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Minimum energy control of fractional descriptor positive discrete-time linear systems with bounded inputs

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Abstract: Necessary and sufficient conditions for the positivity and reachability of fractional descriptor positive discrete-time linear systems are established. The minimum energy control problem for the descriptor positive systems with bounded inputs is formulated and solved. Sufficient conditions for the existence of solution to the minimum energy control problem are given. Procedure for computation of optimal input sequences and minimal value of the performance index is proposed.

1. INTRODUCTION

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in the monographs (Farina, Rinaldi, 2000; Kaczorek, 2001). Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc..

Mathematical fundamentals of the fractional calculus are given in the monographs (Oldham, Spanier, 1974; Ostalczyk, 2008; Podlubny, 1999). The positive fractional linear systems have been investigated in (Kaczorek, 2008a, 2009c, 2011b, 2012). Stability of fractional linear 1D discrete–time and continuous-time systems has been investigated in (Busłowicz, 2008; Dzieliński, Sierociuk, 2008; Kaczorek, 2012) and of 2D fractional positive linear systems in (Kaczorek, 2009a). The notion of practical stability of positive fractional discrete-time linear systems has been introduced in (Kaczorek, 2008b). Controllability and observability of linear electrical circuits have been addressed in (Kaczorek, 2011a). Some recent interesting results in fractional systems theory and its applications can be found in (Dzieliński, *et. all*, 2009; Kaczorek, 2008c; Radwan, *et. all*, 2009; Tenreiro Machado, *et. all*, 2006). The minimum energy control problem for standard linear systems has been formulated and solved by Klamka, 1991, 1983, 1976 and for 2D linear systems with variable coefficient in (Kaczorek, Klamka, 1986). The controllability and minimum energy control problem of fractional discrete-time linear systems has been investigated by Klamka, 2010. The minimum energy control of fractional positive continuous-time linear systems has been addressed in (Kaczorek, 2013b) and of descriptor positive discrete-time linear systems in (Kaczorek, 2013a). Necessary and sufficient conditions for the minimum energy control of positive discrete-time systems with bounded inputs have been proposed in (Kaczorek, 2013c).

In this paper necessary and sufficient conditions for the positivity and reachability of the fractional descriptor systems will be established and the minimum energy control problem with bounded inputs will be formulated and solved.

The paper is organized as follows. In section 2 the reduction of the fractional descriptor linear systems by the use of the shuffle algorithm to equivalent standard system is addressed. In section 3 the solution to the standard equivalent fractional system is given and conditions for the positivity of the fractional descriptor system are established. Necessary and sufficient conditions for the reachability of the positive fractional descriptor systems are given in section 4. The minimum energy control problem with bounded inputs is formulated and solved in section 5. Concluding remarks are given in section 6.

The following notation will be used: $\Re$ - the set of real numbers, $\Re^{n\times m}$ - the set of $n\times m$ real matrices, $\Re_+^{n\times m}$ - the set of $n\times m$ matrices with nonnegative entries and $\Re_+^n = \Re_+^{n\times 1}$, x^T - the transpose of the vector x, I_n - the $n\times n$ identity matrix.

2. REDUCTION OF THE FRACTIONAL DESCRIPTOR SYSTEMS TO STANDARD SYSTEMS

Consider the descriptor discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i,\ 0<\alpha<1,\ i\in Z_+ = \{0,1,...\} \tag{2.1}$$

where

$$\Delta^\alpha x_i = \sum_{j=0}^{i} c_j x_{i-j},\ c_j = (-1)^j \binom{\alpha}{j},$$

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j=0 \\ \dfrac{\alpha(\alpha-1)...(\alpha-j+1)}{j!} & \text{for } j=1,2,... \end{cases} \tag{2.2}$$

is the α order fractional difference of the state vector, $x_i\in\Re^n$, $u_i\in\Re^m$ are the state and input vectors and $E, A\in\Re^{n\times n}$, $B\in\Re^{n\times m}$.

It is assumed that $\det E = 0$ and the pencil $Ez - A$ is regular, i.e.

$$\det[Ez - A] \neq 0 \tag{2.3}$$

for some $z\in C$ (the field of complex numbers).

Substitution of (2.2) into (2.1) yields

$$Ex_{i+1} = A_\alpha x_i - Ec_2 x_{i-1} - Ec_3 x_{i-2} - \ldots - Ec_i x_1 - Ec_{i+1} x_0 + Bu_i \quad (2.4)$$

where $i \in Z_+$ and $A_\alpha = A + E\alpha$.

