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Decentralized predictive control for 1D cascaded systems of conservation laws

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Abstract: In this work we investigate the predictive control problem for partial differential equations interconnected through their boundary conditions. More specifically, we consider systems of cascaded 1D hyperbolic equations. The main obstacle for applying model predictive control methodologies for such systems is the huge computational effort required to solve the corresponding optimization problems. Therefore, we present some results on decomposition methods. The obtained sub-problems are solved in parallel and considered as subsystems controlled by decentralized interacting agents. The proposed approach is validated in simulation for a free surface water transportation system made of interconnected canal reaches interconnected through actuated gates.
Keywords: Model predictive control, Distributed control, Hyperbolic equations, Open Channel Systems, Saint-Venant Equations

1. INTRODUCTION

Because of their spatially distributed nature, many physical systems, such as chemical processes, irrigation systems or transportation networks may be preferably modeled as infinite-dimensional systems described by partial differential equations (PDEs). The control design for such systems is generally based on one of the two following approaches. The first one, called the indirect approach, begins with an approximation of the PDEs by ODEs (also called reduced-order models) to which a finite-dimensional control synthesis is applied. The advantage of this approach is the availability of control synthesis techniques for ODEs. However, the approximation by ODEs sometimes results in important losses in the qualitative behaviour of the solution or in the dynamical properties of the original PDEs. This usually motivates the second approach - called the “direct approach” - where the control synthesis is directly derived from the infinite dimensional system realization. In the direct approach, the control is numerically approximated only at the implementation stage.

Our objective here is to study the direct approach for linear hyperbolic systems of conservation laws, which are used to model many interesting physical problems in various fields such as gas dynamics (see Serre [1999]), road traffic (see Colombo et al. [2011]), air traffic (see Bayen et al. [2006]), transport-reaction processes (see Dubljevic et al. [2005]), and free surface or pressurized water transportation systems (see Georges [2009]). We are particularly interested in the model predictive control (MPC - also known as receding horizon control) in which, the control action is obtained by solving repeatedly, on-line, a finite horizon open-loop optimal control problem. Among the advantages of MPC, one can mention the ability to obtain a guaranteed stability, to handle constraints, to incorporate forecast information and to minimize a given criterion. The approach was well studied for finite-dimensional systems, even in the nonlinear case (see e.g. Findeisen et al. [2003] and Rawlings and Mayne [2009]). Some extension to infinite-dimensional systems was also investigated, as in Ito and Kunisch [2002]. But the latter work is concerned only with the case of distributed control. In Christofides and Daoutidis [1997] and Dubljevic et al. [2005], the authors proposed MPC approach for parabolic systems but the control synthesis was based on a finite-dimensional approximation of the PDEs. An infinite-dimensional MPC for boundary control of nonlinear Saint-Venant equations was considered in Georges [2009], and solved by calculus of variations approach. Our recent work (Pham et al. [2010, 2012]) established the stability of MPC for a single linear hyperbolic system as well as for a cascaded network of such systems.

The usefulness of the MPC approach for these problems related to *interconnected infinite-dimensional systems* is however limited by the required computational effort when the control action for the whole network is calculated in a centralized manner by a single controller. This obstacle can be tackled by using the so-called distributed MPC configuration or decomposition-coordination approach in which the optimization problem of the entire system is divided into several sub-problems, each of them being allocated to a local controller (sometimes referred to as an ”agent”). The global optimal control action is then obtained by exchanging information between these agents. This is currently a living topic in the MPC community and several results have been established (see e.g. Scattolini [2009], Stewart et al. [2010], Christofides et al. [2013] or Liu et al. [2010]), mostly in the finite-dimensional case and very few studies (Georges [2009]) consider this infinite-dimensional case.

In this paper, we consider several algorithms of distributed model predictive control (DMPC) for a system of cascaded hyperbolic equations, with an application to the control of cascaded reaches of an irrigation channels through the connecting sliding gates. These systems are well suited to the demonstration of the advantages of DMPC related to complexity issues, to the need for an optimal management strategy and to the many operational constraints for such water systems. The DMPC was in fact considered in several works for water transport systems (see Carpentier and Cohen [1993], Fawal et al. [1998], Zarate-Florez et al. [2012] and Igreja et al. [2012]), but with a finite-dimensional model.

