Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Global Swarming while Preserving Connectivity via Lagrange-Poincarè Equations

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Abstract: In this paper, we exploit symmetry properties of multi-agent robot systems to design control laws that preserve connectivity while swarming. We start by showing that the connectivity controller is invariant under the action of the special Euclidean group $SE(3)$ and therefore is amenable to reduction of the dynamics by this action. We then utilize the reduced Euler-Lagrange equations that split the Euler-Lagrange equations for the multi-agent system into horizontal and vertical parts. The invariance of the connectivity controller implies that its control effort has zero vertical component. We then use the resulting vertical equations of motion to design a control law that asymptotically stabilizes the centroid and the orientation of the swarm at a desired pose.

Keywords: multiagent, horizontal-vertical, swarming, symmetry

1. INTRODUCTION

Mobile robot networks afford an inexpensive and robust method for achieving certain coverage tasks or cooperative missions. Many of the algorithms employed to achieve such tasks depend on communication between any two robots and hence require connectivity of the communication network. As a result, the problem of maintaining connectivity in mobile robot networks is an active area of research. For many applications, the edges or links in the mobile robot network are functions of the relative positions of nodes in the network. Thus, the connectivity of the network is affected by the motion of the robots, and the motion controllers must maintain connectivity in addition to achieving other goals.

One of the most important goals of a multi-agent mobile network is coverage or surveillance of a given area. This requires the agents to swarm or move in formation along a desired path/-trajectory. In other words, it is desired that the centroid of the formation move along a specified desired trajectory. In addition when avoiding contact with the environment is an issue, we may also want to specify a desired orientation trajectory of the multi-agent system.

A review of different methods to control and maintain connectivity can be found in Zavlanos et al. [2011]. Connectivity can be maintained in a centralized or decentralized manner. One of the simplest methods of ensuring connectivity is to assume that the network is initially connected, and that existing edges in the network are maintained for all time Ji and Egerstedt [2007], Zavlanos and Pappas [2005]. Another centralized method for maintaining connectivity in a group of mobile robots is to maximize the second smallest eigenvalue of the graph Laplacian Kim and Mesbahi [2006], when the edge strengths are non-increasing functions of the distance between robots. The resulting graph is always connected Kim and Mesbahi [2006]. This method is effective for solving rendezvous problems, and can be extended to other applications Zavlanos et al. [2011].

This paper addresses both control problems, i.e. swarming and connectivity maintenance. The connectivity controller is applied as in Satici et al. [2013], This controller achieves and maintains connectivity. We show that this connectivity controller is invariant under the action of the special Euclidean group $SE(3)$. This becomes important when we introduce the splitting of the Euler-Lagrange equations into horizontal and vertical parts where the horizontal part governs the behavior of the multi-agents system's internal configuration, i.e., the position of the agents relative to each other, while the vertical part govern the behavior of the agent formation as a rigid body. The fact that the connectivity control is invariant under the $SE(3)$ action implies that the control effort expended by this controller has no vertical part.

After the splitting of the Euler-Lagrange equations into horizontal and vertical parts, we make use of the vertical part of the resulting equations to impose asymptotically convergent swarming behavior into the system. In other words, we use the available control input in the vertical direction to asymptotically stabilize the desired multi-agent centroid and orientation trajectory. We provide simulation results where we take three-agents each of whose configuration space is $\mathbb{R}^3$ and show that they can be made to swarm with the desired centroid and orientation while the connectivity measure is increased to a desired value.

One of the contributions of this paper is that it presents a framework in which swarming behavior of agents each living in R^3 can be cast into a canonical form globally. In other words, as opposed to Michael et al. [2006], this work does not assume that the configuration space of the robots can be written as the product of a base space S and a group G. Although given in its coordinate form, the Lagrange-Poincaré equations, or the reduced Euler-Lagrange equations, hold globally. This means, one can potentially find a coordinate system whose domain of validity is larger that the one used in this paper and use this coordinate system to achieve the desired swarming behavior. The fact that inputs in charge of swarming behavior operates along the directions tangent to the group orbits, means that any existing controller that acts in the horizontal space is unaffected provided the kinetic energy (metric) of the original system is group invariant. We exploit this property to employ our earlier connectivity controller to simultaneously achieve a desired connectivity measure.

2. BACKGROUND

In this section we introduce various definitions and background material that we will use in the sequel. We also present the final splitting of the original equations of motion, called the Lagrange-Poincaré equations.

2.1 Preliminaries and definitions

We consider N agents, each of whose configuration q_i is an element of $\mathbb{R}^3$. These agents are all simple mechanical systems with kinetic energy $K_i(q_i, \dot{q}_i) = \frac{1}{2}m_i\|\dot{q}_i\|^2$, where $m_i > 0$ is the mass of the i^{th} agent and $\dot{q}_i \in T_{q_i}\mathbb{R}^3 \cong \mathbb{R}^3$ is its velocity at the position q_i. Its potential energy is given by the action of the gravity so that the Lagrangian is given by

$$L_i(q_i, \dot{q}_i) = \frac{1}{2}m_i\|\dot{q}_i\|^2 - m_i g\,(q_i \cdot e_3) \tag{1}$$

where g is the gravitational acceleration and e_3 is the third standard basis vector of $\mathbb{R}^3$. The Euler-Lagrange equations are

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L_i}{\partial \dot{q}_i} - \frac{L_i}{q_i} = m_i\ddot{q}_i - m_i g e_3 = u_i \tag{2}$$

where $u_i \in T^*\mathbb{R}^3 \cong T\mathbb{R}^3$ is the control force on agent i. Now, let us assume we can use our controls to cancel out the gravitational acceleration. Thus, we use the controls to arrive at the Lagrangian $\tilde{L}_i = \frac{1}{2}m_i\|\dot{q}_i\|^2$ which will be our initial Lagrangian for each agent from now on.

