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Integral Input-to-State Stability for Interconnected Discrete-Time Systems

Navid Noroozi * **Alireza Khayatian** ** **Saeed Ahmadizadeh** *** **Hamid Reza Karimi** ****

* *Dept. of Electrical Engineering, Sheikh Bahaei University, Isfahan, Iran (Tel: +98-311-6816761; e-mail: navidnoroozi@gmail.com).*
** *School of Electrical and Computer Engineering, Shiraz University, Shiraz, Iran (e-mail: khayatia@shirazu.ac.ir)*
*** *Department of Electrical and Electronic Engineering, University of Melbourne, Parkville, 3010, Victoria, Australia (e-mail: sahmadizadeh@student.unimelb.edu.au)*
**** *Department of Engineering, Faculty of Engineering and Science, University of Agder, N-4898 Grimstad, Norway (e-mail: hamid.r.karimi@uia.no)*

Abstract: In this paper, we investigate integral input-to-state stability for interconnected discrete-time systems. The system under consideration contains two subsystems which are connected in a feedback structure. We construct a Lyapunov function for the whole system through the nonlinearly-weighted sum of Lyapunov functions of individual subsystems. We consider two cases in which we assume that one of subsystems is integral input-to-state stable and the other is either input-to-state stable or only integral input-to-state stable.

1. INTRODUCTION

Many attempts have been made on stability analysis of interconnected nonlinear systems over the last few decades. It is still challenging to analyze the stability of interconnected systems with nonlinearities and, consequently it would be interesting to establish the universal stability conditions which are applicable to a wide range of nonlinear systems. Small-gain theorem and its variants are integral to methods of stability analysis of feedback interconnected systems. Among them, Lyapunov-based small-gain theorems have been given conspicuously attention. The Lyapunov-based small-gain theorem based on input-to-state stability (ISS) property was formulated by Jiang et al. (1996) for continuous-time systems. This was extended to parameterized discrete-time systems in (Laila and Nešić, 2004). Recently, small-gain theorems for hybrid systems have been reported (cf. (Liu et al., 2012; Dashkovskiy and Mironchenko, 2013; Liberzon et al., 2014) and references therein).

A variant of ISS property was introduced in (Sontag, 1998) extending $\mathcal{L}_2$ stability to nonlinear systems. This generalization is called integral input-to-state stability (iISS) which has been studied further for continuous-time systems and discrete-time in (Angeli et al., 2000) and (Angeli, 1999), respectively. It has been demonstrated that iISS is a broader notion rather than ISS. Results on iISS for feedback interconnected continuous-time systems have been presented in (Ito, 2007; Ito and Jiang, 2009; Ito et al., 2010, 2013). Although, small-gain theorems on iISS have been investigated for continuous-time systems in depth, iISS for interconnected discrete-time systems has not been provided yet. With which this paper is primarily concerned (see motivations in Subsection 3.2 below).

In analogy with the results on iISS for feedback interconnected continuous-time systems, this paper investigates iISS for a feedback interconnection of parameterized discrete-time systems based on changing supply functions. Particularly, small-gain conditions providing iISS for feedback interconnected systems consisting of two subsystems. We construct the iISS Lyapunov function for the whole system through the nonlinearly-weighted sum of Lyapunov functions of individual subsystems, which is called the sum-type construction. Next we provide iISS for the feedback interconnected system when both subsystems are integral input-to-state stable. Then results on iISS for a feedback interconnection of ISS and iISS systems is investigated. Moreover, we note that our results show 0-global asymptotic stability (0-GAS) for the feedback interconnected system as it is equivalent to iISS for discrete-time systems (Angeli, 1999).

The rest of this paper is organized as follows: notations together with notions of ISS and iISS for discrete-time systems are provided in Section 2. The main results are given in Section 3. Section 4 provides the concluding remarks.

2. PRELIMINARIES

In this section, we give notations and review notions of ISS and iISS for discrete-time systems.

2.1 Notation

- $\mathbb{R}_{\geq 0}$ and $\mathbb{Z}_{\geq 0}$ are the nonnegative real and nonnegative integer numbers, respectively.
- $|.|$ denotes the standard Euclidean norm.
- Given a function $\varphi\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$, $\mathcal{L}_\infty$-norm of the function φ is denoted by $\|\varphi\|$. That is, $\|\varphi\| = \sup\{|\varphi(k)| : k \in \mathbb{Z}_{\geq 0}\} < \infty$. The set of all such functions is denoted by $\mathcal{L}_\infty$.
- A function $\gamma\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is positive definite function ($\gamma \in \mathcal{PD}$) if it is zero at zero and positive elsewhere.
- A function $\alpha\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero and strictly increasing. It is of class-$\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if $\alpha \in \mathcal{K}$ and also $\alpha(s) \to \infty$ if $s \to \infty$.

- A function $\beta\colon \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if for each $t \geq 0$ $\beta(.,t) \in \mathcal{K}$ and, for each $s \geq 0$, $\beta(s,.)$ is decreasing and $\beta(s,t)$ is converging to zero as $t \to \infty$.
- Composition of two functions $\gamma_1(.)$ and $\gamma_2(.)$, which are from $\mathbb{R}$ to $\mathbb{R}$, is denoted by $\gamma_1 \circ \gamma_2$.

Consider the following family of parameterized discrete-time systems

$$x(t+1) = g_T(x(t), u(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^n$, inputs or controls $u\colon \mathbb{Z}_{\geq 0} \to \mathbb{R}^m$, $u \in \mathcal{L}_\infty$ and T is a parameter (perhaps a sampling time). We assume that $g_T(.,.)$ is continuous and $g_T(0,0) = 0$. For each ξ and $u \in \mathcal{L}_\infty$, $x(.,\xi,u)$ denotes the trajectory of the system (1) with the initial value $x(0) = \xi$ and the input u. We borrow some definitions from (Jiang and Wang, 2001) and (Angeli, 1999), which are required later.