The paper is organized as follows. In the first section the considered systems of cascaded hyperbolic equations are defined. A stability result for the centralized MPC of such systems is recalled (Pham et al. [2010, 2012]). In the second section, different decomposition approaches (namely the price decomposition and the prediction decomposition proposed by Cohen and Zhu [1984]) are considered and applied to obtain a decentralized MPC scheme. The convergence of these algorithms to the global optimum is discussed. In the third section, the obtained results are applied to the example of a cascaded network of irrigation channel reaches. Simulation results are given in order to validate the here proposed theory. We conclude by a comparison in terms of performance between the different considered approaches.

2. MPC FOR SYSTEMS OF COUPLED HYPERBOLIC EQUATIONS

2.1 Systems of coupled hyperbolic equations

We consider a network of N ($N \geq 2$) cascaded hyperbolic systems:

$$\left\{ \partial_t \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \partial_x \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} + \begin{pmatrix} c_i & d_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} \right. , \quad x \in [0, L],\ t > 0, i = 1, ..., N \tag{1}$$

where $(\alpha_i(\cdot, t), \beta_i(\cdot, t)) \in [\mathbf{L}_2(0, L)]^2$ is the state whereas the constants a_i, b_i, c_i and d_i are parameters of the subsystem i. In this study, we limit ourselves to the case of strictly hyperbolic systems, namely $a_i > 0 > b_i$ ($i = 1, ..., N$). Here t and x classically stand for time and space coordinates, and ∂_t, ∂_x denote the partial derivatives w.r.t. t, x respectively.

The initial condition must be specified at $t = 0$:

$$\alpha_i(\cdot, 0) = \alpha_i^0, \quad \beta_i(\cdot, 0) = \beta_i^0. \tag{2}$$

The boundary condition at junctions i ($i = 2, \ldots, N$) can generally be supposed to have the following form (see Courant and Hilbert [1962]):

$$\begin{pmatrix} \alpha_{i-1}(L, t) \\ \beta_i(0, t) \end{pmatrix} = \begin{pmatrix} m_i^{11} & m_i^{12} \\ m_i^{21} & m_i^{22} \end{pmatrix} \begin{pmatrix} \beta_{i-1}(L, t) \\ \alpha_i(0, t) \end{pmatrix} + \begin{pmatrix} b_i^1 \\ b_i^2 \end{pmatrix} g_i. \tag{3}$$

The first and the last junction have the same form:

$$\begin{aligned} \beta_1(0, t) &= m_1^{22}\alpha_1(0, t) + b_1^2 g_1, \\ \alpha_N(L, t) &= m_{N+1}^{11}\beta_N(L, t) + b_{N+1}^1 g_{N+1}. \end{aligned} \tag{4}$$

where m_i^{jk}, $b_i^{1,2}$ are appropriate constants. In the sequel, for the sake of simplicity, we adopt the notation $\alpha_{i,0} = \alpha_i(0, t)$, $\alpha_{i,L} = \alpha_i(L, t)$ and similarly for β_i.

We add an integrator to inputs g_i as follows:

$$\dot{g}_i = u_i, \quad i = 1, ..., N + 1 \tag{5}$$

In such a way, system (1)-(5) can be rewritten in the abstract form (see e.g. Curtain and Zwart [1995], or Pham et al. [2010, 2012]):

$$\begin{aligned} \dot{z}(t) &= \mathcal{A}z(t) + \mathcal{B}u(t), \quad t > 0 \\ z(0) &= z^0, \end{aligned} \tag{6}$$

where $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup, $\mathcal{B}$ is a linear bounded operator. The new state z and the new control u are determined by

$$\begin{aligned} z &= \begin{pmatrix} g, \\ v - \mathbf{B}g \end{pmatrix}, \quad v = (\alpha_1 \cdots \alpha_N \quad \beta_1 \cdots \beta_N)^T \\ u &= (u_1 \cdots u_{N+1})^T, \end{aligned} \tag{7}$$

with appropriate bounded operator $\mathbf{B}$ (see Pham et al. [2012] for more detail). In this form, we can employ the C_0-semigroup theory (see Curtain and Zwart [1995]) to establish the well-possedness as well as the existence of the optimal control for system (1)-(5).

