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Robust Adaptive Backstepping Control of Second-order Nonlinear Systems with Non-triangular Structure Uncertainties ⋆

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Abstract: In this paper, we consider a class of second-order uncertain nonlinear systems. Besides parametric uncertainties, a non-parametric uncertainty may exist in every state equation or channel with its effects bounded by a known function. Such a bounding function is allowed to depend on all system states which means that the actual system does not need to meet the triangular structure. Therefore the currently available backstepping technique cannot be used to design controllers. To overcome such difficulty, a new backatepping-based robust adaptive control scheme is proposed. It is shown that the proposed scheme can ensure all signals in the closed-loop system bounded.

Keywords: Backstepping, Adaptive control, Nonlinear system, Uncertainties, Non-triangular structure

1. INTRODUCTION

In recent twenty years, backstepping technique has been wildly used in the design of controllers for nonlinear systems, see Krstic (1995) and Zhou (2007) for example. Various kinds of uncertainties including unknown parameters, non-parametric modeling errors and unknown external disturbance are taken into account in the design and analysis of control systems. An essential function of an adaptive controller is to handle uncertainties caused by unknown system parameters by employing an online estimator to identify them real time. As for unknown external disturbance, it is normally assumed bounded by an unknown constant. This unknown constant bound can also be estimated in adaptive controller design, see for example in Zhou (2007) and Zhou (2004). On the other hand, a non-parametric modeling error is often represented as an unknown nonlinear function of system states and inputs. Its effect may be assumed bounded by a known function. But to use the existing backstepping technique, such a bounding function should meet the requirement of a semi-strict feedback form, see for examples Jang (1998) Yao (1997) Yao (2001) Cai (2011) and Cai (2013). All these requirements on uncertainties imply the system model should eventually meet the triangular structure. However, modeling errors may exist in every state equation or channel and their bounding functions may be dependent on all system states. Thus the triangular structure condition cannot be met and this results in that the currently available backstepping technique cannot be used to design controllers.

On the other hand, it is well known that many practical systems such as electrical motor systems, typical mechanical systems, gun control systems of tanks and missile systems can be described by second-order differential equations, see Xu (2001), Xu (2007), Zang (2007) and Kim (2004) for examples. So this paper aims at the design of an adaptive controllers for second-order systems with modeling errors that do not meet triangular structure condition. In the proposed controller design, the effects of uncertainties caused by unknown modeling errors are not considered in every step. Instead, their effects are accumulated to the last step for compensation by selecting appropriate control law and parameter update law. Throughout the whole design procedure, it is important to maintain a linear relation between the state vector $x = (x_1, x_2)$ and its transformed vector $z = (z_1, z_2)$ obtained by coordinate change. It is shown that the proposed adaptive controller ensures global stability of the closed loop system and thus is robust to the modeling errors.

The remaining part of the paper is organized as follows. In Section 2, we present the class of system to be controlled and formulate our control problem. In Section 3, we propose an adaptive control scheme based on the general procedure of backstepping technique and establish system stability. Simulation results are given in Section 4 for illustration. Finally, Section 5 concludes the paper.

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2. SYSTEM MODELS AND PROBLEM STATEMENT

We consider the following class of second-order nonlinear systems.

$$\begin{aligned} \dot{x}_1 &= x_2 + \Delta_1(x_1, x_2, u, t) \\ \dot{x}_2 &= g_0(x) + g(x)^T\theta + bu + \Delta_2(x_1, x_2, u, t) \\ y &= x_1 \end{aligned} \quad (1)$$

where $x = (x_1, x_2)^T$ is system state and $u \in R$ is input, $y = x_1$ is output, $g_0(x) \in R, g(x) \in R^p$ are known functions and $b \in R, \theta \in R^p$ are unknown parameters. $\Delta_i(x_1, x_2, u, t)(i = 1, 2)$ are unknown nonlinear functions representing modeling errors.

Assumption 1: There exists a positive constant σ_i such that

$$|\Delta_i(x_1, x_2, u, t)| \leq \sigma_i ||(x_1, x_2)||_2 \qquad (i = 1, 2) \quad (2)$$

Note that σ_i can be interpreted as the gain or strength of the modeling errors.

