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A new filtering approach for continuous-time linear systems with delayed measurements

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Abstract: This paper introduces a new filter for linear continuous-time stochastic systems with delayed measurements. The approach is inspired by an observer designed for deterministic systems. The obtained solution is suboptimal and does not use distributed integration terms with advantages in terms of computational load. The relationship between the delay bound and the variance of the estimation error is formally characterized and confirmed by a numerical example.

Keywords: Continuous time filters; System state estimation; Filtering techniques; Filtering theory; Kalman filters; Hilbert spaces.

1. INTRODUCTION

The problem of state estimation for deterministic systems in presence of delayed measurements has attracted considerable attention in the last years and several approaches are now available for linear and nonlinear systems with fixed or time-varying delays. A significantly smaller number of authors have studied the problem for stochastic systems with delay. In the area of optimal filtering, the attempt to extend the classical Kalman-Bucy filter in presence of delayed measurements has been pursued along two approaches. The first one is to use the so called reorganized innovation analysis with the orthogonal projection lemma (Lu et al. [2005], Zhang et al. [2006]). The second approach uses the general expression of the Itô differential of the optimal estimate proposed in Pugachev and Sinitsyn [2001] to derive optimal filtering equations similar to the Kalman-Bucy ones (Basin and Zuniga [2004], Basin et al. [2007]).

A recent work that generalizes these approaches is Kong et al. [2013]. The basic idea is to transform the continuous-time system with delayed measurements into a system with delay-free measurements by directly solving the stochastic equation. The optimal filter for the state is naturally given by the conditional expectation over the delay free measurements. It is concluded that time delay in the observation simply leads to an additional term in the error variance equations. These approaches provide optimal filters and can cope with several kinds of systems, *i.e.* single or multiple delays, delays in state and measurements, etc. However they share the main drawback that the filter equations contain distributed terms, whose evaluation is computationally challenging, particularly in presence of deterministic input or unstable system matrices.

In this paper we propose a new suboptimal filter based on the minimum estimation variance framework. Our filter uses only local instantaneous terms, thus avoiding the computational complexities of the distributed terms of optimal filters even in presence of deterministic input. Delay is supposed to be bounded and known. The filter is inspired by the observer for deterministic systems with delayed measurements proposed in Cacace et al. [2013]. Moreover, a precise characterization of the relationship between the delay bound and the variance of the estimation error is provided.

The paper is organized as follows. Section 2 states the problem formulation and gives some preliminary definitions. The proposed approach is developed in Section 3. Technical proofs are provided in Section 4. Section 5 presents the results of a numeric example and Section 6 summarizes the conclusions of the paper.

2. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Notation

Given a real number α, the symbol $\mathbb{C}_{<\alpha}$ ($\mathbb{C}_{\leq\alpha}$) denotes the set of all complex numbers $s \in \mathbb{C}$ such that $\Re(s) < \alpha$ ($\Re(s) \leq \alpha$). $\sigma(A)$ denotes the spectrum of a real square matrix A. I_n is the identity matrix in $\mathbb{R}^n$. $\lambda_{max}(P)$ indicates the maximum eigenvalue of the symmetric nonnegative matrix P. Given $x \in \mathbb{R}^n$, the euclidean norm in $\mathbb{R}^n$ is denoted by $\|x\|$. Given a positive number δ,

$\mathcal{C}_\delta^n$ denotes the space of the continuous functions that maps $[-\delta, 0]$ into $\mathbb{R}^n$, with the uniform convergence norm, denoted by $\|\cdot\|_\infty$. $W^{1,2}$ indicates the space of absolutely continuous functions form $[-\delta, 0]$ into $\mathbb{R}^n$. $L_2^{\delta,n}$ denotes the space of Lebesgue measurable square integrable functions mapping $[-\delta, 0]$ into $\mathbb{R}^n$, with the norm denoted by $\|\cdot\|_2$. Given a linear operator T which maps a normed space H_1 into a second normed space H_2, $\|T\|$ denotes the operator norm defined as $\sup_{x \in H_1} \|Tx\|_{H_2}/\|x\|_{H_1}$.

2.2 Problem formulation

In this work we deal with the class of linear stochastic systems with delayed measurements, described by

$$\dot{x}(t) = Ax(t) + Bu(t) + F\omega(t) \quad (1)$$

$$\bar{y}(t) = Cx(t - \delta_t) + G\omega(t) \quad (2)$$

where $x(t) \in \mathbb{R}^n$ is the system state, $u(t) \in \mathbb{R}^p$ is the deterministic input, $\omega(t) \in \mathbb{R}^s$ is the noise term, and $\bar{y}(t) \in \mathbb{R}^q$ is the output signal. The latter is a function of the state x at the time $t - \delta$, where δ is the measurement delay. We assume that δ is bounded and known. The noise term ω is supposed to be a standard white process, such that $E[\omega(t)\omega^T(t)] = I_s$. Independence of the state and observation noises is assumed, that is, $FG^T = 0$. Finally, we assume that for $t \in [-\delta, 0]$, $x(t) = \phi(t)$, where $\phi \in \mathcal{C}_\delta^n := \mathcal{C}([-\delta, 0]; \mathbb{R}^n)$. Note that this also means that $\phi \in L_2^{\delta,n}$.