The full configuration space consisting of all N agents is the product space of each individual agent's configuration space, i.e., $Q = \prod_{i=1}^{N}\mathbb{R}^3$. The Euclidean group $SE(3)$ acts on the configuration space Q as follows. Consider the simplex C in 3-dimensions whose vertices are formed by the positions of the individual agents. The special Euclidean group acts on this simplex as a rigid body, i.e., rotates and translates it. We express this mathematically by taking an element $g = (R, p) \in SE(3)$ and forming the following map $\Phi : SE(3) \times Q \to Q$

$$\Phi_g(q) = (Rq_1 + (I-R)\bar{q} + p, \ldots, Rq_N + (I-R)\bar{q} + p) \tag{3}$$

where I is the 3×3 identity matrix and $\bar{q} = \frac{1}{N}\sum_{i=1}^{N} q_i$ is the centroid of the simplex C. Let $\xi = (\omega, v) \in \mathfrak{g}$ be an element of the Lie algebra of the Lie group $SE(3)$ acting on the configuration space Q. Then the **infinitesimal generator** $\xi_Q \in C^\infty(Q)$ of this group action (3) is given by

$$\begin{aligned}\xi_Q(q) &= \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \exp(\epsilon\xi)\cdot q \\ &= (\omega \times (q_1 - \bar{q}) + v, \ldots, \omega \times (q_N - \bar{q}) + v)\end{aligned} \tag{4}$$

For each $q \in Q$, the **locked inertia tensor**, is the map $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$, defined by

$$\langle \mathbb{I}(q)\eta, \zeta\rangle = \langle\!\langle \eta_Q(q), \zeta_Q(q)\rangle\!\rangle \tag{5}$$

where $\langle\!\langle \cdot, \cdot\rangle\!\rangle$ is the kinetic energy metric defined on TQ. The locked inertia tensor specifies the inertia of the system whose internal degrees of freedom are frozen. In other words, if the relative distances of the robots are constrained to be constant, then the locked inertia tensor is the inertia tensor resulting body.

Associated with the action of the group $SE(3)$ on Q is a momentum map $\mathbf{J} : TQ \to \mathfrak{g}^*$ defined by

$$\langle \mathbf{J}(q, \dot{q}), \xi\rangle = \langle\!\langle \dot{q}, \xi_Q(q)\rangle\!\rangle, \quad \forall \xi \in \mathfrak{g} \tag{6}$$

The **mechanical connection** $A : TQ \to \mathfrak{g}$ is then found by the relation

$$A(q, \dot{q}) := \mathbb{I}(q)^{-1}(\mathbf{J}(q, \dot{q})) \tag{7}$$

The **horizontal space** of the connection A is given by

$$\mathrm{hor}_q = \{(q, \dot{q}) \in T_qQ : \mathbf{J}(q, \dot{q}) = 0\} \tag{8}$$

which is the subspace of the tangent space at $q \in Q$ that is metric orthogonal to the orbits of the group action. The **vertical space** consists of vectors that are mapped to zero by the projection map $\pi : Q \to Q/G$

$$\mathrm{ver}_q = \{\xi_Q(q) \in T_qQ : \xi \in \mathfrak{g}\} \tag{9}$$

These two subspaces of T_qQ are complementary, i.e., any a vector $(q, \dot{q}) \in T_qQ$ can be decomposed as

$$\dot{q} = \mathrm{hor}_q\dot{q} + \mathrm{ver}_q\dot{q} \tag{10}$$

where

$$\mathrm{ver}_q\dot{q} = [A(q, \dot{q})]_Q(q) \qquad \text{and} \qquad \mathrm{hor}_q\dot{q} = \dot{q} - \mathrm{ver}_q\dot{q} \tag{11}$$

2.2 Connectivity controller

In Satici et al. [2013], we derived a control law for agents whose dynamics are given by first-order integrators. Here, we use the same connectivity controller in order to keep the agents at the desired connectivity measure. In that work, it was shown that the controller achieves the desired connectivity measure and no more, i.e., it shuts off after the desired connectivity measure has been obtained. This is a desired behavior because it allows one to achieve additional criteria, such as formation control.

Following Satici et al. [2013], we define a weighted graph $G = (V, W)$, where $V = \{1, \ldots, N\}$ is the set of nodes and $W : V \times V \times \mathbb{R}_+ \to \mathbb{R}_+$ is the function that determines the weight of the edges. If $w_{ij}(t) := W(i, j, t) = 0$, then there is no connection between nodes i and j. These edge weights give rise to the graph *Laplacian* $\mathcal{L} \in \mathbb{R}^{N\times N}$ defined as

$$\mathcal{L}(t) = \begin{cases} -w_{ij}(t) & \text{if } i \neq j \\ \sum_{k\neq i} w_{ik}(t) & \text{if } i = j \end{cases} \tag{12}$$

The Laplacian gives us a measure of the connectivity of the graph G since the number of connected components in the graph is equal to the number of zero eigenvalues of $\mathcal{L}$. Thus, for the graph to be connected, at most one eigenvalue of $\mathcal{L}$ can be zero. The second smallest eigenvalue $\lambda_2(\mathcal{L})$ thus becomes an indicator of connectivity in the graph.

The Laplacian $\mathcal{L}$ can be converted to a *reduced Laplacian* matrix $\mathcal{M} \in \mathbb{R}^{N-1\times N-1}$, whose eigenvalues are the largest $N-1$ eigenvalues of $\mathcal{L}$. The matrix $\mathcal{M}$ is given by

$$\mathcal{M} = \mathcal{P}^T \mathcal{L} \mathcal{P} \tag{13}$$

where $\mathcal{P} \in \mathbb{R}^{N\times N-1}$ satisfies $\mathcal{P}^T\mathbf{1} = 0$ and $\mathcal{P}^T\mathcal{P} = I_{N-1}$. Thus, the determinant of $\mathcal{M}$ vanishes if and only if $\lambda_2(L)$ vanishes.

The connectivity control law for first-order agents is defined to be the gradient of the potential function

$$V_c(q) = \frac{\bar{\alpha}^2 - \det(\mathcal{M})^2}{\underline{\alpha}^2 - \det(\mathcal{M})^2} \tag{14}$$

where $\bar{\alpha}$ denotes the desired value of the connectivity measure and $\underline{\alpha}$ is a lower bound on the connectivity measure below which we do not want our measure to drop. The entries of the Laplacian matrix and consequently the reduced Laplacian matrix $\mathcal{M}$ are populated by the weights between pairs of agents. This weighting between two robots is a monotonic function of the Euclidean distance $\|q_i - q_j\|$ between the positions of the pair of robots.

