_p]$ and $B_r = \mathcal{B} \cap \mathcal{R}^* = [b_{p+1}, ..., b_m]$. Define the $m \times m$ polynomial matrix $\Phi_m^{S,e}[s]$ by:

$$\Phi_m^{S,e}[s] = \begin{bmatrix} \Phi_{m1}^{S,e}[s] & (0) \\ \Phi_{m2}^{S,e}[s] & \Phi_{m3}^{S,e}[s] \end{bmatrix}, where \tag{10}$$

- $\Phi_{m1}^{S,e}[s] = diag\{s^{n_{ie}}\}_p$
- $\Phi_{m3}^{S,e}[s] = diag\{s^{\sigma^i}\}_{m-p}$
- $\Phi_{m2}^{S,e}[s]$ comes from $\Phi_{m2}^{e}[s] = \left[\, \Phi_{m2,1}^{e} \middle| \Phi_{m2,2}^{e} \,\right]$ by eliminating, in each column $i \leq p$, monomials of degree less than or equal to n_{ie} by a left biproper transformation.
- $\Phi_m^{S,e}[s]$ has the structure of an interactor.

4.1 Shifted system

Definition 3. The shifted system $\Sigma_m^S(C_m^S, A_m^S, B_m^S)$ associated with Σ_m is the invertible system of which $\Phi_m^{S,e}[s]$, equation (10), is the extended interactor.

Extend $\mathcal{X}$ by an dynamic extension $\mathcal{X}_a$, of dimension $n_a = \sum_{i=1}^{p-k} \Delta_i$, composed of $p - k$ independent controllable and observable chains of integrators of lengths $\Delta_1, ..., \Delta_{p-k}$ with entries $w_1(t), ..., w_{p-k}(t)$ in $\mathcal{U}_a$ and outputs $z_1(t), ..., z_{p-k}(t)$ in $\mathcal{Y}_a$. The state, control and output spaces of Σ_m complemented by this extension are $\mathcal{X}_S = \mathcal{X} \oplus \mathcal{X}_a$, $\mathcal{U}_S = \mathcal{U} \oplus \mathcal{U}_a$ and $\mathcal{Y}_S = \mathcal{Y} \oplus \mathcal{Y}_a$.
The shifted system is obtained by replacing the entries $u_b(t)$ of B_b by $z_i(t)$ for $i \leq p-k$. Then:

$$A_m^S = \left[\begin{array}{c|c|c|c} A_m & A_1^S & \ldots & A_{p-k}^S \\ \hline (0) & J_{\Delta_1} & (0) & (0) \\ \hline & & \ddots & \\ (0) & (0) & \ddots & (0) \\ \hline (0) & \ldots & (0) & J_{\Delta_{p-k}} \end{array}\right], where \tag{11}$$

J_{Δ_i} are upper $\Delta_i \times \Delta_i$ Jordan's blocks and the $n \times \Delta_i$ matrices $A_i^S = \left[\, b_{i(n\times 1)} \middle| (0)_{n\times\Delta_i - 1} \,\right]$.

$$B_m^S = \left[\, B_a^S \middle| B_s^S \middle| B_r^S \,\right] =$$

$$\left[\begin{array}{c|c|c} (0)_{(n\times p-k)} & B_{s(n\times k)} & B_{r(n\times m-p)} \\ \hline B_{a(n_a\times p-k)} & (0) & (0) \end{array}\right], where \tag{12}$$

$B_a = diag\{b_{a,i}\}_{p-k}$, $b_{a,i} = [0 \ldots 0\ 1]^T$ being $\Delta_i \times 1$ vectors.

$$C_m^S = \left[\, C_{m(p\times m)} \middle| (0)_{(p\times n_a)} \,\right]. \tag{13}$$

Then,(cf Subsection 3.2), the interactor of Σ_m^S is given by

$$\Phi_m^S[s] = \left[\, diag\{s^{n_{ie}}\}_p \,\right], \tag{14}$$

and the corresponding extended interactor of Σ_m^S is:

$$\Phi_m^{S,e}[s] = \left[\begin{array}{c|c} diag\{s^{n_{ie}}\}_p & (0)_{p\times m-p} \\ \hline \Phi_{m,2}^{S,e} & diag\{s^{\sigma_i}\}_{m-p} \end{array}\right], where \quad (15)$$

$\Phi_{m2}^{S,e}[s]$ comes from $\Phi_{m2}^{e}[s]$ by eliminating, in each column $j \leq p$, monomials of degree less than or equal to n_{je}.