Remark 1: Note that existing results on adaptive control of nonlinear systems by using backstepping techniques normally require the systems to satisfy the following conditions

- The unknown modeling error is considered only in the last state equation of system model, for examples Zhou (2007) Cai (2011) Cai (2013). Namely, it only exists in the channel of $\dot{x}_n$.
- Although unknown modeling errors existing in every state equation are considered, certain strong requirements are imposed, such as $|\Delta_i(x_1, \cdots, x_n, u, t)| \leq \alpha_i(x_1, \cdots, x_i)$ in Jang (1998) Yao (1997) Yao (2001) where $\alpha_i(x_1, \cdots, x_i)$ is known This requirement indicates that the bounding functions must satisfy a semi-strict feedback form, or triangular structure of system states. Therefore this essentially implies that the actual system also needs to satisfy such a structure. As seen in (2), this requirement is no longer needed in this paper.

Assumption 2: Unknown parameter $b \neq 0$ and $sign(b)$ is known.

Our control problem is to design an adaptive controller for system (1) such that all the signals in the closed loop system are bounded in the presence of modeling errors satisfying Assumption 1. A condition related to the strength of the modeling error is to be established. Such a condition will make the choices of positive design parameters possible.

3. DESIGN AND ANALYSIS OF ADAPTIVE CONTROLLERS

3.1 Controller design

To carry out the design of control law and adaptive law, we first make coordinate changes by following the procedure of backstepping.

$$\begin{aligned} z_1 &= y \\ z_2 &= x_2 - \alpha_1 \end{aligned} \quad (3)$$

where α_1 is a the virtual control.

Step 1: From (3) the derivative of z_1 can be rewritten as

$$\dot{z}_1 = \dot{x}_1 = z_2 + \alpha_1 + \Delta_1(x_1, x_2, u, t) \quad (4)$$

Consider the following Lyapunov function

$$V_1 = \frac{1}{2} z_1^2 \quad (5)$$

The derivative of V_1 is

$$\begin{aligned} \dot{V}_1 &= z_1 \dot{z}_1 \\ &= z_1(z_2 + \alpha_1 + \Delta_1(x_1, x_2, u, t)) \\ &= z_1(z_2 + \alpha_1) + z_1 \Delta_1(x_1, x_2, u, t) \end{aligned} \quad (6)$$

Then the virtual control α_1 can be chosen as

$$\alpha_1 = -c_1 z_1 - \frac{1}{4e_{11}} z_1 \quad (7)$$

where c_1, e_{11} are positive constants.

From Assumption 1 and (7), the derivative of V_1 satisfies

$$\begin{aligned} \dot{V}_1 &\leq z_1(z_2 + \alpha_1) + \sigma_1 |z_1| ||(x_1, x_2)||_2 \\ &\leq z_1(z_2 + \alpha_1) + \frac{z_1^2}{4e_{11}} + e_{11}\sigma_1^2 ||(x_1, x_2)||_2^2 \\ &= -c_1 z_1^2 + z_1 z_2 + e_{11}\sigma_1^2 ||(x_1, x_2)||_2^2 \end{aligned} \quad (8)$$

Step 2: From (3), the derivative of z_2 is given as

$$\begin{aligned} \dot{z}_2 &= g_0(x) + g(x)^T\theta + bu + \Delta_2(x_1, x_2, u, t) - D_{\alpha_1}\dot{x}_1 \\ &= g_0(x) + g(x)^T\theta + bu + \Delta_2(x_1, x_2, u, t) \\ &\quad - D_{\alpha_1}(x_2 + \Delta_1(x_1, x_2, u, t)) \end{aligned} \quad (9)$$

where $D_{\alpha_1} = \frac{\partial \alpha_1}{\partial x_1}$ is a constant and

$$D_{\alpha_1} = -(c_1 + \frac{1}{4e_{11}}) \quad (10)$$

Now the control law and update law of unknown parameters can be designed.