Definition 1. The discrete-time system (1) is input-to-state stable (ISS) if there exist $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that, for all $\xi \in \mathbb{R}^n$, all $u \in \mathcal{L}_\infty$ and all $j \in \mathbb{Z}_{\geq 0}$, the solution $x(j,\xi,u)$ satisfies

$$|x(j,\xi,u)| \leq \beta(|\xi|, j) + \gamma(\|u\|). \tag{2}$$

Definition 2. The discrete-time system (1) is integral input-to-state stable (iISS) if there exist $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ such that, for all $\xi \in \mathbb{R}^n$ and all $u \in \mathcal{L}_\infty$, and all $j \in \mathbb{Z}_{\geq 0}$ the following holds

$$|x(j,\xi,u)| \leq \beta(|\xi|, j) + \sum_{t=0}^{t=j-1} \gamma(|u(t)|). \tag{3}$$

Definition 3. A continuous function $V\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a common ISS-Lyapunov function for the whole family of (1) if there exist functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, and $\alpha \in \mathcal{K}_\infty$, $T^* > 0$ such that the following hold for all $T \in (0, T^*)$, $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$

$$\underline{\alpha}(|\xi|) \leq V(\xi) \leq \overline{\alpha}(|\xi|), \tag{4}$$

$$V(g_T(\xi,u)) - V(\xi) \leq T\left[-\alpha(|\xi|) + \sigma(|\mu|)\right]. \tag{5}$$

Definition 4. A continuous function $V\colon \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called a common iISS-Lyapunov function for the whole family of (1) if there exist functions $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$, $\sigma \in \mathcal{K}$, and $\alpha \in \mathcal{PD}$, $T^* > 0$ such that the following hold for all $T \in (0, T^*)$, $\xi \in \mathbb{R}^n$ and $\mu \in \mathbb{R}^m$

$$\underline{\alpha}(|\xi|) \leq V(\xi) \leq \overline{\alpha}(|\xi|), \tag{6}$$

$$V(g_T(\xi,u)) - V(\xi) \leq T\left[-\alpha(|\xi|) + \sigma(|\mu|)\right]. \tag{7}$$

Proposition 1. (Jiang and Wang, 2001) The discrete-time system (1) is ISS if and only if it admits a common smooth ISS-Lyapunov function.

Proposition 2. (Angeli, 1999) The discrete-time system (1) is iISS if and only if it admits a common iISS-Lyapunov function.

Proposition 3. (Angeli, 1999) The discrete-time system (1) is iISS if and only if the zero solution of the system (1) is globally asymptotically stable (GAS), that is to say, the 0-input system

$$x(t+1) = g_T(x(t), 0) \tag{8}$$

is GAS.

Remark 1. It should be noted that Jiang and Wang (2001) and Angeli (1999) provided ISS and iISS for a nonparameterized discrete-time system with respect to the origin, respectively. However, it is easy to see that the similar results hold for a compact set. Hence, the results are applicable to the family of parameterized discrete-time systems (1), as well.

3. MAIN RESULTS

This section first addresses iISS small-gain theorems for a feedback interconnected discrete-time system, which includes two subsystems, based upon changing supply functions. Throughout this paper, we refer to the system which is not ISS but iISS as strictly iISS.

Consider the following family of interconnected discrete-time systems

$$\Sigma_1\colon x_1(t+1) = g_{1_T}(x_1(t), x_2(t), u_1(t))$$
$$(x_1(t), x_2(t), u_1(t)) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_1} \tag{9}$$

$$\Sigma_2\colon x_2(t+1) = g_{2_T}(x_1(t), x_2(t), u_2(t))$$
$$(x_1(t), x_2(t), u_2(t)) \in \mathbb{R}^{n} \times \mathbb{R}^{n_2} \times \mathbb{R}^{m_2} \tag{10}$$

where $x := (x_1, x_2)$, $u := (u_1, u_2)$, $g_T := (g_{1_T}, g_{2_T})$, and $n := n_1 + n_2$. Suppose that g_T is continuous and $g_T(0,0,0) = 0$. We assume that each subsystem Σ_i with $i \in \{1,2\}$ is either ISS or strictly iISS. So we make the following assumption.

Assumption 1. Let $V_i\colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ be the corresponding common iISS(ISS)-Lyapunov function to Σ_i with $i \in \{1,2\}$, that is,

(1) There exist functions $\overline{\alpha}_i, \underline{\alpha}_i \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}_i(|\xi_i|) \leq V_i(\xi_i) \leq \overline{\alpha}_i(|\xi_i|) \quad \forall \xi_i \in \mathbb{R}^{n_i}, \tag{11}$$

(2) There exist $\alpha_i \in \mathcal{PD}(\alpha_i \in \mathcal{K}_\infty)$, $\sigma_i, \sigma_{u_i} \in \mathcal{K}$ and $T^* > 0$ such that

$$V_i(g_{i_T}(\xi, \mu_i)) - V_i(\xi_i) \leq T[-\alpha_i(|\xi_i|) + \sigma_i(|\xi_{3-i}|)$$
$$+ \sigma_{u_i}(|\mu_i|)] \quad \forall(\xi, \mu_i) \in \mathbb{R}^n \times \mathbb{R}^{m_i}, \forall T \in (0, T^*). \tag{12}$$

We look for conditions under which the closed-loop system (9)-(10), which is denoted by Σ, is iISS, as well. To this end, we pursue similar procedures as those for continuous-time systems in (Ito, 2007). We indeed generalize the results in (Ito, 2007) for discrete-time systems. The results in (Ito, 2007) are based on the approach of changing supply functions proposed in (Sontag and Teel, 1995). On the other hand, the discrete-time counterpart of (Sontag and Teel, 1995) is given in (Nešić and Teel, 2001). Hence, we rely on the method of changing supply functions given in (Nešić and Teel, 2001). Define $\rho_i\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by

$$\rho_i(s) := \int_0^s \lambda_i(\tau)\mathrm{d}\tau \tag{13}$$

where $\lambda_i\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a nondecreasing and continuous function. We call λ_i a scaling function. Obviously, $\rho \in \mathcal{K}_\infty$ and ρ is continuously differentiable on $[0,\infty)$. Let $\hat{V}_i\colon \mathbb{R}^{n_i} \to \mathbb{R}_{\geq 0}$ with $i \in \{1,2\}$ be

$$\hat{V}_i(x_i) := \rho_i \circ V_i(x_i) \tag{14}$$

in which $V_i(x_i)$ is the Lyapunov function to the ith subsystem. For convenience, let $\Delta\hat{V}_i := \rho_i(V_i(g_{i_T}(x, u_i))) - \rho_i(V_i(x_i))$. Also, $V_{cl}\colon \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}_{\geq 0}$ is defined by

$$V_{cl}(x) := \hat{V}_1(x_1) + \hat{V}_2(x_2)$$
$$= \int_0^{V_1(x_1)} \lambda_1(\tau)\mathrm{d}\tau + \int_0^{V_2(x_2)} \lambda_2(\tau)\mathrm{d}\tau. \tag{15}$$

The form (15) called the sum-type construction is used throughout this paper to verify iISS for the interconnected system Σ. [1].