The principal aim here is to design a filtering system for estimating the state of (1)-(2).

2.3 Preliminary definitions

Definition 1. (α-*exp* stability). Consider a linear delay system with the form

$$\dot{z}(t) = A_0 z(t) + A_1(t) z(t - \delta) \quad (3)$$

with $z(t) = \phi_z(t)$ for $t \in [-\delta, 0]$. For a given real number $\alpha > 0$, system (3) is said to be α-*exp stable* if there exists a $\gamma > 0$ such that

$$\|z(t)\| \leq e^{-\alpha t} \gamma \|\phi_z\|_2, \quad \forall t \geq 0, \quad \forall \phi \in \mathcal{C}_\delta^n.$$

System (3) is said 0-*exp* stable if it is asymptotically stable.

Definition 2. (Maximal Delay for α-*exp* stability). For a given $\alpha > 0$ the maximal delay for α-*exp* stability, denoted Δ_α, is the supremum among all $\delta > 0$ such that system (3) is α-*exp* stable. If system (3) is α-*exp* stable for any $\delta \in [0, \infty)$, then $\Delta_\alpha = \infty$. Δ_0 indicates the maximal delay for 0-*exp* stability.

2.4 Steady-state Kalman-Bucy gain

As well known, the Kalman-Bucy filter provides the minimum variance estimate $\hat{x}(t)$ for a linear stochastic system of the form:

$$\dot{x}(t) = Ax(t) + Bu(t) + F\omega(t)$$

$$y(t) = Cx(t) + G\omega(t)$$

with the same mentioned properties for system (1)-(2). The filtering equation is

$$\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(t)\left(y(t) - C\hat{x}(t)\right)$$

where

$$K(t) = \hat{P}(t) C^T \left(GG^T\right)^{-1}$$

and $\hat{P}(t) \in \mathbb{R}^{n \times n}$, defined as the covariance matrix of the estimation error $\hat{e}(t) := x(t) - \hat{x}(t)$, is the solution of the matrix differential equation

$$\dot{\hat{P}}(t) = A\hat{P}(t) + \hat{P}(t)A^T - \hat{P}C^T \left(GG^T\right)^{-1} C\hat{P}(t) + FF^T.$$

Under the hypothesis that the pair (A, F) is controllable and the pair (A, C) is observable, $\hat{P}(t)$ converges to a unique positive definite symmetric matrix $\hat{P}_\infty$, solution of the steady-state Riccati equation

$$A\hat{P}_\infty + \hat{P}_\infty A^T - \hat{P}_\infty C^T \left(GG^T\right)^{-1} C\hat{P}_\infty + FF^T = 0$$

from which the steady-state Kalman gain K_∞ can be obtained by

$$K_\infty = \hat{P}_\infty C^T \left(GG^T\right)^{-1}.$$

Such a gain matrix can be used to replace the time-varying gain $K(t)$ in (4) for obtaining the steady-steady Kalman-Bucy filter

$$\dot{\tilde{x}}_s(t) = A\tilde{x}_s(t) + Bu(t) + K_\infty\left(y(t) - C\tilde{x}_s(t)\right) \quad (4)$$

which has two key properties listed in the following lemma.

Lemma 1. Consider the steady-state Kalman-Bucy filter (4), then:

(i) the estimate $\tilde{x}_s(t)$ is *asymptotically* optimal, i.e. $lim_{t \to \infty} E[(x(t) - \tilde{x}_s(t))(x(t) - \tilde{x}_s(t))^T] = \hat{P}_\infty$;
(ii) $\sigma(A - K_\infty C) \in \mathbb{C}_{<0}$, i.e. the filtering system (4) is asymptotically stable.

3. PROPOSED APPROACH

We here propose a filtering system having the form:

$$\dot{\tilde{x}}(t) = A\tilde{x}(t) + Bu(t) + \tilde{K}\left(\bar{y}(t) - C\tilde{x}(t - \delta)\right) \quad (5)$$

with

$$\tilde{K} = e^{A_\infty \delta} K_\infty, \quad \text{and} \quad A_\infty = A - K_\infty C.$$

For $t \in [-\delta, 0]$, a pre-shape function $\tilde{x}(t) = \tilde{\phi}(t) \in \mathcal{C}_\delta^n$ is assigned to the filter. Note that from property (ii) in Lemma 1 results that $\sigma(A_\infty) \in \mathbb{C}_{<0}$. The estimation error $\tilde{e}(t) := x(t) - \tilde{x}(t)$ obeys the following delay differential equation

$$\dot{\tilde{e}}(t) = A\tilde{e}(t) - \tilde{K}C\tilde{e}(t - \delta) + \left(F - \tilde{K}G\right)\omega(t), \quad (6)$$

with $\tilde{e}(t) = \phi_{\tilde{e}}(t) := \phi(t) - \tilde{\phi}(t)$ for $t \in [-\delta, 0]$, $\phi_{\tilde{e}} \in L_2^{\delta,n}$. The corresponding error covariance matrix is defined as

$$\tilde{P}(t) = E\left[\tilde{e}(t)\tilde{e}^T(t)\right], \quad (7)$$

with initial the initial value

$$\tilde{P}(0) = E\left[\left(x(0) - \tilde{\phi}(0)\right)\left(x(0) - \tilde{\phi}(0)\right)^T\right].$$

We are interested in characterizing this covariance matrix. To do this, in the following Lemmas we recall some results of Cacace et al. [2013] applied to the deterministic part of the error system (6), which has the form of a linear delay system as (3).