Control Law:

$$\begin{aligned} u &= \hat{\rho}\bar{u} \\ \bar{u} &= \alpha_2 - g(x)^T\hat{\theta} - g_0(x) \end{aligned} \quad (11)$$

$$\alpha_2 = -z_1 - c_2 z_2 + D_{\alpha_1}x_2 - [\frac{1}{4e_{22}} z_2 + \frac{1}{4e_{21}} (D_{\alpha_1})^2 z_2]$$

where c_2, e_{21}, e_{22} are positive constants and $\hat{\rho}$ is an estimate of parameter $\rho = \frac{1}{b}$, $\hat{\theta}$ is an estimate of θ.

Update Laws:

$$\begin{aligned} \dot{\hat{\theta}} &= \Gamma_\theta(g(x)z_2 - k_\theta(\hat{\theta} - \theta_0)) \\ \dot{\hat{\rho}} &= -\gamma(sign(b)\bar{u}z_2 + k_\rho(\hat{\rho} - \rho_0)) \end{aligned} \quad (12)$$

where $\gamma, k_\rho, k_\theta, \rho_0, \theta_0$ are positive constants and Γ_θ is a positive definite matrix.

3.2 Stability Analysis

We now consider the following Lyapunov function

$$V = \frac{1}{2} z_1^2 + \frac{1}{2} z_2^2 + \frac{1}{2} \tilde{\theta}^T \Gamma_\theta^{-1} \tilde{\theta} + \frac{|b|}{2\gamma} \tilde{\rho}^2 \quad (13)$$

where $\tilde{\theta} = \theta - \hat{\theta}$ and $\tilde{\rho} = \rho - \hat{\rho}$ denoting parameter estimation errors. Note that

$$
\begin{aligned}
z_2\Delta_2(x_1,x_2,u,t) &\leq |z_2|\sigma_2||(x_1,x_2)||_2 \\
&\leq \frac{1}{4e_{22}}z_2^2 + e_{22}(\sigma_2||(x_1,x_2)||_2)^2 \\
-D_{\alpha_1}z_2\Delta_1(x_1,x_2,u,t) &\leq |z_2D_{\alpha_1}|\sigma_1||(x_1,x_2)||_2 \qquad (14)\\
&\leq \frac{(D_{\alpha_1})^2}{4e_{21}}z_2^2 + e_{21}(\sigma_1||(x_1,x_2)||_2)^2
\end{aligned}
$$

we have

$$
\begin{aligned}
z_2\dot{z}_2 &= z_2(g_0(x) + g(x)^T\theta + bu + \Delta_2(x_1,x_2,u,t) \\
&\quad -D_{\alpha_1}(x_2 + \Delta_1(x_1,x_2,u,t))) \qquad (15)\\
&\leq z_2(g_0(x) + g(x)^T\theta + bu - D_{\alpha_1}x_2) + \frac{1}{4e_{22}}z_2^2 + e_{22} \\
&\quad (\sigma_2||(x_1,x_2)||_2)^2 + \frac{(D_{\alpha_1})^2}{4e_{21}}z_2^2 + e_{21}(\sigma_1||(x_1,x_2)||_2)^2
\end{aligned}
$$

From (8) (13) and (15), the derivative of V satisfies

$$
\begin{aligned}
\dot{V} &\leq -c_1z_1^2 + z_1z_2 + e_{11}\sigma_1^2||(x_1,x_2)||_2^2 + z_2(g_0(x) + g(x)^T\theta \\
&\quad +bu - D_{\alpha_1}x_2) + \frac{1}{4e_{22}}z_2^2 + e_{22}(\sigma_2||(x_1,x_2)||_2)^2 + z_2^2 \\
&\quad \frac{(D_{\alpha_1})^2}{4e_{21}} + e_{21}(\sigma_1||(x_1,x_2)||_2)^2 - \tilde{\theta}^T\Gamma_\theta^{-1}\dot{\hat{\theta}} - \frac{|b|}{\gamma}\tilde{\rho}\dot{\hat{\rho}} \qquad (16)
\end{aligned}
$$

Note that

$$
bu = b\hat{\rho}\bar{u} = b(\rho - \tilde{\rho})\bar{u} = \bar{u} - b\tilde{\rho}\bar{u} \qquad (17)
$$