With the equations of motion given by

$$m\ddot{q} = u \tag{15}$$

We choose the control input u as

$$u = -(V_c)_{*q} + \tilde{u} \tag{16}$$

where we regard $(V_c)_{*q}$ as an element of T_q^*Q and where $\tilde{u}$ is an additional control term to be designed. If we relabel $\tilde{u}$ as u again, then we arrive at the control system

$$m\ddot{q} + (V_c)_{*q} = u \tag{17}$$

Note that this is still a Lagrangian dynamical system with the Lagrangian function $L(q,\dot{q}) = \frac{m}{2}\sum_{i=1}^N \|\dot{q}_i\|^2 + V_c(q)$. Therefore, when unforced, the system of differential equations (17) is invariant with respect to the action of $SE(3)$ if and only if $V_c(q)$ is invariant under the same action Olver [2000].

Proposition 1. The connectivity potential function $V_c : Q \to Q$ is invariant under the action of $SE(3)$ on Q.

Proof. $V_c(q)$ is invariant under the action of $SE(3)$ on Q if and only if $\det(\mathcal{M}(q))$ is invariant under this action. If $w_{ij}(q) = w_{ij}(\Phi_g(q))$, then $\det(\mathcal{M}(\Phi_g(q))) = \det(\mathcal{M}(q))$. Since the weight w_{ij} are continuous monotonic functions of the distance between two robots, this condition holds whenever $\|\Phi_g(q_i) - \Phi_g(q_j)\| = \|q_i - q_j\|$. Finally, the following simple calculation proves the assertion

$$\begin{aligned}
&\|\Phi_g(q_i) - \Phi_g(q_j)\|^2 \\
&= \|Rq_i - (I-R)\bar{q} + p - (Rq_j + (I-R)\bar{q} + p)\|^2 \\
&= \|R(q_i - q_j)\|^2 = (q_i - q_j)^T R^T R (q_i - q_j) = \|q_i - q_j\|^2
\end{aligned}$$

3. CALCULATIONS FOR THE NECESSARY OBJECTS

In this section, we calculate the quantities that we have defined in the last section for our system of N-agents, each of which resides in the Euclidean 3-space. In addition, we shall write the infinitesimal generator, the locked inertia tensor in coordinates chosen for the Lie algebra $\mathfrak{g}$. We start with the calculation of the locked inertia tensor, defined in equation (5).

Proposition 2. For the action of $SE(3)$ on $Q = \mathbb{R}^{3N}$ as given in (3), the locked inertia tensor $\mathbb{I}(q) : \mathfrak{g} \to \mathfrak{g}^*$ is given by

$$\mathbb{I}(q) = m \begin{bmatrix} \sum_{i=1}^N \|z_i\|^2 I_3 - z_i z_i^T & 0 \\ 0 & N I_3 \end{bmatrix} \tag{18}$$

where $z_i = q_i - \bar{q}$, 0 and I_3 are the 3×3 zero and identity matrix, respectively.

Proof.

Let $\eta = (\omega, v)$ and $\zeta = (\omega', v')$ be two elements of $\mathfrak{g}$. We calculate the right hand side of (5) by taking $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ as the usual Euclidean inner product on $\mathbb{R}^{3N}$.

$$\begin{aligned}
&\langle\!\langle \eta_Q(q), \zeta_Q(q) \rangle\!\rangle \\
&= m\sum_{i=1}^N (\omega\times(q_i-\bar{q}))\cdot(\omega'\times(q_i-\bar{q})) \\
&+ (\omega\times(q_i-\bar{q}))\cdot v' + v\cdot(\omega'\times(q_i-\bar{q})) + v\cdot v' \\
&= m\sum_{i=1}^N (\omega\cdot\omega')\|q_i-\bar{q}\|^2 - (\omega\cdot(q_i-\bar{q}))\left((q_i-\bar{q})\cdot\omega'\right) \\
&\quad + (q_i-\bar{q})\cdot(v\times\omega' + v'\times\omega) + v\cdot v'
\end{aligned} \tag{19}$$

where $\cdot$ represents the dot product. We have used the cyclic invariance of the triple product and the Binet-Cauchy identity that reads for any $a,b,c,d \in \mathbb{R}^3$,

$$(a\times b)\cdot(c\times d) = (a\cdot c)(b\cdot d) - (a\cdot d)(b\cdot c)$$

Now, the second term in the summation in equation (19) can be written as

$$(\omega\cdot(q_i-\bar{q}))\left((q_i-\bar{q})\cdot\omega'\right) = z_i z_i^T \omega\cdot\omega'$$

while the third term in the same equation (19) can be written as

$$(q_i-\bar{q})\cdot(v\times\omega' + v'\times\omega) = \hat{z}_i(v+v')\cdot(\omega+\omega')$$

where $\hat{z}_i$ is the skew-symmetric matrix constructed from the components of z_i. Thus, combining these equations together, we see that (19) is equivalently written as

$$\begin{aligned}
\langle\!\langle \eta_Q(q), \zeta_Q(q) \rangle\!\rangle &= m\sum_{i=1}^N \begin{bmatrix} \|z_i\|^2 I_3 - z_i z_i^T & -\hat{z}_i \\ \hat{z}_i & I_3 \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix} \cdot \begin{bmatrix} \omega' \\ v' \end{bmatrix} \\
&= \langle m\sum_{i=1}^N \begin{bmatrix} \|z_i\|^2 I_3 + z_i z_i^T & -\hat{z}_i \\ \hat{z}_i & I_3 \end{bmatrix} \eta, \zeta \rangle
\end{aligned}$$

Finally, we note that $\sum_{i=1}^N z_i = \sum_{i=1}^N q_i - \bar{q} = 0$, so that $\sum_{i=1}^N \hat{z}_i = 0$. As a result, $\mathbb{I}(q)$ reduces to its form given in equation (18).

We next proceed with the calculation of the momentum map (6).