4.2 A convenient formulation of SRMP

SRMP will be solved if it is possible to replace the $p-k$ independent external chains of DMP by $p-k$ independent chains extracted from Σ. A chain of length $\mathcal{L}$ is actived only at its beginning by a function $f_a(t)$ and generate $f_e(t)$, i.e. $f_e^{(\mathcal{L})}(t) = f_a(t)$. Let a set of q chains of integrators defined by $f_e^{i(\mathcal{L}_i)}(t) = f_a^i(t), i \leq q$. For SRMP, we require that each function $f_a^i(t)$ contains at least one input of Σ. These chains are independent if, $\forall j \neq i$, the term containing the inputs in $f_a^i(t)$ is not a linear combination (l.c.) of corresponding terms in $f_a^j(t)$, and if $f_e^i(t)$ is not a l.c. of $f_e^j(t)$. The following Lemma characterizes internal chains that will increase the infinite structure without changing the E.O.: it is a new formulation for SRMP, generalizing in a non trivial way Lemma 4.2 in (Herrera H and Lafay [1993])valid when $k = p-1$.

Lemma 1. Let a right invertible system Σ_m with $\mathcal{R}^* = \mathcal{V}^*$, k the rank of $W(s)$ and $\left\{\Delta^i\right\}_{(p-k)}$ its decouplability indices (minimal decoupling indices of Σ). Then, SRMP has a solution if and only if it is possible to extract, from $\mathcal{R}^*$ $p-k$, independent chains of integrators of lengths $\Delta_1, ..., \Delta_{p-k}$ described by $f_e^i(t)^{(\Delta_i)} = f_a^i(t), i \leq p-k$ such that:

- (a) The output $f_e^i(t)$ of each chain is only function of $x(t)$ and these $p-k$ functions are independent,
- (b) For $i \leq p-k$, the entry $f_a^i(t)$ of each chain does not contain derivatives of $u_j(t), j = p-k+1, ..., p$.
- (c) For $i \leq p-k$, the entry $f_a^i(t)$ of each chain does not contain derivatives of inputs $u_j(t)$ of order greather than or equal to Δ_j, $j \leq p-k$.

Proof 1. **IF**. Let a state feedback (F, G) which decouples Σ_m. From Dion and Commault [1988] this feedback is equivalent to the precompensator $C(s) = (\mathbb{I}_m - F(s\mathbb{I}_n - A_m)^{-1}B_m)^{-1}G = \begin{bmatrix} W_{1,m}(s) \\ X(s) \end{bmatrix}$, where $X(s)$ is a proper $m-p \times p$ matrix and, noting $d = s^{-1}$, the $p \times p$ proper part $W_{1,m}(s)$ of the interactor of Σ_m can be written as:

$$W_{1,m}(s) = \begin{bmatrix} d^{\Delta_1} & & & & & \\ \hat{\varphi}_{2,1} & \ddots & & (0) & & \\ & & d^{\Delta_{p-k}} & & & \\ & (\hat{\varphi}_{i,j}) & \vdots & 1 & & \\ & & \vdots & (0) & \ddots & \\ \hat{\varphi}_{p,1} & & \hat{\varphi}_{p,p-k} & 0 & \cdots & 1 \end{bmatrix}, \quad (16)$$

and $\partial\hat{\varphi}_{i,j}[d] < \Delta_j = n_{je} - f_j$ for $i \leq p-k$.

$G = \lim_{s\to\infty} W_{1,m}(s) = \left[\begin{array}{c|c} (0)_{p-k\times p-k} & (0) \\ \hline (g_{i,j})_{k\times p-k} & \mathbb{I}_k \end{array}\right]$ where $g_{i,j} \in \mathbb{R}$, (Herrera H [1992]) .