Lemma 2. Consider the deterministic part of the error system (6):

$$\dot{\tilde{e}} = A\tilde{e}(t) - \tilde{K}C\tilde{e}(t - \delta) \quad (8)$$

and suppose that there exists some $\alpha > 0$ such that $\sigma(A_\infty) \in \mathbb{C}_{<-\alpha}$. Then,

(i) all the eigenvalues in $\sigma(A_\infty)$ are roots of the characteristic equation of (8);

(ii) for any $\alpha > 0$ a sufficient condition for α-*exp* stability of (8) with delay δ is that

$$\int_0^\delta \|Ce^{A_\infty t}\tilde{K}\|e^{\alpha t}dt < 1; \qquad (9)$$

(iii) if $\bar{y}(t)$ is scalar and moreover $Ce^{A_\infty t}\tilde{K} > 0\ \forall t \in [0,\ \delta]$, a necessary and sufficient condition for the α-*exp* stability of (8) with delay δ is that

$$\int_0^\delta Ce^{A_\infty t}\tilde{K}e^{\alpha t}dt < 1. \qquad (10)$$

Lemma 3. Under the hypothesis of Lemma 2, let $\bar{\delta}_\alpha \in \mathbb{R}_+$ be such that

$$\int_0^{\bar{\delta}_\alpha} \|Ce^{A_\infty t}\tilde{K}\|e^{\alpha t}dt = 1 \qquad (11)$$

or $\bar{\delta}_\alpha = \infty$, if $\int_0^\infty \|Ce^{A_\infty t}\tilde{K}\|e^{\alpha t}dt = 1$. Then,

(i) $\bar{\delta}_\alpha$ is a lower bound for Δ_α;

(ii) if the output is scalar and $Ce^{A_\infty t}\tilde{K} > 0\ \forall t \in [0,\ \bar{\delta}_\alpha]$, then the bound is strict, i.e. $\bar{\delta}_\alpha = \Delta_\alpha$, and it is the smallest positive solution to the equation

$$\mathcal{K}^T \bar{A}_\alpha^{-1}\left(e^{\bar{A}_\alpha} - I\right) - 1 = 0, \qquad (12)$$

where $\mathcal{K}^T$ and $\bar{A}_\alpha$ are defined as follows

$$\begin{aligned} A_b - B_b K_A^T &= TAT^{-1}, \\ A_b - B_b\tilde{K}^T &= TA_\infty T^{-1}, \\ \mathcal{K}^T &= \tilde{K}^T - K_A^T, \\ \bar{A}_\alpha = A_b - B_b\tilde{K}^T + \alpha I &= TA_\infty T^{-1} + \alpha I. \end{aligned}$$

Therefore, $\bar{\delta}_\alpha = \Delta_\alpha$ depends only on α, $\sigma(A)$ and $\sigma(A_\infty)$, i.e. it does not depend on B or C and it is the same if A is replaced by a similar matrix.

Note that: 1) by (9), Lemma 2 gives a condition to study the α-*exp* stability of the deterministic part of error system (6); 2) point (i) of Lemma 3, which trivially follows from condition (9), defines a lower bound for the maximum delay Δ_α for which the α-*exp* stability of the deterministic part of error system (6) is guaranteed; 3) such a bound can be computed by solving (12).

In the following theorem these conditions are used to derive an upper bound for the error covariance matrix (7).

Theorem 4. Let $\alpha > 0$ be such that $\sigma(A_\infty) \in \mathbb{C}_{<-\alpha}$. If condition (9) on δ holds true, then for all $t > \delta$ it is

$$\lambda_{max}\left(\widetilde{P}(t)\right) \le {M_\alpha}^2\left(p_0 + \frac{\|F_0\|^2}{2\alpha}\right), \qquad (13)$$

where p_0 is a positive constant that depends on the pre-shape function, M_α is a bounded function of α, and $F_0 = F - \widetilde{K}G$.

A proof of this theorem is provided in Section 4.

Corollary 5. In the hypothesis of Theorem 4, let $\bar{\delta}_\alpha \in \mathbb{R}^+$ be the solution of (11). Then (13) holds for all $\delta < \bar{\delta}_\alpha$.

This corollary trivially follows from Theorem 4 since, by Lemma 3, condition (9) is satisfied for all $\delta < \bar{\delta}_\alpha$ and therefore Theorem 4 holds.