Proposition 3. For the action of $SE(3)$ on $Q = \mathbb{R}^{3N}$ as given in (3), the locked momentum map $\mathbf{J} : TQ \to \mathfrak{g}^*$ is given by

$$\mathbf{J} = m\left(\sum_{i=1}^{N} z_i \times \dot{q}_i, \sum_{i=1}^{N} \dot{q}_i\right) \in \mathfrak{g}^* \tag{20}$$

Proof. Let $\xi = (\omega, v) \in \mathfrak{g}$, we have

$$\begin{aligned}\langle\langle \dot{q}_i, \xi_Q(q)\rangle\rangle &= m\sum_{i=1}^{N} \dot{q}_i \cdot (\omega \times (q_i - \bar{q}) + v) \\ &= m\sum_{i=1}^{N} \omega \cdot (z_i \times \dot{q}_i) + v \cdot \dot{q}_i \\ &= \langle m\left(\sum_{i=1}^{N} z_i \times \dot{q}_i, \sum_{i=1}^{N} \dot{q}_i\right), (\omega, v)\rangle\end{aligned}$$

Finally, in our case of $SE(3)$ acting on $Q = \prod_{i=1}^{N} \mathbb{R}^3$, the mechanical connection is calculated as follows:

$$\begin{aligned}&A(q, \dot{q}) := \mathbb{I}(q)^{-1}\left(\mathbf{J}(q, \dot{q})\right) \\ &= \frac{1}{m}\begin{bmatrix}\left(\sum_{i=1}^{N}\|z_i\|^2 I_3 - z_i z_i^T\right)^{-1} & 0 \\ 0 & \frac{1}{N} I_3\end{bmatrix} m \begin{bmatrix}\sum_{i=1}^{N} z_i \times \dot{q}_i \\ \sum_{i=1}^{N} \dot{q}_i\end{bmatrix} \\ &= \begin{bmatrix}\left(\sum_{i=1}^{N}\|z_i\|^2 I_3 - z_i z_i^T\right)^{-1}\left(\sum_{i=1}^{N} z_i \times \dot{q}_i\right) \\ \frac{1}{N}\sum_{i=1}^{N} \dot{q}_i\end{bmatrix}\end{aligned} \tag{21}$$

When we introduce a basis $\{e_a\}_{a=1}^{6}$ of $\mathfrak{se}(3)$, we can express any element ξ of $\mathfrak{se}(3)$ as $\xi = \xi^a e_a$. Corresponding to the representation, the infinitesimal generator $\xi_Q(q)$ can be expressed as $\xi_Q(q) = K_a^i(q)\xi^a \frac{\partial}{\partial q^i}$ (summation convention in force).

If we choose e_a as the constant vector with a 1 at the a^{th} position and 0 everywhere else, and identify these vectors with the basis matrices of $\mathfrak{se}(3)$ as follows

$$e_1 \leftrightarrow \begin{bmatrix}0&0&0&0\\0&0&-1&0\\0&1&0&0\\0&0&0&0\end{bmatrix} \quad e_2 \leftrightarrow \begin{bmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{bmatrix}$$

$$e_3 \leftrightarrow \begin{bmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad e_4 \leftrightarrow \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$$

$$e_5 \leftrightarrow \begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{bmatrix} \quad e_6 \leftrightarrow \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{bmatrix}$$

we get the following components for the Lie algebra element $\xi \in \mathfrak{g}$:

$$\xi = \xi^a e_a = \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3 + v_1 e_4 + v_2 e_5 + v_3 e_6$$

By the definition of the infinitesimal generator (4), we find the following components for $\xi_Q(q)$

$$\begin{aligned}[\xi_Q(Q)]^{3i-2} &= z_{i3}\xi^2 - z_{i2}\xi^3 + \xi^4 \\ [\xi_Q(Q)]^{3i-1} &= -z_{i3}\xi^1 + z_{i1}\xi^3 + \xi^5 \\ [\xi_Q(Q)]^{3i} &= z_{i2}\xi^1 - z_{i1}\xi^2 + \xi^6\end{aligned} \tag{22}$$

so that

$$K^{3i-2\,:3i} = \begin{bmatrix}0 & z_{i3} & -z_{i2} & 1 & 0 & 0 \\ -z_{i3} & 0 & z_{i1} & 0 & 1 & 0 \\ z_{i2} & -z_{i1} & 0 & 0 & 0 & 1\end{bmatrix} \tag{23}$$

for $i \in \{1, \ldots, N\}$. Now, in coordinates, the locked inertia tensor is given by $\mathbb{I}_{ab} = g_{ij}K_a^i K_b^j$, where g_{ij} are the components of the metric tensor relative to coordinates q^i, $i \in \{1, \ldots 3N\}$ on Q. Therefore, in our case, $g_{ij} = m\delta_{ij}$, where δ_{ij} is the Kronecker delta. As a result, we have

$$\mathbb{I}_{ab} = m\sum_{i=1}^{3N} K_a^i K_b^i \tag{24}$$

Moreover, we get the components of the mechanical connection from the definition (7) as $A_j^a = \mathbb{I}^{ab} g_{ij} K_b^i = \mathbb{I}^{ab} K_b^j$.

4. LAGRANGE-POINCARÉ EQUATIONS

We next work out the equations of motion in as much detail as possible for the case $N = 3$. The final equations are quite involved but can be worked out by a symbolic manipulation software like Mathematica Wolfram Research [2012]. When we have more than three agents the same procedure can be followed without modification, but the resulting equations are more involved.

We start by choosing coordinates for the group variables in the trivialization of the principal bundle $\pi : Q \to Q/SE(3)$. Consider the coordinates $(\phi, \theta, \psi, \bar{x}, \bar{y}, \bar{z}) \in \mathbb{R}^6$ on $SE(3)$. The first three components of these coordinates are EulerXYZ angles, i.e.,

$$R = \exp(\phi e_1)\exp(\theta e_2)\exp(\psi e_3)$$

and the last three are the coordinates of the center of mass of the three agents (since we assume $m_i = m$, $\forall i \in \underline{N}$, this is equal to the centroid of the triangle formed by q_1, q_2 and q_3).

For the base space $Q/SE(3)$, we use the coordinates (r_1, r_2, r_3) on $V \subseteq Q/SE(3)$ (V open in $Q/SE(3)$), defined as the distances between each of the agents

$$\begin{aligned}r_1 &= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} \\ r_2 &= \sqrt{(x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2} \\ r_3 &= \sqrt{(x_2 - x_3)^2 + (y_2 - y_3)^2 + (z_2 - z_3)^2}\end{aligned} \tag{25}$$

For any tangent vector $(r, g, \dot{r}, \dot{g}) \in T_{(r,g)}(V \times G)$, we have

$$A(r, g, \dot{r}, \dot{g}) = \mathrm{Ad}_g\left(A_e(r) \cdot \dot{r} + v\right) \tag{26}$$

where A_e is the $\mathfrak{g}$-valued 1-form on R defined by $A_e(r) \cdot \dot{r} = A(r, e, \dot{r}, 0)$ and $v = g^{-1}\dot{g}$.