2.2 Principle of MPC

Let us consider system (6) and recall for it the principle of MPC or Receding Horizon Optimal Control:

- At each time t, we obtain the current state $z(t)$.
- Then, for a given prediction time T and a cost function J, we compute the optimal solution of the problem:
$$\min_{\bar{u} \in \mathbf{L}_2(t, t+T; \mathbb{R}^{N+1})} J(z(t); \bar{u})$$
$$\text{s.t. } \dot{\bar{z}}(\tau) = \mathcal{A}\bar{z}(\tau) + \mathcal{B}\bar{u}(\tau), \forall \tau \in [t, t+T], \bar{z}(t) = z(t),$$
where the notation $\bar{\cdot}$ stands for the predicted variables.
- The first part of the optimal control is applied on the system in period $[t, t + \sigma)$ for a small σ.
- The procedure restarts at $t + \sigma$.

One can remark that since the actual state $z(t)$ is updated at each sampling step, the resulting control $u(t)$ is in fact in a feedback form which makes an advantage of a receding horizon strategy in comparison with an open-loop optimal control.

We intend to employ this strategy to stabilize system (1)-(5) using the following optimization problem:

$$\begin{aligned} \min_u J = &\sum_{i=1}^{N+1} \int_0^T m_i(g_i, u_i) dt + \sum_{i=1}^{N} \int_0^T \int_0^L l_i(\alpha_i, \beta_i) dx dt \\ &+ \sum_{i=1}^{N+1} m_i^f(g_i(T)) + \int_0^L l_i^f(\alpha_i(\cdot, T), \beta_i(\cdot, T)) dx, \\ &\text{s.t. (1)-(5)} \end{aligned} \tag{8}$$

The stage cost functions m_i and l_i and the terminal cost functions m_i^f and l_i^f are taken in quadratic form:

$$\begin{aligned} m_i(g_i, u_i) &= q_i g_i^2 + r_i u_i^2,\ l_i(\alpha_i, \beta_i) = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}^T Q_i \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, \\ m_i^f(g_i) &= q_i^f g_i(T)^2,\ l_i^f(\alpha_i, \beta_i) = \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}^T Q_i^f \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}, \end{aligned} \tag{9}$$

Using the transformation (7), the optimization problem (8) can be put in the following form:

$$\min_u J = \int_0^T \langle z(t), Mz(t) \rangle dt + \langle u, Ru \rangle + \langle z(T), M^f z(T) \rangle,$$
s.t. (6),

with appropriate positive definite operators M, R and M_f. In this form, we can show that there exists an optimal solution (see Curtain and Zwart [1995]), which guarantees the feasibility at each sampling instant. In addition, we can choose the weighting parameters q_i, r_i, Q_i, q_i^f and Q_i^f (which inspired from the Lyapunov function proposed by Coron et al. [2007]) in order that the closed-loop system by MPC is asymptotically stable at the origin (see Pham et al. [2010]).

It is however difficult to solve the above optimization problem with a centralized control structure due to the computational complexity and to the robustness of the controller. In the next section, we will consider some algorithms in order to get a distributed control scheme.

3. DECOMPOSITION-COORDINATION APPROACH

3.1 Price decomposition and Prediction decomposition

Let us recall firstly the principle of these approaches for a general optimization problem. Consider the following problem:

$$\min_{u,v} J(u,v) = \sum_{i=1}^{N} J_i(u_i, v_i), \tag{10}$$

$$\text{s.t. } \theta_i(u,v) = v_i - \sum_{j\neq i} H_{ij}(u_j, v_j) = 0,\ i = 1, ..., N \tag{11}$$

where $u = (u_1, ..., u_N)$ is the decision variable, and $v = (v_1, ..., v_N)$ is interaction variable. The term $H_{ij}(u_j, v_j)$ represents the influence of sub-system j to sub-system i.