Theorem 4 states that for any strictly positive α such that $\sigma(A_\infty) \in \mathbb{C}_{<-\alpha}$, (9) can be used to study the existence of a bound for the error covariance matrix $\widetilde{P}(t)$; whereas Corollary 5 establishes that, under the same hypothesis about A_∞, for all $\delta < \bar{\delta}_\alpha$ the error covariance matrix admits a bound, whose expression is given in (13). Using these results we can express the relationship between δ and $\widetilde{P}(t)$ in the following way.

Corollary 6. If the pair (A, C) is observable and the pair (A, F) is controllable, the error covariance matrix $\widetilde{P}(t)$ of the filter (5) for the system (1)–(2) has bounded norm if the constant delay δ is such that

$$\int_0^\delta \|Ce^{A_\infty t}K_\infty\|dt < 1.$$

In the conditions of point (ii) of Lemma 3, this condition is necessary, i.e. $\int_0^\delta \|Ce^{A_\infty t}K_\infty\|dt > 1$ implies that $\|\widetilde{P}(t)\|$ is not bounded.

Proof. Definition (11) implies that $\bar{\delta}_0 > \bar{\delta}_\alpha$ for any $\alpha > 0$. This means that $\bar{\delta}_0$ is the superior element of the delay values δ for which the boundedness of the error covariance matrix is guaranteed. The delay bound is therefore obtained by setting $\alpha = 0$ in (11). ■

Remark 7. It can be shown that the bound in (13) decreases when α increases. From definition (11) it results that $\bar{\delta}_\alpha$ monotonically decreases when α increases. Therefore, assuming that $\delta = \bar{\delta}_\alpha$, it follows that the bound of the error covariance matrix bound (13) decreases with δ, as it could be expected.

4. PROOFS

Theorem 4 provides a boundedness condition for the covariance matrix $\widetilde{P}(t)$ of the error system (6), which has the form of a linear stochastic system with state delay. In order to prove this theorem, we need to represent system (6) with the *infinite dimensional state-space representation*, which is often used for managing such a class of systems (e.g. Germani et al. [2000]). In the next section we describe this representation and the proof of Theorem 4 is given in the following Section 4.2.

4.1 Infinite dimensional state-space model of linear delay systems

Consider a linear delay system with the form:

$$\dot{z}(t) = A_0 z(t) + A_1 z(t-\delta) + F_0\omega(t), \qquad (14)$$

where $z(t) \in \mathbb{R}^n$ and ω belongs to the Hilbert space $L_2([0, t_f]; \mathbb{R}^s)$ equipped with standard Gaussian cylinder measure (that corresponds to model ω as a standard white-process). Variable z in the interval $[-\delta, 0]$ is assumed to be generated as follows:

$$z(\theta) = \phi_z(\theta) + \int_{-\delta}^0 k(\theta, \tau)\bar{\omega}(\tau)d\tau \qquad (15)$$

where ϕ_z is uniformly bounded and so it belongs to $L_2^{\delta,n}$, the process $\bar{\omega}$, independent of ω, belongs to $L_2^{\delta,\bar{s}}$ equipped with the standard Gaussian cylinder measure, and the kernel $k(\theta, \tau)$ is integrable for $\tau \in [-\delta, 0]$.

System (14) can be rewritten in state-space form in the Hilbert space $M_2 := \mathbb{R}^n \times L_2^{\delta,n}$, endowed with the following inner product:

$$\left(\begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix}, \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \end{bmatrix} \right)_{M_2} := \mathbf{x}_0^T \mathbf{y}_0 + \int_{-\delta}^{0} \mathbf{x}_1^T(\theta)\mathbf{y}_1(\theta) d\theta.$$

In M_2 system (14) assumes the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{F}\omega(t), \quad \mathbf{x}(0) = \begin{bmatrix} \phi_z(0) \\ \phi_z \end{bmatrix} + \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \end{bmatrix} \bar{\omega} \quad (16)$$

where the component $\mathbf{x}_1(t) \in \mathbb{R}^n$ is the current state $z(t)$ and $\mathbf{x}_2(t) \in L_2^{\delta,n}$ is the state trajectory into the interval $[t-\delta, t]$. The operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \to M_2$ is defined as

$$\mathbf{A} : \ \mathbf{A} \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} A_0 \mathbf{x}_0 + A_1 \mathbf{x}_1(-\delta) \\ \dfrac{d}{d\theta}\mathbf{x}_1 \end{bmatrix}$$

with domain

$$\mathcal{D}(\mathbf{A}) := \left\{ \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} \mid \begin{matrix} \mathbf{x}_0 \in \mathbb{R}^n \\ \mathbf{x}_1 \in W^{1,2} \end{matrix}, \ \ \mathbf{x}_0 = \mathbf{x}_1(0) \right\}$$

and $\mathbf{F} : \mathbb{R}^s \to M_2$ is defined as

$$\mathbf{F} : \ \mathbf{F}\omega(t) = \begin{bmatrix} F_0 \omega(t) \\ 0 \end{bmatrix}.$$