So, the feedback $u(t) = Fx(t) + Gv(t)$ is given by:

$$u_i(t) = \begin{cases} F_i x(t), \quad i = 1, ..., p-k & (a) \\ F_i x(t) + \sum_{j=1}^{p-k} g_{i,j} v_j(t) + v_i(t), & \\ \quad i = p-k+1, .., p & (b) \end{cases} \quad (17)$$

As $\Sigma_{m,(F,G)}(C_m, A_m + B_m F, B_m G)$ is assumed to be decoupled *without changing the E.O.*, we have:

$$y_i^{(n_{ie})}(t) = \begin{cases} u_i^{(\Delta_i)}(t) = v_i(t), i = 1, ...p-k \\ u_i^{(0)}(t) = v_i(t), i = p-k+1, ..., p. \end{cases} \quad (18)$$

Therefore, equations (18) and (17a) describe $p-k$ chains of lengths $\Delta_1, ..., \Delta_{p-k}$ generating independent functions $F_i x(t)$. Moreover, from (17b), these chains are not activated by derivatives of $u_{p-k+1}(t), ..., u_p(t)$ and for i and $j \leq p-k$, the ith chain is not actived by derivatives of $v_j(t), i \neq j$. But, by (17b), if the chains should only make Σ_m regularly decouplable, the ith chain can be activated by derivatives of $u_i(t)$ of order less than Δ_i for $i \leq p-k$, because each chain must generate a function of $x(t)$, say: $f_a^i(t) = f_e^{i(\Delta_i)}(t)$ and $f_a^i(t) = F_i x(t)$.

From Loiseau [1988], since any increasing of infinite structure can only be done using entries of $\mathcal{R}^*$, entries $v_1(t), ..., v_{p-k}(t)$ should contain independent l.c. of entries of $\mathcal{R}^*$.

Only if: Assume that these $p-k$ independent chains can be constructed. For $i \leq p-k$, each chain is characterized by its length Δ_i, its entry $f_a^i(t)$ and its output $f_e^i(t)$ linked by $f_e^{i(\Delta_i)}(t) = f_a^i(t)$. As $f_e^i(t)$ is uniquely a function of $x(t)$, we can write $f_e^i(t) = F_i x(t)$, where $F_i \in \mathbb{R}^{1\times n}$, and as the chains are independent, rank $\bar{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_{p-k} \end{bmatrix} = p-k$.

Now, as the chains are extracted from $\mathcal{R}^*$, each function $f_a^i(t)$ contains at least one l.c. of the entries $u_{p+1}(t), ..., u_m(t)$ and these l.c. are independent: this implies that dim $\mathcal{B} \cap \mathcal{R}^* \geq p-k$, (a necessary and sufficient condition for DMP). For static decoupling, we must add: $\sum_{i=1}^{p-k} \Delta_i \leq \sum_{i=1}^{m-p} \sigma_i$. By assumption, no derivative of $u_{p-k+1}(t), ..., u_p(t)$ appears in functions $f_a^i(t)$. However there may be terms which depend on $x(t)$ and/or on $u_{p-k+1}(t), ..., u_p(t)$, and/or for $i \leq p-k$, on derivatives of entries $u_i(t)$ of order less than Δ_i, (from the fact that $f_e^i = F_i x(t)$). So, for $i \leq p-k$, the general form of $f_a^i(t)$ is:

$$f_{a1}^i(u_{p+1}(t), ..., u_m(t)) + f_{a2}^i(x(t), u_{p-k+1}(t), ..., u_p(t)) + \sum_{j=1}^{p-k} f_{a3,j}^i(u_j^{(1)}(t), .., u_j^{(\Delta_j-1)}(t)). \quad (19)$$

f_{a1}^i are nonzero and independent functions.

As $u_i^{(j)}(t) = y_i^{(f_i+j)}(t) = y_i^{(n_{ie}-\Delta_i+j)}(t)$, the Laplace's transform of each function $f_a^i(t)$ is:

$$f_{a1}^i(u_{p+1}(s).., u_m(s)) + f_{a2}^i(x(s), u_{p-k+1}(s).., u_p(s)) + \sum_{j=1}^{p-k} \Psi_{i,j}[s] y_j(s), \quad (20)$$

where $\partial\Psi_{i,j}[s] < n_{je}$.