In order to deal with the constraints, we use the augmented Lagrangian, which can be viewed as a mix of Lagrangian and penalty method:

$$L_c(u,v,p) = \sum_{i=1}^{N} J_i(u_i, v_i) + \langle p_i, \theta_i(u,v)\rangle + \frac{c}{2}\|\theta_i(u,v)\|^2$$

where c is a positive constant and p_i is the multiplier associated with constraint (11). The original constrained optimization problem is now equivalent to finding a saddle-point of $L_c(u,v,p)$. Thanks to the quadratic term of the constraint, the convexity of the problem is enforced therefore, the convergence of dual algorithms (where we find alternatively $\min\limits_{u,v} L_c(u,v,p)$ with a fixed p then $\max\limits_{p} L_c(u,v,p)$ with (u,v) found in the previous step) is ensured (see Cohen and Zhu [1984]).

By using linearization of the square of the constraint, Cohen [1980] proposed different methods to decompose problem (11) into N sub-problems (each corresponds to control input u_i and can be solved by one agent). These approaches were applied in the context of distributed MPC e.g. by Georges [2006] and Rantzer [2009]. In this paper, we consider the price decomposition and the prediction decomposition.

Price decomposition The algorithm consists of the following steps:

(1) At iteration $k = 0$: choose p_i^0, $i = 1, ..., N$ and $u^0 = (u_1^0, ..., u_N^0)$, $v^0 = (v_1^0, ..., v_N^0)$.
(2) At iteration k: Each agent solves the following problem in (u_i, v_i):
$$\min_{u_i,v_i} J_i(u_i,v_i) + \frac{1}{2\epsilon}\|u_i - u_i^k\|^2 + \frac{1}{2\epsilon}\|v_i - v_i^k\|^2 + \left\langle p_i^k + c\theta_i(u^k,v^k), \frac{\partial\theta_i}{\partial u_i}(u^k,v^k)u_i + \frac{\partial\theta_i}{\partial v_i}(u^k,v^k)v_i \right\rangle$$
Let u_i^{k+1} and v_i^{k+1} be a solution.
(3) Update p_i according to
$$p_i^{k+1} = p_i^k + \rho(\theta_i(u^k,v^k)) \tag{12}$$
(4) If $\|p^{k+1} - p^k\|$ is sufficiently small: stop, otherwise return to step 2 with k replaced by $k+1$

Note that in each sub-problem, a proximal term ($\|u_i - u_i^k\|^2$ and $\|v_i - v_i^k\|^2$) was added in order to enforce convexity and therefore the convergence of the scheme.

Prediction decomposition The algorithm consists of the following step:

(1) At iteration $k = 0$: choose p_i^0, $i = 1, ..., N$ and $u^0 = (u_1^0, ..., u_N^0)$ and $w^0 = (w_1^0, ..., w_N^0)$.
(2) At iteration k: Each agent solves the following problem in (u_i, v_i):
$$\min_{u_i,v_i} J_i(u_i,v_i) + \frac{1}{2\epsilon}\|u_i - u_i^k\|^2 + \left\langle p_i^k + c\theta_i(u^k,v^k), \frac{\partial\theta_i}{\partial u_i}(u^k,v^k)u_i + \frac{\partial\theta_i}{\partial v_i}(u^k,v^k)v_i \right\rangle$$
$$\text{s.t. } v_i = w_i^k \tag{13}$$
Let u_i^{k+1}, v_i^{k+1} be a solution and μ_i^{k+1} the associated multiplier with constrain (13).
(3) Update p_i and w_i according to
$$\begin{aligned} w_i^{k+1} &= w_i^k - \epsilon(\mu_i^{k+1} + p_i^k), \\ p_i^{k+1} &= p_i^k + \rho(\theta_i(u^k,v^k)) \end{aligned} \tag{14}$$
(4) If $\|p^{k+1} - p^k\| + \|w^{k+1} - w^k\|$ is sufficiently small: stop, otherwise return to step 2 with k replaced by $k+1$

In this algorithm, the interaction variables v_i is fixed to its prediction value given by previous iteration. As consequence, problem (13) is in fact minimized only in terms of u_i, which reduces the number of decision variables for each sub-problem.

Note that in the two above algorithms, it is not necessary to have a coordinator since information can be exchanged directly between agents.