Monotonicity of $\lambda_i(.)$ together with the mean value theorem gives for any $i \in \{1,2\}$

$$\rho_i(a) - \rho_i(b) \leq \lambda_i(a)[a-b] \quad \forall(a,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}. \tag{16}$$

[1] The max-type construction is useless to verify iISS when at least one of subsystems is strictly iISS (cf. Theorem 8 in (Ito et al., 2012) for more details).

This property will be invoked below several times. It should be pointed out that one can deduce from (11) and (12) that the following hold

$$V_i(\xi_i) \geq \max\{\underline{\alpha}_i(|\xi_i|), T\alpha_i(|\xi_i|)\} \quad \forall i \in \{1,2\}, \forall \xi_i \in \mathbb{R}^{n_i}, \forall T \in (0,T^*) \tag{17}$$

$$V_i(g_{i_T}(\xi,\mu_i)) \leq V_i(\xi_i) + T\left[\sigma_i(|\xi_{3-i}|) + \sigma_{u_i}(|\mu_i|)\right] \quad \forall i \in \{1,2\}, \forall (\xi,\mu_i) \in \mathbb{R}^n \times \mathbb{R}^{m_i}, \forall T \in (0,T^*), \tag{18}$$

which will be used in the proof of Lemmas 5 and 7 (see below).

3.1 Feedback Interconnections of iISS Subsystems

Let both subsystems Σ_1 and Σ_2 be strictly iISS. In what follows, we show iISS for Σ. It is worthwhile to mention that throughout this paper, we primarily rely on results of Proposition 3 stating that 0-GAS is an equivalence for iISS for discrete-time systems. From now on, by slight abuse of notation, let $V_i := V_i(\xi_i)$ and $V_i(g_{i_T}) := V_i(g_{i_T}(\xi_1,\xi_2,0))$ for each $i \in \{1,2\}$.

Theorem 4. Let $T^* > 0$ be given. Also, let Assumption 1 hold for each $i \in \{1,2\}$. Suppose that $\alpha_i \in \mathcal{K}\backslash\mathcal{K}_\infty$ for any $i \in \{1,2\}$. If for all $T \in (0,T^*)$ there exist $c_1, c_2 > 0$ such that

$$\sigma_2 \circ \underline{\alpha}_1^{-1}(s) \leq c_1\alpha_1 \circ \overline{\alpha}_1^{-1}(s) \qquad \forall s \in \mathbb{R}_{\geq 0}, \tag{19}$$

$$c_2\sigma_1 \circ \underline{\alpha}_2^{-1}(s) \leq \alpha_2 \circ \overline{\alpha}_2^{-1}(s) \qquad \forall s \in \mathbb{R}_{\geq 0}, \tag{20}$$

$$c_1 < c_2, \tag{21}$$

then Σ is iISS.

Proof. To establish iISS for the closed-loop system (9)-(10), we exploit the equivalence between iISS and 0-GAS. So let $u \equiv 0$. Define a Lyapunov function candidate for the closed-loop system by (15). Let strictly positive real numbers T^*, c_1 and c_2 be given. Pick $\lambda_1 = \frac{c_1}{\delta^2}$ and $\lambda_2 = 1$ with $\delta := (\frac{c_1}{c_2})^{\frac{1}{3}}$ and $c_1 < c_2$. Using, in succession, (15), (12) and the fact that $\lambda_2 = 1$, for all $\xi \in \mathbb{R}^n$ and all $T \in (0,T^*)$ we get

$$\begin{aligned}\Delta V_{cl} &:= \rho_1 \circ V_1(g_{1_T}) - \rho_1(V_1) + \rho_2 \circ V_2(g_{2_T}) - \rho_2(V_2)\\ &= \lambda_1[V_1(g_{1_T}) - V_1] + \lambda_2[V_2(g_{2_T}) - V_2]\\ &\leq T\left[\lambda_1[-\alpha_1(|\xi_1|) + \sigma_1(|\xi_2|)] - \alpha_2(|\xi_2|) + \sigma_2(|\xi_1|)\right].\end{aligned} \tag{22}$$

It follows with the fact that $\delta < 1$ and $c_2 = \frac{c_1}{\delta^3}$ that

$$\begin{aligned}\Delta V_{cl} \leq T\,[&-[1-\delta][\lambda_1\alpha_1(|\xi_1|) + \alpha_2(|\xi_2|)] + \sigma_2(|\xi_1|)\\ &- c_1\alpha_1(|\xi_1|) + \delta[c_2\sigma_1(|\xi_2|) - \alpha_2(|\xi_2|)]].\end{aligned} \tag{23}$$

It follows from (11) that

$$\begin{aligned}\Delta V_{cl} \leq T\big[&-[1-\delta][\lambda_1\alpha_1(|\xi_1|) + \alpha_2(|\xi_2|)] + \sigma_2 \circ \underline{\alpha}_1^{-1}(V_1)\\ &- c_1\alpha_1 \circ \overline{\alpha}_1^{-1}(V_1)\\ &+ \delta[c_2\sigma_1 \circ \underline{\alpha}_2^{-1}(V_2) - \alpha_2 \circ \overline{\alpha}_2^{-1}(V_2)]\big].\end{aligned} \tag{24}$$

So it is straightforward to see that the conditions (19)-(20) together with (21) are sufficient to guarantee 0-GAS for Σ. □

3.2 When Things Go Wrong!

We make this observation that Theorem 4 is the discrete-time counterpart of Theorem 4 in (Ito, 2007) when $k = 1$ therein. We would like to see whether we are able to get iISS for the feedback interconnected system (9)-(10) with the same conditions as those in Theorem 4 of (Ito, 2007) for all $k > 0$ therein. So we follow steps in the proof of Theorem 4 in (Ito, 2007). To this end, let $T^* > 0$ be given and let Assumption 1 hold for each $i \in \{1,2\}$ with $\alpha_1, \alpha_2 \in \mathcal{K}\backslash\mathcal{K}_\infty$. As iISS is equivalent to 0-GAS, let $u \equiv 0$.