With control law in (11), we have

$$
\begin{aligned}
\dot{V} &\leq -c_1z_1^2 + z_1z_2 + e_{11}\sigma_1^2||(x_1,x_2)||_2^2 + z_2(g_0(x) + g(x)^T\theta \\
&\quad +\bar{u} - b\tilde{\rho}\bar{u} - D_{\alpha_1}x_2) + \frac{1}{4e_{22}}z_2^2 + e_{22}(\sigma_2||(x_1,x_2)||_2)^2 \\
&\quad +\frac{(D_{\alpha_1})^2}{4e_{21}}z_2^2 + e_{21}(\sigma_1||(x_1,x_2)||_2)^2 - \tilde{\theta}^T\Gamma_\theta^{-1}\dot{\hat{\theta}} - \frac{|b|}{\gamma}\tilde{\rho}\dot{\hat{\rho}} \\
&= -c_1z_1^2 - c_2z_2^2 + e_{11}\sigma_1^2||(x_1,x_2)||_2^2 + e_{21}(\sigma_1||(x_1,x_2)||_2)^2 \\
&\quad +e_{22}(\sigma_2||(x_1,x_2)||_2)^2 - \tilde{\theta}^T\Gamma_\theta^{-1}\dot{\hat{\theta}} - \frac{|b|}{\gamma}\tilde{\rho}\dot{\hat{\rho}} \\
&\quad +z_2g(x)^T\tilde{\theta} - z_2b\tilde{\rho}\bar{u} \\
&\leq -c_1z_1^2 - c_2z_2^2 - \tilde{\theta}^T\Gamma_\theta^{-1}(\dot{\hat{\theta}} - \Gamma_\theta g(x)z_2) - \frac{|b|}{\gamma}\tilde{\rho}(\dot{\hat{\rho}} + \\
&\quad sign(b)\gamma\bar{u}z_2) + e_{11}\sigma_1^2||(x_1,x_2)||_2^2 + e_{21}(\sigma_1||(x_1,x_2)||_2)^2 \\
&\quad +e_{22}(\sigma_2||(x_1,x_2)||_2)^2 \qquad (18)
\end{aligned}
$$

Similar to Zhou (2007), we have

$$
\begin{aligned}
k_\rho\tilde{\rho}(\hat{\rho} - \rho_0) &= -k_\rho(\hat{\rho} - \rho)[\frac{1}{2}(\hat{\rho} - \rho) + \frac{1}{2}(\hat{\rho} + \rho) - \rho_0] \\
&= -\frac{1}{2}k_\rho(\hat{\rho} - \rho)^2 - k_\rho(\hat{\rho} - \rho)[\frac{1}{2}(\hat{\rho} + \rho) - \rho_0] \\
&= -\frac{1}{2}k_\rho\tilde{\rho}^2 - \frac{1}{2}k_\rho(\hat{\rho} - \rho)[(\hat{\rho} + \rho) - 2\rho_0] \\
&= -\frac{1}{2}k_\rho\tilde{\rho}^2 + \frac{1}{2}k_\rho[(\rho - \hat{\rho})(\hat{\rho} + \rho) - 2(\rho - \hat{\rho})\rho_0] \\
&= -\frac{1}{2}k_\rho\tilde{\rho}^2 + \frac{1}{2}k_\rho[\rho^2 - \hat{\rho}^2 - 2\rho_0\rho + 2\hat{\rho}\rho_0] \\
&= -\frac{1}{2}k_\rho\tilde{\rho}^2 + \frac{1}{2}k_\rho[\rho^2 - 2\rho_0\rho - (\hat{\rho} - \rho_0)^2 + \rho_0^2] \\
&\leq -\frac{1}{2}k_\rho\tilde{\rho}^2 + \frac{1}{2}k_\rho(\rho - \rho_0)^2 \qquad (19)
\end{aligned}
$$

and

$$
k_\theta\tilde{\theta}^T(\hat{\theta} - \theta_0) \leq -\frac{1}{2}k_\theta\|\tilde{\theta}\|_2^2 + \frac{1}{2}k_\theta\|\theta - \theta_0\|_2^2 \qquad (20)
$$