3.2 Application to system of cascaded hyperbolic equations

We apply now the above approaches to problem (8) to decompose it into $N+1$ sub-problems. Let us first introduce the interconnection variables of each sub-system as:

$$\begin{aligned} q_{i,+} &= m_{i+1}^{12}\alpha_{i+1,0} + b_{i+1}^1 g_{i+1}, \\ q_{i,-} &= m_i^{21}\beta_{i-1,L}, \end{aligned} \tag{15}$$

These variables $q_{i,+}$ and $q_{i,-}$ play the role of v_i in the general presentation in the previous section. The above relations can be seen as constraints for the optimization problem:

$$\begin{aligned} \theta_{i,+}(q_{i,+}, \alpha_{i+1,0}, g_{i+1}) &= q_{i,+} - m_{i+1}^{12}\alpha_{i+1,0} - b_{i+1}^1 g_{i+1} = 0, \\ \theta_{i,-}(q_{i,-}, \beta_{i-1,L}) &= q_{i,-} - m_i^{21}\beta_{i-1,L} = 0, \end{aligned}$$

In the sequel, the dependance of $\theta_{i,+}$ and $\theta_{i,-}$ to their arguments will be omitted for the sake of clarity. Let us denote $q_i = (q_{i,+} \quad q_{i,-})^T$, $\theta_i = (\theta_{i,+} \quad \theta_{i,-})^T$ and $p_i = (p_{i,+} \quad p_{i,-})^T$ the associated multiplier to θ_i. Then the boundary conditions (3) can be rewritten as:

$$\begin{aligned} \alpha_i(L,t) &= m_{i+1}^{11}\beta_i(L,t) + q_{i,+}, \\ \beta_i(0,t) &= q_{i,-} + m_i^{22}\alpha_i(0,t) + b_i^2 g_i \end{aligned} \tag{16}$$

Price decomposition The decomposition scheme is as follows:

(1) At iteration $k = 0$: Choose p_i^0, u_i^0 and q_i^0. Simulate the sub-system i to get α_i^0, β_i^0 and g_i^0.
(2) At iteration k: Solve in parallel
For $i = 1, ..., N$

$$\begin{aligned} &\min_{u_i,q_i} \int_0^T m_i(g_i,u_i)dt + \int_0^T \int_0^L l_i(\alpha_i,\beta_i)dxdt \\ &+ \int_0^T (\frac{1}{2\epsilon}(u_i - u_i^k)^2 + \frac{1}{2\epsilon}\|q_i - q_i^k\|^2)dt \\ &+ \int_0^T [C_{1,i}^k q_{i,+} + C_{2,i}^k q_{i,-} + C_{3,i}^k \alpha_{i,0} + C_{4,i}^k \beta_{i,L} + C_{5,i}^k g_i]dt \\ &+ m_i^f(g_i(T),u_i(T)) \\ &\text{s.t. (1) and (16)} \end{aligned} \tag{17}$$

with

$$\begin{aligned} C_{1,i}^k &= [p_{i,+}^k + c\theta_{i,+}(q_{i,+}^k, \alpha_{i+1,0}^k, g_{i+1}^k)] \\ C_{2,i}^k &= [p_{i,-}^k + c\theta_{i,-}(q_{i,-}^k, \beta_{i-1,L}^k)], \\ C_{3,i}^k &= -m_i^{12}[p_{i-1,+}^k + c\theta_{i-1,+}(q_{i-1,+}^k, \alpha_{i,0}^k, g_i^k)], \\ C_{4,i}^k &= -m_{i+1}^{21}[p_{i+1,-}^k + c\theta_{i+1,-}(q_{i+1,-}^k, \beta_{i,L}^k)], \\ C_{5,i}^k &= -b_i^1[p_{i-1,+}^k + c\theta_{i-1,+}(q_{i-1,+}^k, \alpha_{i,0}^k, g_i^k)], \end{aligned}$$

For the last control input ($i = N + 1$):

$$\begin{aligned} &\min_{u_i} \int_0^T [m_i(g_i,u_i) + \frac{1}{2\epsilon}(u_i - u_i^k)^2 + C_{5,i}^k g_i]dt \\ &\quad + m_i^f(g_i(T),u_i(T)), \\ &\text{s.t. } \dot{g}_i = u_i \end{aligned} \tag{18}$$