Define a Lyapunov function candidate for the closed-loop system by (15). Let $\lambda_1 \in \mathbb{R}_{>0}$ be given. Using (14) and (12), we get for all $\xi \in \mathbb{R}^n$ and all $T \in (0,T^*)$

$$\begin{aligned}\Delta\hat{V}_1 &= \rho_1 \circ V_1(g_{1_T}) - \rho_1(V_1) = \lambda_1\left[V_1(g_{1_T}) - V_1\right]\\ &\leq \lambda_1 T\left[-\alpha_1(|\xi_1|) + \sigma_1(|\xi_2|)\right].\end{aligned} \tag{25}$$

It follows with adding and subtracting $\delta\lambda_1 T\alpha_1(|\xi_1|)$ that

$$\Delta\hat{V}_1 \leq T\left[-[1-\delta]\lambda_1\alpha_1(|\xi_1|) + \lambda_1\sigma_1(|\xi_2|) - \lambda_1\delta\alpha_1(|\xi_1|)\right]. \tag{26}$$

Let the scaling function λ_2 be nondecreasing and continuous on $\mathbb{R}_{\geq 0}$. From the mean value theorem there exists some point z on the line segment joining V_2 to $V_2(g_{2_T})$ such that the following holds

$$\Delta\hat{V}_2 = \rho_2 \circ V_2(g_{2_T}) - \rho_2(V_2) = \lambda_2(z)\left[V_2(g_{2_T}) - V_2\right]. \tag{27}$$

Using (12), we get

$$\Delta\hat{V}_2 \leq T\lambda_2(z)\left[-\alpha_2(|\xi_2|) + \sigma_2(|\xi_1|)\right]. \tag{28}$$

Exploiting Young's inequality yields

$$\Delta\hat{V}_2 \leq T\left[-\lambda_2(z)\alpha_2(|\xi_2|) + \frac{1}{p\kappa^p}[\lambda_2(z)]^p + \frac{\kappa^q}{q}[\sigma_2(|\xi_1|)]^q\right] \tag{29}$$

for some point z on the line segment joining V_2 to $V_2(g_{2_T})$, for any $T \in (0,T^*)$, for all $(\xi_1,\xi_2) \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, and any $\kappa \in \mathbb{R}_{>0}$ and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. It follows with adding and subtracting $T\delta\alpha_2(|\xi_2|)\lambda_2(z)$ that

$$\begin{aligned}\Delta\hat{V}_2 \leq T\Bigg[&-[1-\delta]\alpha_2(|\xi_2|)\lambda_2(z) + \frac{1}{p\kappa^p}[\lambda_2(z)]^p\\ &-\delta\alpha_2(|\xi_2|)\lambda_2(z) + \frac{\kappa^q}{q}[\sigma_2(|\xi_1|)]^q\Bigg].\end{aligned} \tag{30}$$

Combining (26) and (30) gives

$$\begin{aligned}\Delta V_{cl} \leq T\,[&-[1-\delta]\,\lambda_1\alpha_1(|\xi_1|) - [1-\delta]\,\alpha_2(|\xi_2|)\lambda_2(z)\\ &+ \lambda_1\sigma_1(|\xi_2|) + \frac{1}{p\kappa^p}[\lambda_2(V_2(g_2))]^p\\ &-\delta\alpha_2(|\xi_2|)\lambda_2(z) + \frac{\kappa^q}{q}[\sigma_2(|\xi_1|)]^q - \lambda_1\delta\alpha_1(|\xi_1|)\Bigg].\end{aligned} \tag{31}$$

It follows from (11) that

$$\begin{aligned}\Delta V_{cl} \leq T[&-[1-\delta]\lambda_1\alpha_1(|\xi_1|) - [1-\delta]\alpha_2(|\xi_2|)\lambda_2(z)\\ &+ \lambda_1\sigma_1 \circ \underline{\alpha}_2^{-1}(V_2) + \frac{1}{p\kappa^p}[\lambda_2(z)]^p\\ &- \delta\alpha_2 \circ \overline{\alpha}_2^{-1}(V_2)\lambda_2(z)\\ &+ \frac{\kappa^q}{q}[\sigma_2 \circ \underline{\alpha}_1^{-1}(V_1)]^q - \lambda_1\delta\alpha_1 \circ \overline{\alpha}_1^{-1}(V_1)].\end{aligned}$$

So the following conditions guarantee 0-GAS for Σ

$$\left[\frac{\kappa^q}{q}\right][\sigma_2 \circ \underline{\alpha}_1^{-1}(V_1)]^q - \lambda_1\delta\alpha_1 \circ \overline{\alpha}_1^{-1}(V_1) \leq 0 \qquad \forall\, V_1 \in \mathbb{R}_{\geq 0}, \tag{32}$$

$$\frac{1}{p\kappa^p}[\lambda_2(z)]^p - \delta\alpha_2 \circ \overline{\alpha}_2^{-1}(V_2)\lambda_2(z) + \lambda_1\sigma_1 \circ \underline{\alpha}_2^{-1}(V_2) \leq 0 \qquad \forall\,(V_2,z) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}. \tag{33}$$