With (19) (20) and update laws (12), we obtain

$$
\begin{aligned}
\dot{V} &\leq -\sum_{i=1}^{2}c_iz_i^2 + \tilde{\theta}^Tk_\theta(\hat{\theta} - \theta_0) + |b|\tilde{\rho}k_\rho(\hat{\rho} - \rho_0) \\
&\quad +\sum_{i=1}^{2}\sum_{j=1}^{i}e_{ij}\sigma_j^2||(x_1,x_2)||_2^2 \\
&\leq -\sum_{i=1}^{2}c_iz_i^2 - \frac{|b|}{2}k_\rho\tilde{\rho}^2 - \frac{1}{2}k_\theta\|\tilde{\theta}\|_2^2 \\
&\quad +\sum_{i=1}^{2}\sum_{j=1}^{i}e_{ij}\sigma_j^2||(x_1,x_2)||_2^2 + \Pi \qquad (21)
\end{aligned}
$$

where

$$
\Pi = \frac{1}{2}k_\theta\|\theta - \theta_0\|_2^2 + \frac{|b|}{2}k_\rho(\rho - \rho_0)^2
$$

Now we are at the position to give our main result.

Theorem 1. Consider the closed loop system consisting of system (1), controller (11) and update laws (12). Under Assumptions 1 and 2, the following results held

- All signals in the closed-loop system are globally bounded if σ_1 satisfies
$$\sigma_1 < \sqrt{1-\varepsilon} \qquad (22)$$
where ε is an arbitrarily small positive number.
- The output y satisfies
$$lim_{t\to\infty}|y(t)| \leq \sqrt{\frac{1}{F_1}\Pi}$$

Proof.

- From (3) and virtual control given in (7), we have
$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 + (c_1 + \frac{1}{4e_{11}})z_1
\end{aligned} \qquad (23)
$$
Namely,
$$x = B(c_1,e_{11})z \qquad (24)$$
where $z = (z_1,z_2)^T$ and $B = \begin{pmatrix} 1 & 0 \\ -(c_1 + \frac{1}{4e_{11}}) & 1 \end{pmatrix}$. It is clear that
$$
\begin{aligned}
||x||_2 &\leq ||B(c_1,e_{11})||_F \cdot ||z||_2 \\
&= \Big(2 + (c_1 + \frac{1}{4e_{11}})^2\Big)^{\frac{1}{2}}||z||_2 \qquad (25)
\end{aligned}
$$
Similar to Wen (1999), letting $\chi = (z_1,\cdots z_n,\tilde{\rho},\tilde{\theta}^T)^T$ and from (21) (25), we have

$$\begin{aligned}\dot{V} \leq &-\sum_{l=1}^{2} c_l z_l^2 - \frac{|b|}{2}k_\rho \tilde{\rho}^2 - \frac{1}{2}k_\theta \|\tilde{\theta}\|_2^2 \\ &+\sum_{i=1}^{2}\sum_{j=1}^{i} e_{ij}\sigma_j^2 ||(x_1,x_2)||_2^2 + \Pi \\ \leq &-\sum_{l=1}^{2} c_l z_l^2 - \frac{|b|}{2}k_\rho \tilde{\rho}^2 - \frac{1}{2}k_\theta \|\tilde{\theta}\|_2^2 \\ &+\sum_{i=1}^{2}\sum_{j=1}^{i} e_{ij}\sigma_j^2 \Big(2 + (c_1 + \frac{1}{4e_{11}})^2\Big)||z||_2^2 + \Pi \\ = &-\sum_{l=1}^{2}\Big[c_l - \sum_{i=1}^{2}\sum_{j=1}^{i} e_{ij}\sigma_j^2 \Big(2 + (c_1 + \frac{1}{4e_{11}})^2\Big)\Big] z_l^2 \\ &-\frac{|b|}{2}k_\rho \tilde{\rho}^2 - \frac{1}{2}k_\theta \|\tilde{\theta}\|_2^2 + \Pi \\ \leq &-F_1||\chi||_2^2 + \Pi \end{aligned} \tag{26}$$