The Hilbert-Schmidt operator

$$\mathcal{L} = \begin{bmatrix} \mathcal{L}_0 \\ \mathcal{L}_1 \end{bmatrix},$$

which defines the stochastic initial state $\mathbf{x}(0)$, descends from definition (15) and it is defined as

$$\mathcal{L}_0 : L_2^{\delta,\bar{s}} \to \mathbb{R}^n; \qquad \mathcal{L}_0 \omega = \int_{-\delta}^{0} k(0,\tau)\bar{\omega}(\tau) d\tau;$$

$$\mathcal{L}_1 : L_2^{\delta,n} \to W^{1,2}; \quad \mathcal{L}_1 \omega(\theta) = \int_{-\delta}^{0} k(\theta,\tau)\bar{\omega}(\tau) d\tau.$$

The mean value and nuclear covariance of the initial state $\mathbf{x}_0$ are

$$\bar{\mathbf{x}}_0 = \begin{bmatrix} \phi_z(0) \\ \phi_z \end{bmatrix}, \quad \mathbf{P}_0 = \mathcal{L}\mathcal{L}^*.$$

The operators in the next proposition must be well defined to be used to our purposes.

Proposition 8. The operators $\mathbf{F}^*$, $\mathbf{F}\mathbf{F}^*$, and $\mathbf{A}^*$ are as follows:

$$\mathbf{F}^* : M_2 \to \mathbb{R}^s, \quad \mathbf{F}^* \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} F_0^T \mathbf{x}_0 \\ 0 \end{bmatrix}; \quad (17)$$

$$\mathbf{F}\mathbf{F}^* : M_2 \to M_2, \quad \mathbf{F}\mathbf{F}^* \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \end{bmatrix} = \begin{bmatrix} F_0 F_0^T \mathbf{x}_0 \\ 0 \end{bmatrix}; \quad (18)$$

$$\mathbf{A}^* : \mathcal{D}(\mathbf{A}^*) \to M_2, \ \mathbf{A}^* \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1(0) + A_0^T \mathbf{y}_0 \\ -\dfrac{d}{d\theta}\mathbf{y}_1 \end{bmatrix}, \quad (19)$$

with dense domain

$$\mathcal{D}(\mathbf{A}^*) := \left\{ \begin{bmatrix} \mathbf{y}_0 \\ \mathbf{y}_1 \end{bmatrix} \middle| \begin{matrix} \mathbf{y}_0 \in \mathbb{R}^n, \quad A_1^T \mathbf{y}_0 = \mathbf{y}_1(-\delta) \\ \mathbf{y}_1 \in W^{1,2} \end{matrix} \right\}. \quad (20)$$

Only (19) and (20) require a not trivial proof, that can be obtained as a particular case of Proposition 2.2 in Germani et al. [2000].

The representation is now complete and it enables us to write the solution of (16) as

$$\mathbf{x}(t) = \mathbf{T}(t)\mathbf{x}(0) + \int_0^t \mathbf{T}(t-\tau)\mathbf{F}\omega(\tau) d\tau,$$

where $\mathbf{T}(t)$ is the semigroup generated by $\mathbf{A}$. Moreover, it can be proved that the nuclear covariance operator $\mathbf{P}(t)$ evolves in the Hilbert space of Hilbert-Schimdt operators with the following equation

$$\mathbf{P}(t) = \mathbf{T}(t)\mathbf{P}_0\mathbf{T}^*(t) + \int_0^t \mathbf{T}(t-\tau)\mathbf{F}\mathbf{F}^*\mathbf{T}^*(t-\tau) d\tau. \quad (21)$$

We finally define the operator $\mathbf{\Pi}_n^0 : M_2 \to \mathbb{R}^n$ as

$$\mathbf{\Pi}_n^0 \mathbf{x} = \mathbf{x}_0,$$

which simply extracts the first component of the state $\mathbf{x} \in M_2$, and we introduce the useful property stated by the following Lemma.

Lemma 9. Consider the deterministic part of system (16) (i.e. assume $\mathbf{F} = 0$). If there exist $\alpha, \ \gamma > 0$ such that $\|\mathbf{\Pi}_n^0 \mathbf{x}(t)\| \le \gamma e^{-\alpha t} \|\phi_z\|_2$, $\forall t \ge 0$, then there exists a uniformly bounded $M_\alpha > 0$ such that $\|\mathbf{T}(t)\| \le M_\alpha e^{-\alpha t}$, $\forall t > \delta$.

Proof. For any $t \ge \delta$,

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_0(t) \\ \mathbf{x}_1(t) \end{bmatrix}$$

where $\mathbf{x}_0(t) = \mathbf{\Pi}_n^0 \mathbf{x}(t) \in \mathbb{R}^n$ and $\mathbf{x}_1(t) \in L_2^{\delta,n}$, i.e. $\mathbf{x}_1(t) = \mathbf{x}_1(t,\theta)$ with $\theta \in [-\delta, 0]$. Moreover, by the definition of $\mathcal{D}(\mathbf{A})$, $\mathbf{x}_1(t,\theta) = \mathbf{x}_0(t-\theta) = \mathbf{\Pi}_n^0 \mathbf{x}(t-\theta)$. Therefore,

$$\|\mathbf{T}(t)\|^2 = \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{\|\mathbf{T}(t)\mathbf{x}(0)\|_{M_2}^2}{\|\mathbf{x}(0)\|_{M_2}^2} = \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{\|\mathbf{x}(t)\|_{M_2}^2}{\|\mathbf{x}(0)\|_{M_2}^2}$$
$$= \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{\|\mathbf{\Pi}_n^0 \mathbf{x}(t)\|^2 + \int_{-\delta}^0 \|\mathbf{\Pi}_n^0 \mathbf{x}(t-\theta)\|^2 d\theta}{\|\mathbf{x}(0)\|_{M_2}^2}.$$