Let $c_1 := \left[\frac{q\lambda_1\delta}{\kappa^q}\right]$, $k := q$ and $s := V_1$. So it follows from (32) that

$$[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k \leq c_1\alpha_1 \circ \overline{\alpha}_1^{-1}(s) \quad \forall s \in \mathbb{R}_{\geq 0}, \tag{34}$$

which is identical to the condition (47-a) in (Ito, 2007).
Let $w := V_2$ and let $\lambda_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be

$$\lambda_2(z) := \kappa^q \delta^{q-1} \left[\alpha_2 \circ \overline{\alpha}_2^{-1}(w)\right]^{q-1}. \quad (35)$$

Substituting (35) into (33) gives

$$\sigma_1 \circ \underline{\alpha}_2^{-1}(w) \leq \frac{[\kappa\delta]^q}{q\lambda_1}[\alpha_2 \circ \overline{\alpha}_2^{-1}(w)]^q \quad \forall w \in \mathbb{R}_{\geq 0}, \quad (36)$$

which is the same as (47-b) in (Ito, 2007) by setting $c_2 := \frac{(\kappa\delta)^q}{q\lambda_1}$ and $k := q$. As seen the conditions (34) and (36) are the same as the pair of conditions (47) in (Ito, 2007). So one might naively conclude that the system Σ is iISS under these conditions. However, the main problem is that the equation (35) is not well-defined because the argument of the scaling function λ_2 is different from one on the right-hand side of the equation.
It should be noted that the equation (35) becomes well-defined as $T \to 0$ because z tends to w. In this case, the equation (35) equals to its continuous-time counterpart (cf. (50) in (Ito, 2007) for more details). So we recover results of Theorem 4 in (Ito, 2007).

3.3 Feedback Interconnections of iISS Subsystems (continuation)

Here we provide new conditions under which iISS holds for the system Σ when both subsystems are strictly iISS. To give these results, we rely on Lemma 5 whose proof is removed for space reason.

Lemma 5. Let $T^* > 0$ and $\lambda_i(.)$ be a nondecreasing function with $\lambda_i(a+b) \leq \lambda_i(a) + \lambda_i(b)$ for all $a, b \in \mathbb{R}_{\geq 0}$ and $\lambda_i(s) \leq \epsilon \left[\frac{s}{T}\right]^{q-1}$ for all $T \in (0, T^*)$, some $\epsilon > 0$, for some $q > 1$, and for all $s \in \mathbb{R}_{\geq 0}$. Also, let Assumption 1 hold with $\mu \equiv 0$. Then there exist positive numbers $\kappa > 0$, p and q with $\min\{p, q\} > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$ such that for any $\tau_i > 1$, for each $i \in \{1, 2\}$, for all $T \in (0, T^*)$ and for all $\xi \in \mathbb{R}^n$ the following dissipation inequality holds

$$\Delta\hat{V}_i \leq T\left[-\left[\frac{\tau_i - 1}{\tau_i}\right]\alpha_i(|\xi_i|)\lambda_i(\frac{V_i}{\tau_i}) + \left[\frac{1}{p\kappa^p}\right][\lambda_i(V_i)]^p + \left[\frac{\kappa^q}{q} + \epsilon\right][\sigma_i(|\xi_{3-i}|)]^q\right]. \quad (37)$$

Theorem 6. Let T^*, $\lambda_2(.)$ and τ_2 come from Lemma 5. Also, let Assumption 1 hold for each $T \in (0, T^*)$ and for any $i \in \{1, 2\}$ with $\alpha_1, \alpha_2 \in \mathcal{K}\backslash\mathcal{K}_\infty$. If there exist $\hat{\alpha}_2 \in \mathcal{K}$ and $c_1, c_2 > 0$, and $k > 1$ such that

$$\left[\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)\right]^k \leq \alpha_2 \circ \overline{\alpha}_2^{-1}(s)\left[\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(\frac{s}{\tau_2})\right]^{k-1} \quad \forall s \in \mathbb{R}_{\geq 0}, \quad (38)$$

$$[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k \leq c_1\alpha_1 \circ \overline{\alpha}_1^{-1}(s) \quad \forall s \in \mathbb{R}_{\geq 0}, \quad (39)$$

$$c_2\sigma_1 \circ \underline{\alpha}_2^{-1}(s) \leq [\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)]^k \quad \forall s \in \mathbb{R}_{\geq 0}, \quad (40)$$

then Σ is iISS.

Proof. As iISS is equivalent to 0-GAS, let $u \equiv 0$. Also, let $T^* > 0$ be given and $T \in (0, T^*)$. Define a Lyapunov function candidate for the closed-loop system by (15).
Let $\lambda_1 \in \mathbb{R}_{>0}$. Using (14) and (12), we get for all $\xi \in \mathbb{R}^n$ and for all $T \in (0, T^*)$

$$\begin{aligned}\Delta\hat{V}_1 &= \rho_1 \circ V_1(g_{1_T}) - \rho_1(V_1) = \lambda_1[V_1(g_{1_T}) - V_1] \\ &\leq T\lambda_1[-\alpha_1(|\xi_1|) + \sigma_1(|\xi_2|)].\end{aligned} \quad (41)$$

Let λ_2 come from Lemma 5. It follows from Lemma 5 that for all $\xi \in \mathbb{R}^n$ and all $T \in (0, T^*)$

$$\Delta\hat{V}_2 \leq T\left[-\left[\frac{\tau_2 - 1}{\tau_2}\right]\alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2}) + \left[\frac{1}{p\kappa^p}\right][\lambda_2(V_2)]^p + \left[\frac{\kappa^q}{q} + \epsilon\right][\sigma_2(|\xi_1|)]^q\right] \quad (42)$$

Combining (41) and (42) gives

$$\begin{aligned}\Delta V_{cl} \leq T\Bigg[&-[1-\delta]\left[\lambda_1\alpha_1(|\xi_1|) + \left[\frac{\tau_2 - 1}{\tau_2}\right]\alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2})\right] \\ &+ \lambda_1\sigma_1(|\xi_2|) + \left[\frac{1}{p\kappa^p}\right][\lambda_2(V_2)]^p \\ &+ \left[\frac{\kappa^q}{q} + \epsilon\right][\sigma_2(|\xi_1|)]^q \\ &-\delta\left[\frac{\tau_2 - 1}{\tau_2}\right]\alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2}) - \lambda_1\delta\alpha_1(|\xi_1|)\Bigg].\end{aligned} \quad (43)$$