where $F_1 = min\{\Big[c_l - \sum_{i=1}^{2}\sum_{j=1}^{i} e_{ij}\sigma_j^2 \Big(2 + (c_1 + \frac{1}{4e_{11}})^2\Big)\Big], \frac{|b|}{2}k_\rho, \frac{1}{2}k_\theta\}$. If we choose design parameters c_1, c_2, e_{11} satisfy that

$$c_l > \sum_{i=1}^{2}\sum_{j=1}^{i} e_{ij}\sigma_j^2 \Big(2 + (c_1 + \frac{1}{4e_{11}})^2\Big) \quad (l = 1, 2) \tag{27}$$

all the signals of closed-loop system are bounded from (26) and thus the stability of closed-loop system is established. To analyze the existence of design parameters $c_1, c_2, e_{11}, e_{21}, e_{22}$ from condition (27), we note that

$$V \leq F_2||\chi||_2^2 \tag{28}$$

where $F_2 = max\{\frac{1}{2}, \frac{1}{2}\lambda_{max}(\Gamma_\theta^{-1}), \frac{1}{2\gamma}\}$ and $\lambda_{max}(\Gamma_\theta^{-1})$ is the maximum eigenvalue of matrix Γ_θ^{-1}. Then we have

$$\dot{V} \leq -\frac{F_1}{F_2}V + \Pi \tag{29}$$

Similar to Zhou (2007) and Zhou (2004), by direct integrations of the differential inequality, we obtain

$$V \leq V(0)e^{-\frac{F_1}{F_2}t} + \frac{F_2}{F_1}\Pi \tag{30}$$

Now from (27), the design parameters should satisfy that

$$\begin{cases} c_1 > \sum_{i=1}^{2}\sum_{j=1}^{i} e_{ij}\sigma_j^2 \Big(2 + (c_1 + \frac{1}{4e_{11}})^2\Big) \\ c_2 > \sum_{i=1}^{2}\sum_{j=1}^{i} e_{ij}\sigma_j^2 \Big(2 + (c_1 + \frac{1}{4e_{11}})^2\Big) \end{cases} \tag{31}$$

From (31), it is clear that when c_1 is determined, c_2 can be chosen easily. So the key lies in how select c_1 is chosen to ensure the first inequality. This inequality can be rewritten as

$$\begin{aligned}&[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]c_1^2 + \Big(\frac{[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]}{2e_{11}} \\ &-1\Big)c_1 + [(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]\Big(2 + \frac{1}{16(e_{11})^2}\Big) < 0\end{aligned} \tag{32}$$

It is a quadratic inequalities of c_1. By Vietas formula which shows the relationship between roots and coefficients, (32) has positive solution when $e_{11}, e_{21}, e_{22}, \sigma_1, \sigma_2$ meet one of the following conditions:

$$(*) \begin{cases} \Delta > 0 \\ \Pi_p < 0 \end{cases} or(**) \begin{cases} \Delta > 0 \\ \Sigma_s > 0 \\ \Pi_p > 0 \end{cases} or(***) \begin{cases} \Delta = 0 \\ \Sigma_s > 0 \end{cases} \tag{33}$$

where Δ is the discriminant of roots of unary quadric equation and Σ_s, Π_p represent the sum and product of two solutions, respectively. Note that $\Pi_p = \Big(2 + \frac{1}{16(e_{11})^2}\Big) > 0$, so the first case (*) is impossible. From the second and third cases, we have

$$\Sigma_s = \frac{1 - \frac{[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]}{2e_{11}}}{[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]} > 0 \tag{34}$$

By $[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2] > 0$, we need

$$1 - \frac{[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]}{2e_{11}} > 0 \tag{35}$$

Namely,

$$1 > \frac{[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]}{2e_{11}} \tag{36}$$

Then

$$[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2] < 2e_{11} \tag{37}$$

By $\Delta \geq 0$, we have

$$\begin{aligned}&\Big(\frac{[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2]}{2e_{11}} - 1\Big)^2 - 4\Big((e_{11}+e_{21})\sigma_1^2 \\ &+e_{22}\sigma_2^2\Big)^2\Big(2 + \frac{1}{16(e_{11})^2}\Big) \geq 0\end{aligned} \tag{38}$$