Let u_i^{k+1}, q_i^{k+1} a solution and α_i^{k+1}, β_i^{k+1}, g_i^{k+1} the associated trajectory.
(3) Send ($\alpha_{i,0}^{k+1}$,g_i^{k+1},$q_{i,-}^{k+1}$) to sub-system $(i-1)$ and send ($\beta_{i,L}^{k+1}$, $q_{i,+}^{k+1}$) to sub-system $(i + 1)$.
(4) Each agent updates the multiplier according to $p_i^{k+1} = p_i^k + \rho\theta_i^{k+1}$ and constants $C_{j,i}^{k+1}$, $j = 1, ..., 5$ using received information from neighbors.
(5) Stop if $\|p^{k+1} - p^k\|$ is below a desired threshold. Otherwise, make $k \leftarrow k + 1$ and return to step 2.

Prediction decomposition The algorithm is almost the same the one above, except that in step 2, each agent solves:

$$\begin{aligned} &\min_{u_i} \int_0^T m_i(g_i,u_i)dt + \int_0^T \int_0^L l_i(\alpha_i,\beta_i)dxdt \\ &+ \int_0^T [\frac{1}{2\epsilon}(u_i - u_i^k)^2 + C_{3,i}^k \alpha_{i,0} + C_{4,i}^k \beta_{i,L} + C_{5,i}^k g_i]dt \\ &+ m_i^f(g_i(T),u_i(T)) \\ &\text{s.t. (1), (16) and } q_i = w_i^k \end{aligned} \tag{19}$$

and in step 4, we need to update w_i and p_i:

$$\begin{aligned} w_i^{k+1} &= w_i^k - \epsilon(p_i^k + \mu_i^{k+1}), \\ p_i^{k+1} &= p_i^k + \rho\theta_i^{k+1} \end{aligned} \tag{20}$$

where μ_i^{k+1} is the multiplier associated with the constraint $q_i = w_i^k$.

Convergence Thanks to the convexity and the well-posedness of the global problem and of each sub-problem, with sufficiently small ϵ and ρ, the above decomposition schemes converge to global optimum of (8) (see Cohen and Zhu [1984] and Cohen [1980]).

Thanks to this distributed scheme, the robustness is improved since a local controller's default does not lead to the disfunctioning of the whole system. Moreover, some reorganization mechanisme can be added to each controller in order to adapt themselves in case of actuator fault (see Pham et al. [2014]).

4. APPLICATION TO A NETWORK OF OPEN-CHANNEL SYSTEM

In order to illustrate the proposed control technique and the numerical implementation scheme, we present here an application to the linearized model of an irrigation channel consisting of N cascaded pools. Each pool is usually described by a set of two partial differential equations (PDEs) named Saint-Venant equations, which represent the mass and the momentum conservation (see Georges [2009]):

$$\begin{aligned} &B_i\partial_t h_i + \partial_x Q_i &&= 0 \\ &\partial_t Q_i + \partial_x\left(\frac{Q_i^2}{B_i h_i} + \frac{1}{2}B_i g h_i^2\right) &&= gB_i h_i(I_i - J(Q_i,h_i)), \\ &(x,t) \in [0,L] \times [0,\infty), && i = 1, ..., N, \end{aligned} \tag{21}$$

where h_i denotes the water depth, Q_i the discharge, g the gravitational acceleration, B_i the channel width, I_i the slope and J the friction term.

Interconnections between pools are subject to a set of $N+1$ sliding gate equations:

$$Q_{g_i} = K_i^2\Theta_i^2(t)2g(h_{us}^{g_i} - h_{ds}^{g_i}), i = 1, ..., N + 1, \tag{22}$$

and $N - 1$ discharge conservation constraints:

$$Q_i(L,t) = Q_{i+1}(0,t), \quad i = 1, ..., N - 1, \tag{23}$$

where Q_{g_i} is the discharge through the gate, K_i the gate coefficient, Θ_i the opening, $h_{us}^{g_i}$ and $h_{ds}^{g_i}$ are the water levels at upstream and at downstream respectively.