It follows from (11) that

$$\begin{aligned}\Delta V_{cl} \leq T\Big[&-[1-\delta]\Big[\lambda_1\alpha_1(|\xi_1|) + \Big[\frac{\tau_2 - 1}{\tau_2}\Big]\alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2})\Big] \\ &+ \lambda_1\sigma_1 \circ \underline{\alpha}_2^{-1}(V_2) + \left[\frac{1}{p\kappa^p}\right][\lambda_2(V_2)]^p \\ &+ \Big[\frac{\kappa^q}{q} + \epsilon\Big][\sigma_2 \circ \underline{\alpha}_1^{-1}(V_1)]^q \\ &- \delta\Big[\frac{\tau_2 - 1}{\tau_2}\Big]\alpha_2 \circ \overline{\alpha}_2^{-1}(V_2)\lambda_2(\frac{V_2}{\tau_2}) \\ &- \lambda_1\delta\alpha_1 \circ \overline{\alpha}_1^{-1}(V_1)\Big].\end{aligned}$$

So sufficient conditions under which 0-GAS for Σ preserves are

$$\left[\frac{\kappa^q}{q} + \epsilon\right][\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^q - \lambda_1\delta\alpha_1 \circ \overline{\alpha}_1^{-1}(s) \leq 0 \quad \forall s \in \mathbb{R}_{\geq 0}, \quad (44)$$

$$\begin{aligned}&\lambda_1\sigma_1 \circ \underline{\alpha}_2^{-1}(s) + \left[\frac{1}{p\kappa^p}\right][\lambda_2(s)]^p \\ &- \delta\left[\frac{\tau_2 - 1}{\tau_2}\right]\alpha_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(\frac{s}{\tau_2}) \leq 0 \quad \forall s \in \mathbb{R}_{\geq 0}.\end{aligned} \quad (45)$$

Pick $c_1 := \left[\frac{\kappa^q}{q} + \epsilon\right]^{-1}\lambda_1\delta$ and $k := q$. So it follows from (44) that

$$[\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^k \leq c_1\alpha_1 \circ \overline{\alpha}_1^{-1}(s) \quad \forall s \in \mathbb{R}_{\geq 0}, \quad (46)$$

which is identical to (39).
Now we need to find $\lambda_2(.)$ so that the condition (45) holds. But the problem is that arguments of the functions λ_2 in (45) are different. To solve this matter, we impose the condition (38). Let $\hat{\alpha}_2 \in \mathcal{K}\backslash\mathcal{K}_\infty$ so that the following holds

$$\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(s) \leq \alpha_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(\frac{s}{\tau_2}) \quad \forall s \in \mathbb{R}_{\geq 0}. \quad (47)$$

It follows from (47) that

$$\begin{aligned}&\lambda_1\sigma_1 \circ \underline{\alpha}_2^{-1}(s) + \left[\frac{1}{p\kappa^p}\right][\lambda_2(s)]^p \\ &- \delta\left[\frac{\tau_2 - 1}{\tau_2}\right]\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(s) \leq 0 \quad \forall s \in \mathbb{R}_{\geq 0}.\end{aligned} \quad (48)$$

Now arguments of the functions λ_2 in (48) are the same (both functions are with s). Taking the derivative of (48) with respect to λ_2 and equating to zero give

$$\lambda_2(s) = \kappa^q \delta^{q-1} \left[\frac{\tau_2 - 1}{\tau_2}\right]^{q-1} [\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)]^{q-1} \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{49}$$

Substituting (49) into (48) gives

$$\frac{q\lambda_1 \tau_2^q}{[\kappa\delta[\tau_2 - 1]]^q} \sigma_1 \circ \underline{\alpha}_2^{-1}(s) \leq [\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)]^q \quad \forall s \in \mathbb{R}_{\geq 0}, \tag{50}$$

which is identical to (40) by setting $c_2 := \frac{q\lambda_1\tau_2^q}{[\kappa\delta[\tau_2-1]]^q}$ and $k := q$. Also, substituting (49) into (47) yields

$$\left[\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)\right]^k \leq \alpha_2 \circ \overline{\alpha}_2^{-1}(s) \left[\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(\frac{s}{\tau_2})\right]^{k-1} \quad \forall s \in \mathbb{R}_{\geq 0}, \tag{51}$$

which is equivalent to (38). This completes the proof. □

Remark 2. Theorem 6 gives a discrete-time version of Theorem 4 in (Ito, 2007) where $k > 1$. One additional condition is required by Theorem 6 in comparison with Theorem 4 in (Ito, 2007). This condition is given to solve problems mentioned in subsection 3.2. The other two conditions (39) and (40) are similar to the pair of conditions (47) in Theorem 4 in (Ito, 2007).

3.4 Feedback Interconnections of iISS and ISS Subsystems

We give iISS for Σ when the first subsystem Σ_1 is ISS and the second one Σ_2 is strictly iISS. We are in the need of lemma below to establish Theorem 8. This lemma is a slight modification of Lemma 1 in (Nešić and Teel, 2001).

Lemma 7. Let $T^* > 0$ be given. Let $\lambda_i(.)$ be a nondecreasing function. Also, let Assumption 1 hold with $\alpha_i \in \mathcal{K}_\infty$ and $\mu \equiv 0$. Then there exists some $\tau_i > 1$ such that for each $i \in \{1, 2\}$, any $T \in (0, T^*)$, and all $\xi \in \mathbb{R}^n$, the following dissipation inequality holds

$$\Delta\hat{V}_i \leq T\Big[- \Big[\frac{\tau_i - 1}{\tau_i}\Big]\alpha_i(|\xi_i|)\lambda_i(\frac{V_i(\xi_i)}{\tau_i}) + \lambda_i \circ \theta_i(|\xi_{3-i}|)\, \sigma_i(|\xi_{3-i}|)\Big] \tag{52}$$

where $\theta_i : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined by

$$\theta_i(s) := \overline{\alpha}_i \circ \alpha_i^{-1} \circ \tau_i\sigma_i(s) + T\sigma_i(s). \tag{53}$$

We are ready to state the main result of this subsection.