From the hypothesis of the Lemma, it follows that

$$\begin{aligned}
\|\mathbf{T}(t)\|^2 &= \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{\|\mathbf{\Pi}_n^0 \mathbf{x}(t)\|^2 + \int_{-\delta}^0 \|\mathbf{\Pi}_n^0 \mathbf{x}(t-\theta)\|^2 d\theta}{\|\mathbf{x}(0)\|_{M_2}^2} \\
&\le \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{\gamma^2 e^{-2\alpha t}\|\phi_z\|_2^2 + \int_{-\delta}^0 \gamma^2 e^{-2\alpha(t-\theta)}\|\phi_z\|_2^2 d\theta}{\|\mathbf{x}(0)\|_{M_2}^2} \\
&= \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{\gamma^2 e^{-2\alpha t}\|\phi_z\|_2^2 + \gamma^2 e^{-2\alpha t}\|\phi_z\|_2^2 \frac{1}{2\alpha}\left(1 - e^{-2\alpha\delta}\right)}{\|\mathbf{x}(0)\|_{M_2}^2} \\
&= \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{{M_\alpha}^2 e^{-2\alpha t}\|\phi_z\|_2^2}{\|\mathbf{x}(0)\|_{M_2}^2} \\
&\le \sup_{x(0)\in\mathcal{D}(\mathbf{A})} \frac{{M_\alpha}^2 e^{-2\alpha t}\left(\|\phi_z(0)\|^2 + \|\phi_z\|_2^2\right)}{\|\mathbf{x}(0)\|_{M_2}^2} \\
&= {M_\alpha}^2 e^{-2\alpha t},
\end{aligned}$$

with ${M_\alpha}^2 = \gamma^2 \left(1 + \frac{1}{2\alpha}\left(1 - e^{-2\alpha\delta}\right)\right)$. ■

Lemma 10. Consider system (16). If there exist $\alpha, M_\alpha > 0$ such that $\|\mathbf{T}(t)\| \le M_\alpha e^{-\alpha t}$ for all $t > \delta$, then

$$\|\mathbf{P}(t)\| \ \le \ {M_\alpha}^2 \left(\frac{\|\mathbf{F}\|^2}{2\alpha} + \|\mathbf{P}_0\| \right), \quad \forall t > \delta.$$

Proof. From (21) and the main hypothesis of the Lemma, it turns out that, for any $t > \delta$,

$$\begin{aligned}\|\mathbf{P}(t)\| &\leq \|\mathbf{T}(t)\|^2\|\mathbf{P}_0\| + \int_0^t \|\mathbf{T}(t-\tau)\mathbf{F}\|^2 d\tau \\ &\leq M_\alpha{}^2 e^{-2\alpha t}\|\mathbf{P}_0\| + \int_0^t M_\alpha{}^2 e^{-2\alpha(t-\tau)}\|\mathbf{F}\|^2 d\tau \\ &= M_\alpha{}^2 e^{-2\alpha t}\|\mathbf{P}_0\| + \frac{M_\alpha{}^2}{2\alpha}\|\mathbf{F}\|^2(1-e^{-2\alpha t}) \\ &\leq M_\alpha{}^2\|\mathbf{P}_0\| + \frac{M_\alpha{}^2}{2\alpha}\|\mathbf{F}\|^2.\end{aligned}$$

■

We have now all the instruments to prove Theorem 4 as follows.

4.2 Proof of Theorem 4

By setting $A_0 = A$, $A_1 = -\widetilde{K}C$, $F_0 = F - \widetilde{K}G$, and $\phi_z \equiv \phi_{\tilde{e}}$, the error system (6) can be rewritten in the form of (14). Under the hypothesis of Theorem 4, from Lemma 2 follows that there exists $\gamma > 0$ such that

$$\|\tilde{e}(t)\| \leq e^{-\alpha t}\gamma\|\phi_{\tilde{e}}\|_2, \ \forall t \geq 0. \tag{22}$$

If we use the infinite dimensional representation (16) for the error system, (22) is equivalent to the following condition:

$$\|\mathbf{\Pi}_n^0\mathbf{x}(t)\| \leq e^{-\alpha t}\gamma\|\phi_{\tilde{e}}\|_2, \ \forall t \geq 0.$$