The given left-invariant Lagrangian $L : TQ \to \mathbb{R}$ induces a reduced Lagrangian $l : TQ/G \to \mathbb{R}$ which is represented in local coordinates corresponding to a local trivialization as

$$l(r^\alpha, \dot{r}^\alpha, \eta^a)$$

Rewriting the reduced Lagrangian $l(r^\alpha, \dot{r}^\alpha, \eta^a)$ in terms of the "locked" angular velocity ξ defined by $\xi^a = A^a_\alpha \dot{r}^\alpha + \eta^a$, we get the locked Lagrangian, i.e.,

$$l_{\text{lock}}(r^\alpha, \dot{r}^\alpha, \xi^a) = l(r^\alpha, \dot{r}^\alpha, \xi^a - A^a_\alpha \dot{r}^\alpha) \tag{27}$$

The Lagrange-Poincaré equations then are written in coordinates as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial l_{\text{lock}}}{\partial \dot{r}^\alpha} - \frac{\partial l}{\partial r^\alpha} = \frac{\partial l_{\text{lock}}}{\partial \xi^d}\left(B^d_{b\alpha}\xi^b + B^d_{\alpha\beta}\dot{r}^\beta\right) + u^h_\alpha \tag{28a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial l_{\text{lock}}}{\partial \xi^b} = c^a_{db}\frac{\partial l_{\text{lock}}}{\partial \xi^a}\xi^d + \frac{\partial l_{\text{lock}}}{\partial \xi^a}B^a_{b\alpha}\dot{r}^\alpha + u^v_b \tag{28b}$$

where c^a_{db} are the structure constants of the Lie algebra $\mathfrak{se}(3)$ of $SE(3)$, $B^d_{b\alpha} = c^d_{ab}A^a_\alpha$ and $B^d_{\alpha\beta} = \frac{\partial A^d_\beta}{\partial r^\alpha} - \frac{\partial A^d_\alpha}{\partial r^\beta} + c^d_{ba}A^b_\beta A^a_\alpha$ are the components of the curvature of A.

In this construction A^a_α, called the **connection coefficients**, are the components of the map A_e.

To express (r, e) in terms of $q = \{(x_i, y_i, z_i)\}^3_{i=1} \in Q$, we first show how the elements of $SE(3)$ in the local trivialization are defined. We define the rotation matrix R as the matrix whose columns are defined by the vectors (u_1, u_2, u_3):

$$\begin{aligned} u_1 &= \frac{q_1 - q_2}{\|q_1 - q_2\|} \\ u_2 &= \frac{q_1 - q_3}{\|q_1 - q_3\|} - \left(\frac{q_1 - q_3}{\|q_1 - q_3\|}\cdot u_1\right)u_1 \\ u_3 &= \frac{u_1 \times u_2}{\|u_1 \times u_2\|} \end{aligned} \tag{29}$$

Therefore, the identity element of $SE(3)$ is defined by the following equations

$$\begin{aligned} q_1 + q_2 + q_3 &= (0, 0, 0) \\ q_1 - q_2 &= (a_1, 0, 0) \\ q_1 - q_3 &= (a_2, b_2, 0) \end{aligned} \tag{30}$$

for some $a_1, a_2, b_2 \in \mathbb{R}$. Solving these equations, we get the relations

$$\begin{aligned} x_1 &= \frac{a_1 + a_2}{3},\ \ x_2 = \frac{-2a_1 + a_2}{3},\ \ x_3 = \frac{a_1 - 2a_2}{3} \\ y_1 &= \frac{b_2}{3},\ \ y_2 = \frac{b_2}{3}, y_3 = \frac{-2b_2}{3} \\ z_1 &= z_2 = z_3 = 0 \end{aligned} \tag{31}$$

Given r_1, r_2, r_3, we can then find the values of a_1, a_2, b_2 as

$$\begin{aligned} a_1 &= r_1 \\ a_2 &= \frac{r_1^2 + r_2^2 - r_3^2}{2r_1} \\ b_2 &= \sqrt{\left(r_2^2 - \left(\frac{r_1^2 + r_2^2 - r_3^2}{2r_1}\right)^2\right)} \end{aligned} \tag{32}$$

Thus, $(r, e) = (r_1, r_2, r_3, 0, 0, 0, 0, 0, 0)$ corresponds to

$$q_0 = \left(\frac{a_1 + a_2}{3}, \frac{b_2}{3}, 0, \frac{-2a_1 + a_2}{3}, \frac{b_2}{3}, 0, \frac{a_1 - 2a_2}{3}, \frac{-2b_2}{3}, 0\right)$$

where a_1, a_2, b_2 are given by (32). Evaluating the components of the mechanical connection $A : TQ \to \mathfrak{g}$ at q_0, we get

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ \frac{1}{3}I_3 & \frac{1}{3}I_3 & \frac{1}{3}I_3 \end{bmatrix} \tag{33}$$

where A_{11}, A_{12}, A_{13} are functions of r_1, r_2, r_3. The matrix $[A_e]$of A_e in these coordinates is then

$$[A_e] = \begin{bmatrix} A_{11} \\ \frac{1}{3}I_3 \end{bmatrix} \tag{34}$$

It is usually not always easy to write down the original Lagrangian $L : TQ \to \mathbb{R}$ in the bundle trivialization. Therefore, we compute the equation (28) using the chain rule. The horizontal equations (28a) can be written