Theorem 8. Let $\lambda_1(.), \theta_1(.), \alpha_1(.), \sigma_1(.)$ and τ_1 come from Lemma 7. Also, let $\lambda_2(.), \alpha_2(.), \sigma_2(.)$ and τ_2 come from Lemma 5. Suppose that $\alpha_2 \in \mathcal{K}\backslash\mathcal{K}_\infty$. If there exist some $\hat{\alpha}_2 \in \mathcal{K}$, $c_1, c_2 > 0$, and $k > 1$ such that for all $s \in \mathbb{R}_{\geq 0}$, the following hold

$$\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)^k \leq \alpha_2 \circ \overline{\alpha}_2^{-1}(s)\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(\frac{s}{\tau_2})^{k-1} \tag{54}$$

$$\max_{w\in[0,s]} \frac{[c_1\sigma_2 \circ \underline{\alpha}_1^{-1}(\tau_1\theta_1(w))]^k}{\alpha_1 \circ \overline{\alpha}_1^{-1}(\tau_1\theta_1(w))} \leq \frac{[c_2\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^k}{\sigma_1(s)}, \tag{55}$$

then Σ is iISS.

Proof. Using the result of Proposition 3, let $u \equiv 0$. Define a Lyapunov function candidate for the closed-loop system by (15). Take any arbitrary $T^* > 0$. Let λ_1 come from Lemma 7. So for all $\xi \in \mathbb{R}^n$ and for all $T \in (0, T^*)$ we get

$$\Delta\hat{V}_1 \leq T \left[- \left[\frac{\tau_1 - 1}{\tau_1}\right] \alpha_1(|\xi_1|)\lambda_1(\frac{V_1}{\tau_1}) + \lambda_1 \circ \theta_1(|\xi_2|)\, \sigma_1(|\xi_2|)\right]. \tag{56}$$

Let λ_2 come from Lemma 5. It follows from Lemma 5 that for all $\xi \in \mathbb{R}^n$ and for all $T \in (0, T^*)$

$$\Delta\hat{V}_2 \leq T \left[- \left[\frac{\tau_2 - 1}{\tau_2}\right] \alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2}) + \left[\frac{1}{p\kappa^p}\right] [\lambda_2(V_2)]^p + \left[\frac{\kappa^q}{q} + \epsilon\right] [\sigma_2(|\xi_1|)]^q\right] \tag{57}$$

Combining (56) and (57) gives

$$\begin{aligned}\Delta V_{cl} \leq T\Big[&- [1 - \delta]\Big\{ \Big[\frac{\tau_1 - 1}{\tau_1}\Big]\alpha_1(|\xi_1|)\lambda_1(\frac{V_1}{\tau_1}) + \Big[\frac{\tau_2 - 1}{\tau_2}\Big]\alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2})\Big\} \\ &+ \lambda_1 \circ \theta_1(|\xi_2|)\, \sigma_1(|\xi_2|) + \Big[\frac{1}{p\kappa^p}\Big][\lambda_2(V_2)]^p \\ &+ \Big[\frac{\kappa^q}{q} + \epsilon\Big][\sigma_2(|\xi_1|)]^q - \delta\Big[\frac{\tau_2 - 1}{\tau_2}\Big]\alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2}) \\ &- \delta\Big[\frac{\tau_1 - 1}{\tau_1}\Big]\alpha_1(|\xi_1|)\lambda_1(\frac{V_1}{\tau_1})\Big].\end{aligned} \tag{58}$$

Given that (53) and (11) yields

$$\begin{aligned}\Delta V_{cl} \leq T\Big[&- [1 - \delta]\Big\{ \Big[\frac{\tau_1 - 1}{\tau_1}\Big]\alpha_1(|\xi_1|)\lambda_1(\frac{V_1}{\tau_1}) + \Big[\frac{\tau_2 - 1}{\tau_2}\Big]\alpha_2(|\xi_2|)\lambda_2(\frac{V_2}{\tau_2})\Big\} + \Big[\frac{1}{p\kappa^p}\Big][\lambda_2(V_2)]^p \\ &+ \lambda_1 \circ \theta_1 \circ \underline{\alpha}_2^{-1}(V_2)\, \sigma_1 \circ \underline{\alpha}_2^{-1}(V_2) \\ &+ \Big[\frac{\kappa^q}{q} + \epsilon\Big][\sigma_2 \circ \underline{\alpha}_1^{-1}(V_1)]^q \\ &- \delta\left[\frac{\tau_2 - 1}{\tau_2}\right] \alpha_2 \circ \overline{\alpha}_2^{-1}(V_2)\lambda_2(\frac{V_2}{\tau_2}) \\ &- \delta\Big[\frac{\tau_1 - 1}{\tau_1}\Big]\alpha_1 \circ \overline{\alpha}_1^{-1}(V_1)\lambda_1(\frac{V_1}{\tau_1})\Big].\end{aligned}$$

So Σ is 0-GAS if for all $\forall s \in \mathbb{R}_{\geq 0}$, the following hold

$$\left[\frac{\kappa^q}{q} + \epsilon\right] [\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^q - \delta \left[\frac{\tau_1 - 1}{\tau_1}\right] \alpha_1 \circ \overline{\alpha}_1^{-1}(s)\lambda_1(\frac{s}{\tau_1}) \leq 0 \tag{59}$$

$$\lambda_1 \circ \theta_1 \circ \underline{\alpha}_2^{-1}(s)\, \sigma_1 \circ \underline{\alpha}_2^{-1}(s) + \left[\frac{1}{p\kappa^p}\right] [\lambda_2(s)]^p - \delta \left[\frac{\tau_2 - 1}{\tau_2}\right] \alpha_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(\frac{s}{\tau_2}) \leq 0. \tag{60}$$