Let us now consider the linearization of the system around a uniform steady state $(\bar{h}_i, \bar{Q}_i)$ which has to satisfy $\bar{Q}_i =$ constant and $J(\bar{h}_i, \bar{Q}_i) = I$. Denote by $\tilde{h}_i = h_i - \bar{h}_i$, $\tilde{Q}_i = Q_i - \bar{Q}_i$ deviation of the state h and Q around this steady state. We obtain then:

$$\begin{aligned} \partial_t\tilde{h}_i &= -B_i^{-1}\partial_x\tilde{Q}_i, \\ \partial_t\tilde{Q}_i &= \zeta\partial_x\tilde{h}_i + \kappa\partial_x\tilde{Q}_i + \rho\tilde{h}_i + \phi\tilde{Q}_i, \end{aligned} \tag{24}$$

with approriate ζ, κ, ρ and ϕ. Let us additionally define:

$$G = \begin{pmatrix} 0 & -B_i^{-1} \\ \zeta & \kappa \end{pmatrix}, \quad H = \begin{pmatrix} 0 & 0 \\ \rho & \phi \end{pmatrix}.$$

In the sub-critical regime (low flow speed), G has two eigenvalues satisfying $a_i = -\frac{\bar{Q}_i}{B_i\bar{h}_i} + \sqrt{g\bar{h}_i} > 0$ and $b_i = -\frac{\bar{Q}_i}{B_i\bar{h}_i} - \sqrt{g\bar{h}_i} < 0$. By applying the transformation

$$\begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = P^{-1}\begin{pmatrix} \tilde{h}_i \\ \tilde{Q}_i \end{pmatrix} \text{ with } P = \begin{pmatrix} 1 & 1 \\ -B_i a_i & -B_i b_i \end{pmatrix}, \tag{25}$$

we have a new system:

$$\partial_t \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \partial_x \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix} + \begin{pmatrix} c_i & d_i \\ c_i & d_i \end{pmatrix} \begin{pmatrix} \alpha_i \\ \beta_i \end{pmatrix}. \qquad (26)$$

The $N+1$ gate equations (22) can also be linearized and combined with the discharge conservation (23) to form a linear boundary condition as in section (3). As a consequence, this system has the form studied previously in 2, hence can be stabilized by the MPC scheme, which will be illustrated below.

4.1 Simulation results

In this section, we present some simulation results carried out with a system of three cascaded pools of the same length $L = 3000m$ and width $B = 4.36m$. The slopes are $I_1 = 2.4 \times 10^{-4}$, $I_2 = 4.2 \times 10^{-4}$ and $I_3 = 6.2 \times 10^{-4}$. The steady state corresponds to $\bar{Q} = 4.1m^3/s$ and $\bar{h}_1 = 1.97m$, $\bar{h}_2 = 1.6m$ and $\bar{h}_3 = 1.4m$. The PDEs are solved with the Lattice Boltzmann Method (see Pham et al. [2010]) with spatial step $\Delta x = 300m$ and $\Delta t = 1s$. The decomposition-coordination scheme uses $c = 7$, $\epsilon = 0.01$ and $\rho = 0.001$. The cost function is formulated with $T = 30s$.

Figure 1 presents the norm of constraints $\sum_{i=1}^{N} \|\theta_i\|^2$ and cost function of the decomposition-coordination schemes, in comparison with a centralized approach, with same turning parameters (which is the step size of the steepest descent method). We can notice that the price decomposition and the prediction decomposition have relatively a similar convergence speed, which is faster than that of the centralized scheme. Nevertheless, the computation time on a Intel Core i7 3.4GHz, 8G RAM PC of the centralized scheme is around $1s$ whereas that of the price decomposition and of the prediction decomposition are $13.2s$ and $12.8s$ respectively. The reason is that agents have to realize several iterations before obtaining the optimal solution of sub-problem (17) or (19). The advantage of a distributed scheme in terms of computation time will be more evident when the number of subsystems increases.