So

$$\begin{aligned}&\Big(\frac{[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2] - 2e_{11}}{2e_{11}}\Big)^2 \geq 4\Big((e_{11}+e_{21})\sigma_1^2 \\ &+e_{22}\sigma_2^2\Big)^2\Big(2 + \frac{1}{16(e_{11})^2}\Big)\end{aligned} \tag{39}$$

From (37) and taking square root of both sides of (39), we obtain

$$\begin{aligned}&2e_{11} - [(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2] \geq 4e_{11}\Big((e_{11}+e_{21})\sigma_1^2 \\ &+e_{22}\sigma_2^2\Big)\sqrt{2 + \frac{1}{16(e_{11})^2}}\end{aligned} \tag{40}$$

Namely,

$$\begin{aligned}&2e_{11} - [(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2] \geq \Big((e_{11}+e_{21})\sigma_1^2 \\ &+e_{22}\sigma_2^2\Big)\sqrt{32(e_{11})^2 + 1}\end{aligned} \tag{41}$$

So

$$[(e_{11}+e_{21})\sigma_1^2 + e_{22}\sigma_2^2] \leq \frac{2e_{11}}{1 + \sqrt{32(e_{11})^2 + 1}} \tag{42}$$

Compared with inequalities (37) and (42), we can have positive design parameters $e_{11}, e_{21}, e_{22}, \sigma_1, \sigma_2$ if (42) is true. Rewrite (42) as

$$[e_{11}\sigma_1^2 + (e_{21}\sigma_1^2 + e_{22}\sigma_2^2)] \leq \frac{2e_{11}}{1 + \sqrt{32(e_{11})^2 + 1}} \tag{43}$$

Note that the right of (43) does not depend on parameters e_{21} and e_{22}. So we can choose these two parameters such that $(e_{21}\sigma_1^2 + e_{22}\sigma_2^2) = e_{11}\varepsilon$, where $\varepsilon > 0$ is an arbitrarily small positive real number. Then we have

$$[e_{11}\sigma_1^2 + e_{11}\varepsilon] \leq \frac{2e_{11}}{1+\sqrt{32(e_{11})^2+1}} \tag{44}$$

$$e_{11}\sigma_1^2 \leq \frac{2e_{11}}{1+\sqrt{32(e_{11})^2+1}} - e_{11}\varepsilon \tag{45}$$

$$\sigma_1^2 \leq \frac{2}{1+\sqrt{32(e_{11})^2+1}} - \varepsilon < 1 - \varepsilon \tag{46}$$

Namely,

$$\sigma_1 < \sqrt{1-\varepsilon} \tag{47}$$

- If

$$|y(t)| > \sqrt{\frac{1}{F_1}\Pi} \tag{48}$$

then

$$\dot{V} \leq -F_1||\chi||_2^2 + \Pi < 0 \tag{49}$$

So V will decrease till $|y(t)| \leq \sqrt{\frac{1}{F_1}\Pi}$.

■

We now derive guidelines for choosing design parameters. Based on (22), we choose small positive numbers ε_1 and ε_2 such that

$$\sigma_1^2 < 1 - \varepsilon_1 - \varepsilon_2 \tag{50}$$

We determine the parameters e_{11}, e_{21} and e_{22} from the following conditions:

$$\frac{2}{1+\sqrt{32(e_{11})^2+1}} \geq 1 - \varepsilon_1,$$
$$(e_{21}\sigma_1^2 + e_{22}\sigma_2^2) \leq e_{11}\varepsilon_2 \tag{51}$$

Then parameter c_1 and c_2 can be selected based on (32) and (31), respectively.