Define the non regular feedback $u(t) = Fx(t) + Gv(t)$ by:

$$F = \left[\begin{array}{c} \bar{F}_{p-k\times n} \\ \hline 0_{k\times n} \\ \hline F_{0\ m-p\times n} \end{array}\right] \text{ and } G = \left[\begin{array}{c|c} 0_{p-k\times k} & 0_{p-k\times p-k} \\ \hline \mathbb{I}_k & 0_{k\times p-k} \\ \hline 0_{\ m-p\times k} & G_{0m-p\times p-k} \end{array}\right].$$

Here $rank \bar{F} = p - k$ and $rank\ G_0 = p - k \leq m - p$.
The possibly non regular feedback (F_0, G_0) acts only on $\mathcal{R}^*$ to create the $p - k$ independent chains of integrators. Noting $B_r = ImB \cap \mathcal{R}^*$ and $x_{\mathcal{R}^*}(t)$ trajectories in $\mathcal{R}^*$, this part of feedback is such that: $F_0 x_{\mathcal{R}^*}(t) + B_r G_0 v_*(t) = \begin{bmatrix} f_{a1}^1(\bullet) \\ \vdots \\ f_{a1}^{p-k}(\bullet) \end{bmatrix}$, where $v_*(t) = \begin{bmatrix} v_{p-k+1}(t) \\ \vdots \\ v_p(t) \end{bmatrix}$.
Then, as $y_i^{(f_i)}(t) = u_i(t) = f_e^i(t)$ for $i \leq p - k$, we obtain: $\mathcal{L}(y_i^{(n_{ie})}(t)) = f_e^i x(s) s^{\Delta_i} = f_a^i(t) = f_{a1}^i(\bullet) + f_{a2}^i(\bullet) + \sum_{j=1}^{p-k} \Psi_{i,j}[s] y_j(s)$ with $\partial \Psi_{i,j}[s] < n_{je}$. Then, outputs $y(s)$ and new entries $v(s)$ are linked by:

$$\left[\begin{array}{c|c} H_{1,1} & (0)_{p-k \times k} \\ \hline H_{2,1} & H_{2,2} \end{array}\right] y(s) = V(s) = \left[\begin{array}{c} V_1(s) \\ \hline V_2(s) \end{array}\right], where \quad (21)$$

1- The $p - k \times p - k$ polynomial matrix $H_{1,1}$ is given by:

$$H_{1,1} = \begin{bmatrix} s^{n_{1e}} - \Psi_{1,1} & & \\ & \ddots & (-\Psi_{i,j}) \\ & & \\ (\varphi^e_{mi,j} - \Psi_{i,j}) & & \ddots \\ & & s^{n_{(p-k)e}} - \Psi_{p-k,p-k} \end{bmatrix}$$

.

2- $H_{2,1} = \left[\varphi^e_{mi,j}[s]\right]$ is a $k \times p - k$ polynomial matrix
3- $H_{2,2}[s] = diag\{s^{n_{(p-k+1)e}}, ..., s^{n_{pe}}\}_k$.
4- $V_1(s) = \begin{bmatrix} f_{a1}^1(\bullet) + f_{a2}^1(\bullet) \\ \vdots \\ f_{a1}^{p-k}(\bullet) + f_{a2}^{p-k}(\bullet) \end{bmatrix}$ and

$$V_2(s) = \begin{bmatrix} v_{p-k+1}(s) = u_{p-k+1}(s) \\ \vdots \\ v_p(s) = u_p(s) \end{bmatrix}.$$

As the polynomial of highest degree of each column of $H[s]$ is the diagonal polynomial, the infinite structure coincides with E.O.: this system is regularly decouplable without increasing E.O.. This ends the proof of Lemma 1.

Remark 2. Functions $f_a^i(t)$ cannot contain derivatives or entries of $\mathcal{R}^*$. If that were the case, for example for the first chain, the effective length of this chain would be less than Δ_1. Indeed, suppose that this chain of length Δ_1 is activated by $u_{p+1}(t)$ and by $u_{p+j}^{(1)}(t)$, for $j \leq m - p$. Then $f_e^{1(\Delta_1)}(t) = f_a^1(t) = u_{p+1}(t) + u_{p+j}^{(1)}(t)$ and $f_e^{1(\Delta_1 - 1})(t) = \int f_a^1(t)dt = \int u_{p+1}(t)dt + \int u_{p+j}^{(1)}(t)dt = f(x(t)) + u_{p+j}(t)$. The effective length of the chain is $\Delta_1 - 1$.

Remark 3. Inputs $u_1(t), ..., u_{p-k}(t)$ are suppressed, while inputs $u_{p-k+1}(t), ..., u_p(t)$ are preserved.

Remark 4. If functions $f_a^i(t)$ do not include derivatives of inputs $u_i(t), i = 1, ..., p - k$, the system is decoupled.