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}}{\partial \dot{r}^\alpha} - \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \xi^a}A^a_\alpha\right) + \left(\frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \xi^a}\right)\left(\frac{\partial A^a_\beta}{\partial q^j}\frac{\partial q^j}{\partial r^\alpha}\right)\dot{r}^\beta \\ &= \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \xi^d}B^d_{b\alpha}\xi^b + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \xi^d}B^d_{\alpha\beta}\dot{r}^\beta + u^h_\alpha \end{aligned} \tag{35}$$

and the vertical equations (28b) can be written

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \xi^b}\right) = c^a_{db}\frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \xi^a}\xi^d + \frac{\partial L}{\partial \dot{q}^i}\frac{\partial \dot{q}^i}{\partial \xi^a}B^a_{b\alpha}\dot{r}^\alpha + u^v_b \tag{36}$$

where u^h, u^v are the horizontal and vertical components of the control input u, and

$$\begin{aligned} B^d_{\alpha\beta} &= \frac{\partial A^d_\beta}{\partial q^i}\frac{\partial q^i}{\partial r^\alpha} - \frac{\partial A^d_\alpha}{\partial q^i}\frac{\partial q^i}{\partial r^\beta} + c^d_{ba}A^b_\beta A^a_\alpha \\ B^d_{b\alpha} &= c^d_{ab}A^a_\alpha \end{aligned} \tag{37}$$

Remark 4. Calculation of the curvature of the mechanical connection $A(q, \dot{q})$ shows that the connection is not flat. This means Kobayashi and Nomizu [1996] that the principal bundle $\pi : Q \to Q/SE(3)$ is not a direct product of spaces, i.e., we cannot globally write $Q = Q/SE(3) \times SE(3)$. Therefore, we need to use the Lagrange-Poincaré equations in order to get a globally meaningful representation of the equations of motion, split into horizontal and vertical variations. The particular parametrization we select for the group and the internal space then may shrink the domain of definition but this is a choice that can be improved upon by the control designer.

These equations of motion are second-order differential equations, governing the evolution of the variables $r_1, r_2, r_3, \phi, \theta, \psi,$

$\bar{x}, \bar{y}, \bar{z}$. Note that the horizontal equations (35) immediately drop out as second order differential equations; however, the vertical equations (36) yield the evolution of the locked angular velocity ξ, which is related to the group velocity $\dot{g}$ through a linear mapping given by equation (48). Note that, although in these particular coordinates the representation may be local, the Lagrange-Poincarè equations (28) are global.

From the standard coordinates $q = \{(x_i, y_i, z_i)\}_{i=1}^N$ to the principal bundle trivialization (r, g), there exist a local diffeomorphism $\Xi : Q \to Q$ such that $(r, g) = \Xi(q)$. Therefore, the relation between the velocities is

$$\begin{bmatrix} \dot{r} \\ \dot{g} \end{bmatrix} = \Xi_{*q} \dot{q} \tag{38}$$

or from the velocity $\dot{q}$ to the "rigid" body velocity, this reads

$$\begin{bmatrix} \dot{r} \\ \eta \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & g^{-1} \end{bmatrix} \Xi_{*q} \dot{q} \tag{39}$$

and from the velocity $\dot{q}$ to the locked angular body velocity ξ, this reads

$$\nu := \begin{bmatrix} \dot{r} \\ \xi \end{bmatrix} = \begin{bmatrix} I & 0 \\ [A_e] & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & g^{-1} \end{bmatrix} \Xi_{*q} \dot{q} \tag{40}$$

For convenience, we define another map $\Psi_{*q} : T_q Q \to T_{\Xi(q)} Q$ by

$$\Psi_{*q} := \begin{bmatrix} I & 0 \\ [A_e] & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & g^{-1} \end{bmatrix} \Xi_{*q} \tag{41}$$

so that (40) simply reads $\nu = \Psi_{*q} \dot{q}$.

Even though the exact equations of motion in these coordinates are too long to fit in this paper, it is in the following form

$$\begin{aligned} \ddot{r} &= f_h(r, \dot{r}, g, \xi) + G_{hh}(r, g) u^h + G_{hv}(r, g) u^v \\ \dot{\xi} &= f_v(r, \dot{r}, g, \xi) + G_{vh}(r, g) u^h + G_{vv}(r, g) u^v \end{aligned} \tag{42}$$

where f_h and f_v along with the input matrices G_{hh}, G_{vv}, G_{vh} and G_{hv} can be computed from equations (35) and (36). In our particular problem, they turn out to be

$$G(r, g) = \begin{bmatrix} G_{hh} & G_{hv} \\ G_{vh} & G_{vv} \end{bmatrix} = \frac{1}{m} \Psi_* \Psi_*^T$$

where we have used the matrix representation of Ψ_* on the right hand side. These expressions are obtainable, precisely because we have access to all the terms in these equations, for instance $\frac{\partial \dot{q}^i}{\partial \xi^b}$ is the i^{th} row of the matrix of $\Psi_{*\Xi(q)}^{-1}$ and similarly every other term is computable from the ordinary rules of differentiation.

5. CONTROL LAW

The Lagrange-Poincarè equations split the Euler-Lagrange equations such that the horizontal set of equations (28a) govern the behavior of the shape (internal) space, i.e., the relative positions of the agents with respect to each other, while the vertical set of equations (28b) govern the behaviour of the virtual rigid body some of whose points consist of the individual agents.

The control law that we design has two purposes

(1) Keep the connectivity of the graph above a certain threshold and increase it to a predefined value
(2) Move the agents as a rigid body from an initial configuration to a desired final configuration

We saw in section 2.2 that the connectivity of the agents depends on the distances between the agents. This was formalized by constructing the elements of the adjacency matrix as a decreasing function of the distance between each individual agent. It was also shown that the determinant of the Laplacian matrix, which is used as the measure of connectivity of the graph, is invariant under the action of the special Euclidean group $SE(3)$. Thus, the connectivity controller as given in Satici et al. [2013] has vanishing vertical component, $u^v = A\,(\text{grad } V_c)$. This is easily shown as follows

$$(V_c \circ \exp(t\xi))(q) = V_c(q) \Rightarrow \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{t=0} (V_c \circ \exp(\epsilon\xi))(q) = 0$$

$$\Rightarrow (V_c)_{*q} \cdot \xi_Q(q) = 0 \Rightarrow \langle\!\langle \text{grad } V_c, \xi_Q \rangle\!\rangle = 0$$

Therefore, independent of the connectivity controller, we may design the vertical component u^v of our control input u to swarm the agents in the desired manner. We shall not touch the horizontal part of the controller when we design the swarming aspect and thus the connectivity task shall be achieved no matter how we choose u^v.