4. SIMULATION STUDIES

To illustrate the effectiveness of the proposed scheme, we consider the following second-order system

$$\dot{x}_1 = x_2 + \Delta_1(x_1, x_2, u, t)$$
$$\dot{x}_2 = (2 + cos(x_1x_2))\theta + u + \Delta_2(x_1, x_2, u, t) \tag{52}$$

where x_1, x_2 are system states and u is the input, $\theta = 2$ is an unknown parameter and $\Delta_1(x_1, x_2, u, t)$, $\Delta_2(x_1, x_2, u, t)$ are modeling errors. Suppose

$$\Delta_1(x_1, x_2, u, t) = 0.1sin(t)\sqrt{x_1^2 + x_2^2}$$
$$\Delta_2(x_2, x_2, u, t) = sin(\sqrt{(x_1^2 + x_2^2)}) \tag{53}$$

Note that $|\Delta_2(x_2, x_2, u, t)| \leq \sqrt{x_1^2 + x_2^2} = ||(x_1, x_2)||_2$ and $\Delta_1(x_1, x_2, u, t)$ satisfies that $\Delta_1(x_1, x_2, u, t) \leq 0.1||(x_1, x_2)||_2$, namely $\sigma_1 = 0.1$. As Δ_1 depends on both x_1 and x_2, system (52) does not meet the triangular structure requirement and thus existing backstepping approach cannot be applied to it. With our proposed scheme, choose $\varepsilon_1 = \varepsilon_2 = 0.01$. Then based on (51), (31) and (32), $c_1 = c_2 = 1, e_{11} = e_{21} = e_{22} = 0.1$. We also select $l_\theta = 1, \theta_0 = 1$ and $\Gamma_\theta = 1$. The initial values are chosen as follows: $x_1(0) = 1.5, x_2(0) = -5, \hat{\theta}(0) = 0, u(0) = 0$. Fig.1 and Fig.2 shows the states x_1 and x_2 which illustrate and verify our theoretical results in Theorem 1.

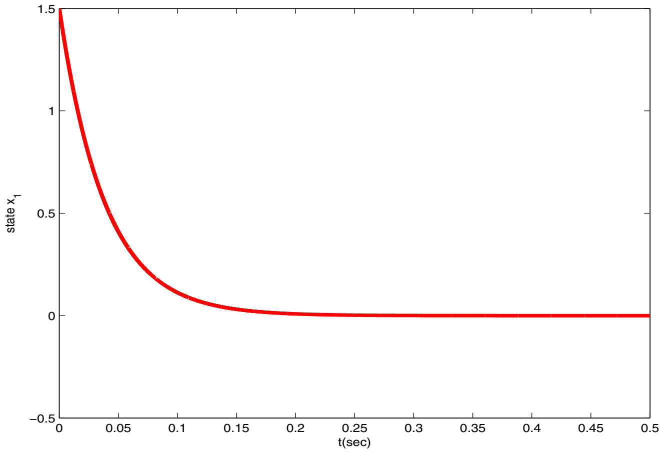

Fig. 1. State $x_1(t)$

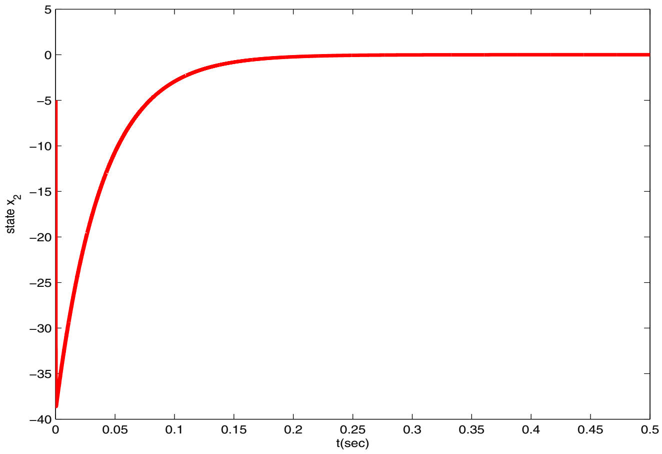

Fig. 2. State $x_2(t)$

5. CONCLUSION

In this paper, we develop a new robust adaptive control scheme based on backstepping approach for uncertain second-order systems with modeling errors in each state equation. With our scheme, it is not necessary for the system to meet the triangular structure required by existing backstepping approaches. It is shown that the proposed controller can ensure that all the signals of the closed-loop system are bounded. Simulation studies also verify the effectiveness of our theoretical results.

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