The inequality (59) gives

$$\lambda_1(\frac{s}{\tau_1}) \geq \frac{\left[\frac{\kappa^q}{q} + \epsilon\right] [\sigma_2 \circ \underline{\alpha}_1^{-1}(s)]^q}{\delta \left[\frac{\tau_1 - 1}{\tau_1}\right] \alpha_1 \circ \overline{\alpha}_1^{-1}(s)} \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{61}$$

We pick the same λ_2 as that in the proof of Theorem 6. So let $\hat{\alpha}_2 \in \mathcal{K}\backslash\mathcal{K}_\infty$ so that the following holds

$$\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(s) \leq \alpha_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(\frac{s}{\tau_2}) \quad \forall s \in \mathbb{R}_{\geq 0} \tag{62}$$

which is equivalent to (54). It follows from (62) that

$$\lambda_1 \circ \theta_1 \circ \underline{\alpha}_2^{-1}(s)\, \sigma_1 \circ \underline{\alpha}_2^{-1}(s) + \left[\frac{1}{p\kappa^p}\right] [\lambda_2(s)]^p - \delta \left[\frac{\tau_2 - 1}{\tau_2}\right] \hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)\lambda_2(s) \leq 0 \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{63}$$

Now arguments of the functions λ_2 in (63) are equal. Let $\lambda_2 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be

$$\lambda_2(s) := \kappa^q \delta^{q-1} \left[\frac{\tau_2 - 1}{\tau_2}\right]^{q-1} [\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1}(s)]^{q-1} \quad \forall s \in \mathbb{R}_{\geq 0}. \tag{64}$$

Substituting (64) into (63) gives

$$\lambda_1 \circ \theta_1(s) \leq \frac{[\kappa\delta]^q}{q} \left[\frac{\tau_2 - 1}{\tau_2}\right]^q \frac{[\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^q}{\sigma_1(s)} \; \forall s \in \mathbb{R}_{\geq 0}. \tag{65}$$

Let $\lambda_1(s) := \max_{w\in[0,s]} \frac{\left[\frac{\kappa^q}{q}+\epsilon\right][\sigma_2\circ\underline{\alpha}_1^{-1}(\tau_1 w)]^q}{\delta\left[\frac{\tau_1-1}{\tau_1}\right]\alpha_1\circ\overline{\alpha}_1^{-1}(\tau_1 w)}$. Obviously, this choice of $\lambda_1(.)$ is nondecreasing and satisfies (61). Pick $c_1 := \frac{\left[\frac{\kappa^q}{q}+\epsilon\right]^{\frac{1}{q}}}{\delta^{\frac{1}{q}}\left[\frac{\tau_1-1}{\tau_1}\right]^{\frac{1}{q}}}$, $c_2 := \frac{[\kappa\delta]}{q^{\frac{1}{q}}}\left[\frac{\tau_2-1}{\tau_2}\right]$ and $k := q$. Let the condition (55) hold. So the condition (55) guarantees that for all $s \in \mathbb{R}_{\geq 0}$, (65) holds

$$\max_{w\in[0,s]} \frac{[c_1\sigma_2 \circ \underline{\alpha}_1^{-1}(\tau_1\theta_1(w))]^q}{\alpha_1 \circ \overline{\alpha}_1^{-1}(\tau_1\theta_1(w))} \leq \frac{[c_2\alpha_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^q}{\sigma_1(s)}. \tag{66}$$

This completes the proof. □

Remark 3. Thanks to monotonicity of $\theta(.)$ and the fact that $\tau_1 > 1$, the following simpler-to-check condition provides a sufficient condition to ensure that for all $s \in \mathbb{R}_{\geq 0}$, (55) holds

$$\max_{w\in[0,s]} \frac{[c_1\sigma_2 \circ \underline{\alpha}_1^{-1}(\tau_1\theta_1(w))]^k}{\sigma_1(s)} \leq \frac{[c_2\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s)]^k}{\sigma_1(s)}. \tag{67}$$

Now we show sufficient conditions to check (67).

Lemma 9. Let the functions $\theta_1, \hat{\alpha}_2, \sigma_i, \underline{\alpha}_i$, and $\overline{\alpha}_i$ for each $i \in \{1, 2\}$ and the constant τ_1 come from Theorem 8. Define $\eta(s) := \frac{[\hat{\alpha}_2\circ\overline{\alpha}_2^{-1}(s)]^k}{\sigma_1\circ\overline{\alpha}_2^{-1}(s)}$ with $k > 1$. Let for some $k > 1$ the following pair of conditions hold

$$\eta(s + \varepsilon) \geq \eta(s) \qquad \forall \varepsilon > 0 \tag{68}$$

$$c_1\sigma_2 \circ \underline{\alpha}_1^{-1}(\tau_1\theta_1(s)) \leq c_2\hat{\alpha}_2 \circ \overline{\alpha}_2^{-1} \circ \underline{\alpha}_2(s) \qquad \forall s \in \mathbb{R}_{\geq 0}. \tag{69}$$

Then the inequality (67) holds.

Proof. It is proved in similar arguments as those in the proof of Lemma 1 in (Ito, 2007) with minor modifications. □

Remark 4. Theorem 8 presents a discrete-time version of Theorems 2 and 3 in (Ito, 2007). By approaching $T \to 0$, Case 1 in the proof of Lemma 7 vanish and the function θ_1 tends to its continuous-time counterpart in Theorems 2 and 3 in (Ito, 2007). Further, it follows from the arguments in subsection 3.2 together with those in the proof of Lemma 7 that the condition (54) and the constant τ_1, which is multiplied by θ_1, in the left-hand side of (55) disappear as $T \to 0$. So we indeed recover results in Theorems 2 and 3 in (Ito, 2007).

4. CONCLUSION

The main purpose of the current work was to study integral input-to-state stability for a feedback interconnection of parameterized discrete-time systems. We considered two different cases of subsystems. The former contained a feedback interconnection of two strictly integral input-to-state stable subsystems. In the latter, one of the subsystems was allowed to be input-to-state stable.

An extension could be integral input-to-state stability for large-scale interconnected systems (cf. (Ito et al., 2013) and references therein). Another future work is to provide integral input-to-state stability for time-delay discrete-time systems in a feedback interconnection (cf. (Dashkovskiy et al., 2012) and references therein).

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