Correspondingly the control inputs are mapped by the pull back Ψ^* as follows

$$u = \Psi_{\Xi(q)}^* \tilde{u} := \Psi_{\Xi(q)}^* \begin{bmatrix} u^h \\ u^v \end{bmatrix} \tag{43}$$

where u is the vector of control inputs in equation (15) applied to each Euclidean component of R^{3N}. In the chosen coordinates, this reads

$$u = \Psi_{*\Xi(q)}^T \begin{bmatrix} u^h \\ u^v \end{bmatrix} \tag{44}$$

where we view $\Psi_{*\Xi(q)}$ as the matrix of the linear mapping between tangent spaces. Since we will set $u^h = 0$ for the development of the swarming controller, we are actually not interested in the first $3N - 6$ columns of the matrix of $\Psi_{*\Xi(q)}^T$.

To ensure connectivity of the agents and increase it to a desired value, we apply the preliminary control law presented in section 2.2. That is, we set the connectivity controller as the gradient of the determinant of reduced Laplacian $\mathcal{M}$ that is constructed from the Laplacian $\mathcal{L}$ by the transformation $\mathcal{M} = \mathcal{P}^T \mathcal{L} \mathcal{P}$, where $\mathcal{P}$ is matrix that satisfies $\mathcal{P}^T \mathcal{P} = I$ and $\mathcal{P}^T \mathbf{1} = 0$. Our final control law shall be the sum of what we get from the swarming component and the connectivity controller, i.e.,

$$u = u_c + \Psi_{*\Xi(q)}^T \begin{bmatrix} 0 \\ u^v \end{bmatrix} \tag{45}$$

In this equation u_c is given by the connectivity controller with added damping in the horizontal direction, i.e.,

$$u_c = (V_c)_{*q} - k_d \Psi^T_{*\Xi(q)} \begin{bmatrix} \dot{r} \\ -A_e(r) \cdot \dot{r} \end{bmatrix} \tag{46}$$

where V_c is the connectivity potential as given in (14) In this potential function, we have two parameters, $\underline{\alpha}$ and $\bar{\alpha}$; the former is a lower bound on the connectivity measure of the network and the latter is the measure that we would like to achieve. Once this measure is achieved, the connectivity controller smoothly shuts down and does not increase it anymore.

On the other hand, we select the vertical control effort u^v so as to render the vertical Lagrange-Poincarè equations asymptotically stable. Working with equation (42), we may select u^v as

$$u^v = G_{vv}^{-1} \left[-k_d \left(\xi - \xi_{ref}\right) - G_{vh}(r,g)u^h - f_v(r,\dot{r},g,\xi)\right] \tag{47}$$

where ξ_{ref} is a reference velocity input that renders the left-invariant group kinematics asymptotically stable; that is, ξ_{ref} is such that if $\xi \to \xi_{ref}$, then the left invariant group kinematics

$$\dot{g} = g\left(\xi - A_e(r) \cdot \dot{r}\right) \tag{48}$$

asymptotically converges, i.e., $g \to g_d$. In the next section 6, we use this control law to stabilize the "rigid-body", or swarming behaviour of three robots, each evolving in the Euclidean 3-space, $\mathbb{R}^3$, while achieving a desired connectivity measure and keeping the overall connectivity intact for all time.

6. SIMULATION

In this section we demonstrate the use of the equations of motion and the control law that we have developed in sections 4 and 5. The setting has three robots each of whose configuration is $\mathbb{R}^3$. They are assumed to move without the action of gravity, or we may assume that we've used active controls to remove the action of gravity. This way the Lagrangian of each robot is equal to its kinetic energy. The Lagrangian of a group of such agents is then

$$L = \frac{m}{2} \sum_{i=1}^{3} \left(\|\dot{x}_i\|^2 + \|\dot{y}_i\|^2 + \|\dot{z}_i\|^2\right) \tag{49}$$

The equations of motion in the original coordinates for each $i \in \{1,2,3\}$ are

$$\begin{aligned} m\ddot{x}_i &= u_{xi} \\ m\ddot{y}_i &= u_{yi} \\ m\ddot{z}_i &= u_{zi} \end{aligned} \tag{50}$$

Viewing the configuration space $Q = \mathbb{R}^9$ as a principal bundle with structure group $SE(3)$, we assign the coordinates (r,g) that we have constructed in section 4 to Q.

Since we do not want to interfere with the horizontal or shape-space dynamics, we set $\tilde{u}_i = 0$ for $i = 1,2,3$ and design $\tilde{u}_j$, for $j = 4,\ldots,9$ such that the centroid $\bar{x},\bar{y},\bar{z}$ and the orientation parametrized by ϕ,θ,ψ converges to a desired location, given by $(\bar{x}_d,\bar{y}_d,\bar{z}_d,\phi_d,\theta_d,\psi_d)$. Now, we want to apply the control law (47) and (48). For that we need to come up with ξ_{ref} which will make $g \to g_d$ asymptotically. To do that, we compute the relation between $\left(\dot{\psi},\dot{\theta},\dot{\phi}\right)$ and $\left(\eta^1,\eta^2,\eta^3\right)$, which turns out to be

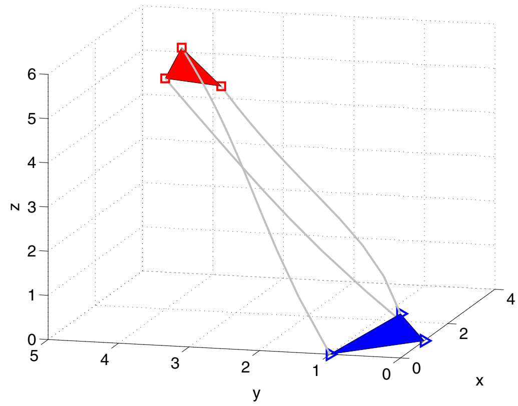

Fig. 1. Implicit plot

$$\begin{bmatrix} \eta^1 \\ \eta^2 \\ \eta^3 \end{bmatrix} = T \begin{bmatrix} \dot{\psi} \\ \dot{\theta} \\ \dot{\phi} \end{bmatrix} \qquad T = \begin{bmatrix} -\sin\theta & 0 & 1 \\ \cos\theta\sin\theta & \cos\phi & 0 \\ \cos\theta\cos\phi & -\sin\phi & 0 \end{bmatrix} \tag{51}$$

and the relation between $(\dot{\bar{x}},\dot{\bar{y}},\dot{\bar{z}})$ and $\left(\eta^4,\eta^5,\eta^6\right)$ is

$$\begin{bmatrix} \eta^4 \\ \eta^5 \\ \eta^6 \end{bmatrix} = R^T \begin{bmatrix} \dot{\bar{x}} \\ \dot{\bar{y}} \\ \dot{\bar{z}} \end{bmatrix} - R^T \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} \tag{52}$$