From which, by using Lemma 9 and Lemma 10, it results that, since $\|\mathbf{F}\|^2 = \|F_0\|^2$, for all $t > \delta$ it is

$$\|\mathbf{P}(t)\| \leq M_\alpha{}^2\left(p_0 + \frac{\|F_0\|^2}{2\alpha}\right), \tag{23}$$

with $p_0 = \|\mathbf{P}_0\|$. Now recall that the covariance operator is defined as the linear bounded operator such that

$$[\mathbf{P}(t)\mathbf{y}, \mathbf{y}]_{M_2} = \int_{M_2} [\mathbf{x}(t), \mathbf{y}]^2 \, d\mu_{M_2}(\mathbf{x}(t)),$$

where μ_{M_2} is the mentioned Gaussian cylinder measure defined into M_2. By choosing the special case $\mathbf{y} = [\mathbf{y}_0 \ \ 0]^T$, (i.e. the component $\mathbf{y}_0$ is identically equal to zero), we have:

$$\begin{aligned}[\mathbf{P}(t)\mathbf{y}, \mathbf{y}]_{M_2} &= \int_{\mathbb{R}^n} \left(\mathbf{x}_0^T(t)\mathbf{y}_0\right)^2 d\mu_{\mathbb{R}^n}(\mathbf{x}_0(t)) \\ &= \mathbf{y}_0^T E\left[\mathbf{x}_0(t), \mathbf{x}_0^T(t)\right]\mathbf{y}_0 = \mathbf{y}_0^T\widetilde{P}(t)\mathbf{y}_0\end{aligned}$$

since, in this case, $\mathbf{x}_0(t) = \mathbf{\Pi}_n^0\mathbf{x}(t) = \tilde{e}(t)$. It is well known that (always assuming $\mathbf{y} = [\mathbf{y}_0 \ \ 0]^T$)

$$[\mathbf{P}(t)\mathbf{y}, \mathbf{y}]_{M_2} \leq \|\mathbf{P}(t)\|\|\mathbf{y}\|^2 = \|\mathbf{P}(t)\|\mathbf{y}_0^T\mathbf{y}_0,$$

from which it follows that

$$\begin{aligned}\lambda_{max}\left(\widetilde{P}(t)\right) &= \sup_{\|\mathbf{y}_0\|\neq 0} \frac{\mathbf{y}_0^T\widetilde{P}(t)\mathbf{y}_0}{\mathbf{y}_0^T\mathbf{y}_0} \\ &= \sup_{\|\mathbf{y}_0\|\neq 0} \frac{[\mathbf{P}(t)\mathbf{y}, \mathbf{y}]_{M_2}}{\mathbf{y}_0^T\mathbf{y}_0} \leq \|\mathbf{P}(t)\|.\end{aligned}$$

The theorem is finally proved by considering last inequality and (23). ■

5. EXAMPLE

Consider system (1)-(2) with

$$A = \begin{bmatrix} -1 & 0 & 0.48 \\ 1 & -0.2 & 0 \\ 1 & -0.1 & -0.3 \end{bmatrix}, \ B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \ F = \begin{bmatrix} 1 & 0 \\ -1 & 0 \\ 0.7 & 0 \end{bmatrix}$$

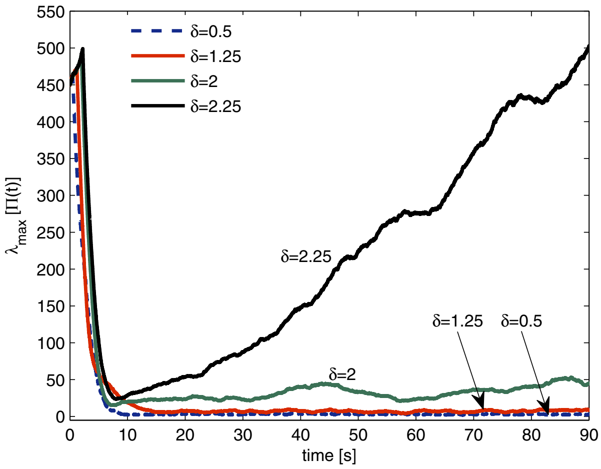

Fig. 1. Sample error covariance matrix averaged over 100 noise realizations for different values of the measurement delay δ.

$$C = [\ 1 \quad 0.1 \quad 0\ ], \ G = [\ 0 \quad 0.5\ ], \ u(t) = \sin\left(\frac{2\pi}{1000}t\right).$$

The pair (A, C) is observable and (A, F) is reachable. Therefore there exists the steady-state Kalman gain

$$K_\infty = [\ 1.6211 \quad -0.7939 \quad 2.1073\ ]^T,$$

which is obtained solving the matrix steady-state Riccati equation, as recalled in Section 2.4. The corresponding eigenvalues of $A_\infty = A - K_\infty C$ are

$$\sigma(A_\infty) = \{-2.281, \ -0.381 + 0.284j, \ -0.381 - 0.284j\}.$$

The hypothesis of Theorem 4 and Corollary 5-6 are satisfied for example with $\alpha = 0.38$. Using (12) with $\alpha = 0$ and a solver of scalar nonlinear equations we have $\bar{\delta}_0 = 2.2137$. Corollary 6 guarantees the existence of a bound for the error covariance matrix $\tilde{P}(t)$ for all $\delta < \bar{\delta}_0$.