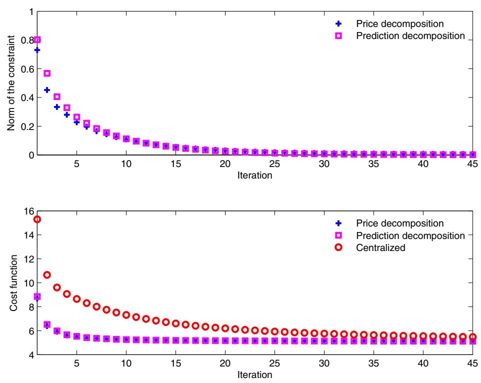

Fig. 1. Convergence of the decomposition-coordination scheme

We consider next the closed-loop system with MPC ($\sigma = 10s$). In order to reduce the communication cost (and the computation time), we limit the number of exchanges between agents to 30. This choice is justified by figure 1 where we can see that with 30 iterations, the decomposition schemes converge already to the optimum. The results are presented in figure 2 and 3 for the price decomposition. The prediction decomposition and the centralized control have a very similar behavior. We can see that $\tilde{h}_i$ and $\tilde{Q}_i$ converge to the origin meaning that the physical variables h_i and Q_i converge to the steady state $\bar{h}_i$ and $\bar{Q}_i$. In top of figure 4, we present the value of cost function in comparison with the optimal cost (obtained by using a centralized scheme with a threshold 10^{-6}). We can see that both decompositions schemes are very close to the optimal cost. The computation time is presented in bottom of figure 4. The prediction decomposition is a little faster than the price decomposition since it has less variables to manipulate (see section 3.2).

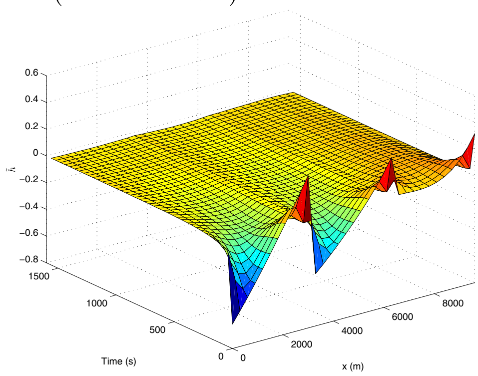

Fig. 2. Evolution of $\tilde{h}$ with MPC

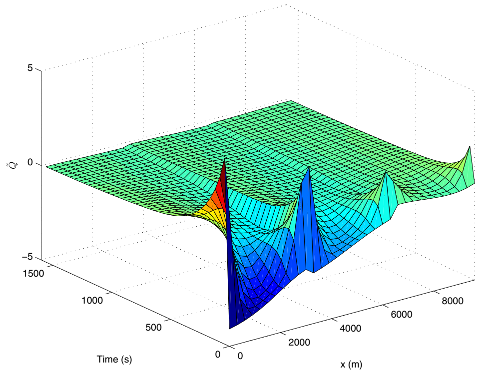

Fig. 3. Evolution of $\tilde{Q}$ with MPC

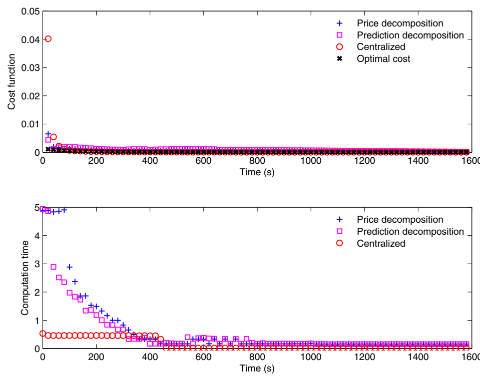

Fig. 4. Cost function and computation time at each sampling step

5. CONCLUSION

In this paper, a distributed MPC was considered for interconnected hyperbolic systems. Different decomposition schemes were presented and applied to a network of open-channel systems. The proposed approaches were validated and compared with a centralized scheme in simulation. The results showed that decomposition schemes have a better performance in terms of convergence speed. Finally, the prediction decomposition is a little more preferable thanks to its smaller computation time.

These results motivate several directions for future works. First of all, the stability of sub-optimal MPC for infinite-dimensional systems (generalization of Scokaert et al. [1999]) must be studied in order to ensure the stability of the closed-loop when the iteration is stopped before the optimum is found. Different implementations of distributed MPC (such as serial or hierarchical structure (Negenborn et al. [2009])) can also be considered. Finally, extension of this approach to other classes of PDEs could be interesting.

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