Now the reference signal for $\left(\dot{\psi},\dot{\theta},\dot{\phi},\dot{\bar{x}},\dot{\bar{y}},\dot{\bar{z}}\right)$ is straightforward:

$$\begin{bmatrix} \dot{\psi}_{ref} \\ \dot{\theta}_{ref} \\ \dot{\phi}_{ref} \\ \dot{\bar{x}}_{ref} \\ \dot{\bar{y}}_{ref} \\ \dot{\bar{z}}_{ref} \end{bmatrix} = -k_p \begin{bmatrix} \psi - \psi_d \\ \theta - \theta_d \\ \phi - \phi_d \\ \bar{x} - \bar{x}_d \\ \bar{y} - \bar{y}_d \\ \bar{z} - \bar{z}_d \end{bmatrix} \tag{53}$$

Using the relations (51) and (52), we convert these references to η_{ref}, which in turn converts to ξ_{ref} through the relation $\xi_{ref} = \eta_{ref} + A_e(r) \cdot \dot{r}$. Finally, the vertical control input that was presented in equation (47) can be implemented and the final control law becomes

$$u = u_c + \Psi^T_{*\Xi(q)} \begin{bmatrix} 0_{3\times 1} \\ u^v_{6\times 1} \end{bmatrix} \tag{54}$$

If we select k_d in equation (47) and k_p in equation (53) such that $k_d \geq 2k_p$ than the whole system becomes fully damped, and we can expect convergence to the desired centroid location and orientation without overshoot. In the simulation, we have selected $m = 1$, $k_p = 2$ and $k_d = 4$ and arrived at the behaviour depicted in Fig. 1.

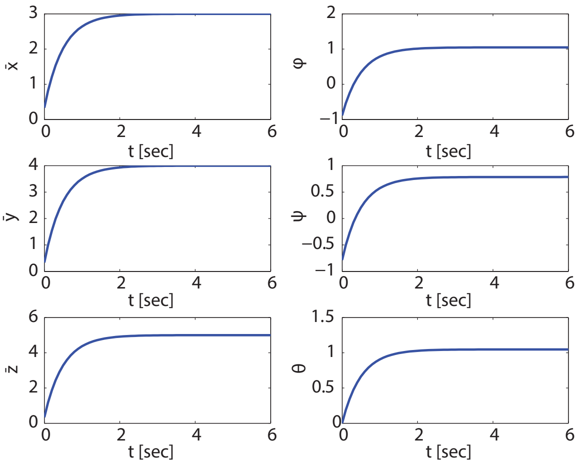

Fig. 2. Convergence of the group coordinates

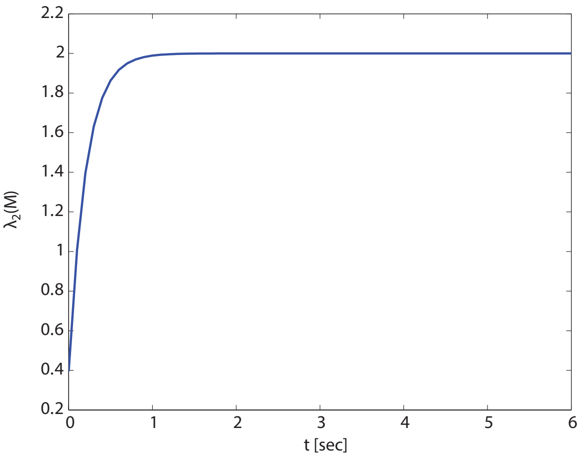

Fig. 3. Connectivity measure $\lambda_2(\mathcal{M})$

Convergence of the centroid and the orientation to their desired location, which is selected to be $(\bar{x}_d, \bar{y}_d, \bar{z}_d, \phi_d, \theta_d, \psi_d) = (3, 4, 5, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{3})$, is shown in Fig. 2. The initial pose was parametrized by $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -0.8861, -\frac{\pi}{4}, 0)$.

Fig. 3 depicts the evolution of the connectivity parameter, the second smallest eigenvalue of the Laplacian $\mathcal{L}$. In this particular simulation the initial value was equal to $\lambda_2(\mathcal{M}(t_0)) = 0.3957$ and it ends up at the desired value $\lambda_2(\mathcal{M}(t_f)) = 2$, not interfering with the convergence of the group coordinates, as was predicted by the theory.

7. CONCLUSION

In this work, we have tackled the problem of swarming multiple agents while preserving connectivity. We have used the theory of reduced Euler-Lagrange equations, or Lagrange-Poincarè equations as was developed by Marsden and Scheurle Marsden and Scheurle [1993]. This development allows us to split the original Euler-Lagrange equations into horizontal and vertical parts where the horizontal equations govern the evolution of the internal space and the vertical components govern the evolution of the swarming behaviour. It is important to note that this splitting makes global sense. Although we have used a particular bundle trivialization to finish the control design, it is possible to use the same equations with a different bundle trivialization with larger domain of validity. The control design phase would then follow exactly as presented.

We have shown that the connectivity controller affects only the horizontal components and so leaves us free to design a swarming component in the resulting split equations of motion. We have exploited this structure to impose an asymptotically convergent swarming behavior. Finally, we presented simulation results that supports the arguments presented in the theoretical part of the paper.

As part of future work, we would like to extend the use of this decomposition for connectivity control and swarming behavior to agents whose configuration space is a Lie group rather than the Euclidean vector space.

ACKNOWLEDGMENTS

This research was partially supported by the National Science Foundation Grants ECCS 07-25433 and CMMI-0856368, by the University of Texas STARs program, and by DGIST R&D Program of the Ministry of Education, Science and Technology of Korea(12-BD-0101).

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