The proposed filter has been applied to the example system, in a 100 realizations Monte Carlo simulation. Euler-Meruyama integration (with $\Delta_{EM} = 0.001$) has been used. The initial conditions are $x(0) = [5\ 20\ 5]^T$ for the real state and $\tilde{x}(t) = [0\ 0\ 0]^T$, with $t \in [0, \delta]$, for the estimated state. Figure 1 depicts the maximum eigenvalues of the sample error covariance matrix, averaged over the 100 noise realizations, defined as

$$\Pi(t) = \frac{1}{100}\sum_{i=1}^{100}\left(x^{(i)}(t) - \tilde{x}^{(i)}(t)\right)\left(x^{(i)}(t) - \tilde{x}^{(i)}(t)\right)^T,$$

where the apex (i) indicates the i-th noise realization. The measurement delay is set to the values $\delta = \{0.5, \ 1.25, \ 2\}$, which are lower than $\bar{\delta}_0 = 2.2137$, and $\delta = 2.25$, which is larger than $\bar{\delta}_0$. As stated by Theorem 4, after the initial transient, in the first three cases $\lambda_{max}(\Pi(t))$ is bounded, whereas in the last case it is not bounded. Moreover, in the former cases the bound grows with δ.

Two different filtering methods have been also tested on this system: the Kalman-Bucy filter without measurement delay (KBF) and the standard optimal Kalman-Bucy predictor (KBP). Obviously, both the filters are not suitable in real applications since the former assumes a null delay and the latter requires the use of distributed terms. In this sense, they can considered as referring cases to assess the

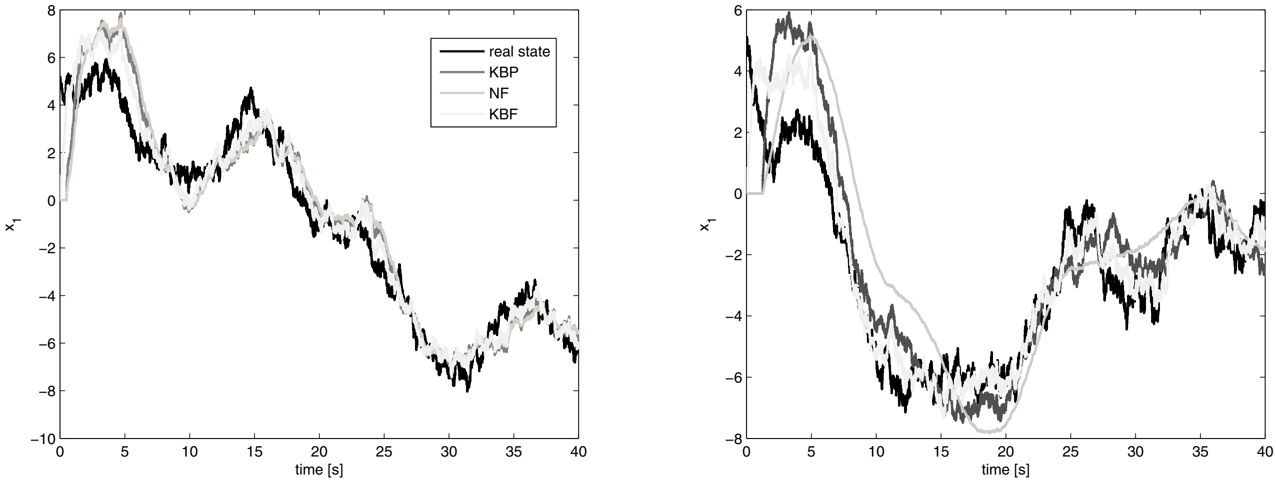

Fig. 2. Real and estimated state component x_1 for one of the 100 noises realization with $\delta = 0.5$ (left) and $\delta = 1.25$ (right).

Table 1. Numerical results: MSE.

	$\delta = 0.5$			$\delta = 1.25$		
	x_1	x_2	x_3	x_1	x_2	x_3
NF	0.730	0.774	1.556	1.546	0.864	4.739
KBP	0.698	0.773	1.472	1.064	0.817	2.896
KBF	0.421	0.740	0.812	0.426	0.723	0.823

performances of the new filter (NF). Table 5 reports the mean square error (MSE) for the three state components of the estimation results obtained with $\delta = 0.5$ and $\delta = 1.25$, with the KBP and the NF, and the corresponding results for the KBF. Obviously, KBF definitely has better performances with respect to both KBP and NF. Moreover, also KBP outperforms the proposed filter. This was expected because of the optimality of KBP. However, KBP and NF clearly have comparable results. Figure 2 reports an example of the estimation results for the first component of the system for two values of delay, $\delta = 0.5$ and $\delta = 1.25$.

6. CONCLUSIONS

A suboptimal filter for linear stochastic system with delayed measurement is proposed in this paper. The filer avoids the computational complexities due to the use of distributed terms of optimal filters. Bound conditions on the delay for assure the boundedness of the error covariance matrix are provided. A numeric example confirms the theoretical results. Future works will be devoted to the extension of the approach to the time-varying delay case.

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