Lemma 1 will help us to derive conditions on Σ_m for ensure that such $p - k$ independent decoupling chains exist.
To characterize the maximal lengths of the chains of $\mathcal{R}^*$ which are not actived by derivatives of entries $u_j(t)$ of order higher or equal to Δ_j , $j \leq p$, it is possible to apply Theorem 4.1 in Herrera H and Lafay [1993] taking into account the following remark:

Remark 5. The chains of $\mathcal{R}^*$ satisfying items (b) and (c) of Lemma1 for Σ_m are the same as the chains of $\mathcal{R}^*$ which are not actived by derivatives of entries $w_1, ..., w_{p-k}$ $u_{p-k+1}, ..., u_p$ for Σ_m^S.

Theorem 4.1 in Herrera H and Lafay [1993] becomes:

Proposition 4. Let $\bar{\Sigma}_m^S$ deduced from Σ_m by reordering B_m^S as $\bar{B}_m^S = \left[\, B_r^S \middle| B_a^S \middle| B_s^S \,\right]$. Let $\{c_i\}_m$ the controllability indices of $(A_m^S, \bar{B}_m^S)$. Then the set $\{\alpha_i\}_{m-p}$ of maximal lenghts of subchains of $\mathcal{R}^*$ not activated by derivatives of inputs $w_1, ..., w_{p-k}, u_{p-k+1}, ..., u_p$ is given by the controllability indices of the entries of $\mathcal{R}^*$:

$$\alpha_i = c_i, \qquad \text{for } i \leq m - p. \quad (22)$$

4.3 Main result: the solution of SRMP

Theorem 2. Let the right invertible system Σ be given with $\mathcal{R}^* = \mathcal{V}^*$ and k the rank at infinity of $W(s)$. Let Σ_m deducted by regular state feedback from Σ such that the infinite structure $\{\Delta_i\}_{(p-k)}$ of $W_m(s)$ of Σ_m is the minimal list of decoupling indices of Σ. Let $\{\alpha_i\}_{(m-p)}$ the controllability indices of the entries of $\mathcal{R}^*$ for the shifted system $\bar{\Sigma}_m^S$ associated with Σ_m. Then SRMP has a solution if and only if, for all $i \geq 1$,

$$\sum_{j=1}^{i} \hat{\alpha}_j \geq \sum_{j=1}^{i} \gamma_j, \quad (23)$$

where $\{\hat{\alpha}_i\}_{\sup \alpha_i}$ the dual list of $\{\alpha_i\}_{(m-p)}$ and $\{\gamma_i\}_{\sup \Delta_i}$ is the dual list of $\{\Delta_i\}_{(p-k)}$.

Proof 2. Sufficiency:

Remark 6. As mentioned in Lafay [2013], Theorem 1 cannot be applied directly , but it suffices to apply it choosing for list $\{n'_i\}_{p-k}$ the list $\{1\}_{p-k}$ and try to obtain $\{1 + \Delta_i\}_{p-k}$. This amounts to build, from the $m-p$ chains of integrators of lengths σ_i of $\mathcal{R}^*$, $p-k$ independent chains of lengths $\{\Delta_i\}$. Note that condition (8) is still always true. So there remains only conditions (9).

Let us now return to SRMP. It is sufficient to replace the dynamic extension of Subsection 4.1 by $p - k$ independent chains of lengths $\{\Delta_i\}_{(p-k)}$ extracted from $\mathcal{R}^*$ and satisfying Lemma 1. This is equivalent to create, from $\bar{\Sigma}_m^S$, $p - k$ independent chains extracted from $\mathcal{R}^*$ not actived by derivatives of entries $w_1, ..., w_{p-k}, u_{p-k+1}, ..., u_p$ (cf Remark 5). By Proposition 4, such chains should be built from the maximal sub-chains of $\mathcal{R}^*$ which are not individually actived by derivatives of $w_1, ..., w_{p-k}, u_{p-k+1}, ..., u_p$.

According to the construction of the shifted system, and Remark 5, integers $\{\alpha_i\}_{m-p}$ represent equivalently the maximal lengths of sub-chains of $\mathcal{R}^*$ not activated by derivatives of inputs $u_{p-k+1}, ..., u_p$ and for $i \leq p - k$ not activated by derivatives of inputs u_i of order greater than or equal to Δ_i. From Remark 6 we obtain (23) which is a sufficient condition for SRMP.
Necessity: The necessity comes from two facts. First: by Proposition 2, if there is a solution to SRMP, some of them require only (p-k) increases of infinite structure. Conditions (9) therefore requires that the minimal list of decoupling indices contains only (p-k)terms. Secondly: If there is no solution with these indices, it does not exist

solution for any other list of decoupling indices coming from other permutations of outputs of Σ.

Corollary 1. The right invertibility of Σ and condition (23) are necessary and sufficient conditions for the general Static Morgan's Problem when $m - p = k$.

5. COMPARISONS

At our knowledge, the more advanced structural results on static Morgan's problem are the following .

5.1 Herrera H and Lafay [1993]

SRMP is solved when $k = p - 1$.

Theorem 3. Let the right invertible system Σ_m be given with $\mathcal{R}^* {=} \mathcal{V}^*$. Suppose $k = p - 1$. Let δ_1 be the (nonzero)infinite structure of the interactor $\Pi_{1m}[s]$, and $\{\alpha_{r,i}\}_{m-p}$ the controllability indices of the pair $(A_m, [B_r|B_s])$ related with the columns of $B_r = \mathcal{B} \cap \mathcal{R}^*$. SRMP has a solution if and only if:

$$\sum_{j=1}^{m-p} \alpha_{r,j} \geq \delta_1. \qquad (24)$$

Let $\Sigma_m(C_m, A_m, B_m)$. Note $B_m = [b_1|B_s|B_r]$ as in Section 4. As $k = p - 1$, the list of decoupling indices contains only one term $\delta_1 = \sum_{i=1}^{p} n_{ie} - \sum_{i=1}^{p} n'_i < n_{1e}$, and is minimal. A static solution will exist if and only if the sum of the lengths of the sub-chains of $\mathcal{R}^*$ not actived by derivatives of entries $(u_2, ..., u_p)$ is greater than or equal to δ_1. Consider now the permuted shifted system $\bar{\Sigma}_m^S$ associated with Σ_m. By (11) and (4.1), we have:

$$A_m^S = \left[\begin{array}{c|c} A_m & A_1^S \\ \hline (0) & J_{\delta_1} \end{array}\right] where \qquad (25)$$

- J_{δ_1} is the upper δ_1 Jordan bloc, and matrix $A_1^S = \left[\, b_{1(n\times 1)} \middle| (0)_{n\times\delta_1 - 1} \,\right]$.
- $B_m^S =$
$$\left[\begin{array}{c|c|c} B_{r(n\times m-p)} & (0)_{(n\times 1)} & B_{s(n\times p-1)} \\ \hline (0) & b_{a(\delta_1\times 1)} & (0) \end{array}\right] with \qquad (26)$$
$b_a = [0 \, \ldots \, 0 \; 1]^T$.

Let us compare the controllability indices $\{\alpha_i\}_{m-1}$ of $(A_m, [B_r|B_s])$ and these $\left\{c_i^S\right\}_m$ of $\bar{\Sigma}_m^S$. By the Brunovsky's procedure, Brunovski [1970], we have $c_1^S \geq n_{1e}$, then

- if $\alpha_i \leq n_{1e}$, then $\alpha_i = c_{i+1}^S$, and
- if $\alpha_i > n_{1e}$, we can have $c_{i+1}^S < \alpha_i$ but $c_{i+1}^S > n_{1e}$.

Then as $\delta_1 \leq n_{1e}$ the two Theorems are equivalent for the existence of a solution for SRMP.

5.2 Zagalak et al. [1998]

The authors consider systems Σ with special conditions on dimensions as $m = 2p$ and $\sum_{i=1}^{p} \delta_i = \sum_{i=1}^{p} \sigma_i$, and mainly without couplings between $\mathcal{R}^*$ and the blocks of infinite structure, ie $\Phi_{m2}^e {=} \left[\, \Phi_{m2,1}^e \middle| \Phi_{m2,2}^e \,\right] {=} [0]$ in the extended interactor of Σ_m. The problem was however difficult although these assumptions seem very simplistic. Certainly, the authors propose factorizations which must have a link with the minimal list of decoupling indices but, unless I have not properly understood their approach, this structural information does never appear explicitly.

6. CONCLUSION

In this paper, we propose a necessary and sufficient conditions for the row by row decoupling problem without modifying the essential orders. Even in this special form, the problem was recognized as structurally difficult. It remains to solve the general problem which is much more difficult because we do not know yet how to define minimal structures for the decoupled system.

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