Theorem 2.1. If the pencil of the fractional descriptor system (2.1) is regular ((2.3) holds), then the system can be reduced by the use of the shuffle algorithm to the standard equivalent form

$$x_{i+1} = \overline{A}_i x_i + \overline{A}_{i-1} x_{i-1} + \ldots + \overline{A}_0 x_0 + \overline{B}_0 u_i + \overline{B}_1 u_{i+1} + \ldots + \overline{B}_q u_{i+q} \quad (2.5a)$$

where

$$\overline{A}_k \in \Re_+^{n\times n},\ k = 0,1,\ldots,i,\ \overline{B}_j \in \Re_+^{n\times m},\ k = 0,1,\ldots,q \quad (2.5b)$$

and q is the number of the shuffles.

Proof. The following elementary row operations will be used (Kaczorek, 1992, 2009b):

1) Multiplication of the i-th row by a real number c. This operation will be denoted by $L[i \times c]$.
2) Addition to the i-th row of the j-th row multiplied by a real number c. This operation will be denoted by $L[i + j \times c]$.
3) Interchange of the i-th and j-th rows. This operation will be denoted by $L[i, j]$.

Performing elementary row operations on the array

$$E \quad A_\alpha \quad -Ec_2 \quad \ldots \quad -Ec_{i+1} \quad B \quad (2.6)$$

or equivalently on (2.4) we get

$$\begin{matrix} E_1 & A_{\alpha 1} & -E_1 c_2 & \ldots & -E_1 c_{i+1} & B_1 \\ 0 & A_{\alpha 2} & 0 & \ldots & 0 & B_2 \end{matrix} \quad (2.7)$$

and

$$E_1 x_{i+1} = A_{\alpha 1} x_i - E_1 c_2 x_{i-1} - E_1 c_3 x_{i-2} - \ldots - E_1 c_{i+1} x_0 + B_1 u_i, \quad (2.8a)$$

$$0 = A_{\alpha 2} x_i + B_2 u_i. \quad (2.8b)$$

Substituting in (2.8b) i by i+1 we obtain

$$-A_{\alpha 2} x_{i+1} = B_2 u_{i+1} \quad (2.9)$$

The equations (2.8a) and (2.9) can be written in the form

$$\begin{bmatrix} E_1 \\ -A_{\alpha 2} \end{bmatrix} x_{i+1} = \begin{bmatrix} A_{\alpha 1} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} -E_1 c_2 \\ 0 \end{bmatrix} x_{i-1} + \ldots + \begin{bmatrix} -E_1 c_{i+1} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_{i+1} \quad (2.10)$$

The array

$$\begin{matrix} E_1 & A_{\alpha 1} & -E_1 c_2 & \ldots & -E_1 c_{i+1} & B_1 & 0 \\ -A_{\alpha 2} & 0 & 0 & \ldots & 0 & 0 & B_2 \end{matrix} \quad (2.11)$$

can be obtained from (2.7) by performing a shuffle.

If the matrix $\begin{bmatrix} E_1 \\ -A_{\alpha 2} \end{bmatrix}$ is nonsingular, then solving the equation (2.10) we obtain the standard system

$$x_{i+1} = \begin{bmatrix} E_1 \\ -A_{\alpha 2} \end{bmatrix}^{-1} \left(\begin{bmatrix} A_{\alpha 1} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} -E_1 c_2 \\ 0 \end{bmatrix} x_{i-1} + \ldots + \begin{bmatrix} -E_1 c_{i+1} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} 0 \\ B_2 \end{bmatrix} u_{i+1} \right). \quad (2.12)$$

If the matrix is singular then performing elementary row operations on (2.11) we obtain

$$\begin{matrix} E_2 & A'_{\alpha 2} & -E_2 c_2 & \ldots & -E_2 c_{i+1} & B'_2 & B_4 \\ 0 & A_{\alpha 3} & 0 & \ldots & 0 & B_3 & B_5 \end{matrix} \quad (2.13)$$

where E_2 has full row rank and $\operatorname{rank} E_2 \geq \operatorname{rank} E_1$.

Substituting in $0 = A_{\alpha 3} x_i + B_3 u_i + B_5 u_{i+1}$ i by i+1 we obtain

$$-A_{\alpha 3} x_{i+1} = B_3 u_{i+1} + B_5 u_{i+2} \quad (2.14)$$

The equations

$E_2 x_{i+1} = A'_{\alpha 2} x_i - E_2 c_2 - \ldots - E_2 c_{i+1} + B'_2 u_i + B_4 u_{i+1}$ and (2.14) can be written as

$$\begin{bmatrix} E_2 \\ -A_{\alpha 3} \end{bmatrix} x_{i+1} = \begin{bmatrix} A'_{\alpha 2} \\ 0 \end{bmatrix} x_i + \begin{bmatrix} -E_2 c_2 \\ 0 \end{bmatrix} x_{i-1} + \ldots + \begin{bmatrix} -E_2 c_{i+1} \\ 0 \end{bmatrix} x_0 + \begin{bmatrix} B'_2 \\ 0 \end{bmatrix} u_i + \begin{bmatrix} B_4 \\ B_3 \end{bmatrix} u_{i+1} + \begin{bmatrix} 0 \\ B_5 \end{bmatrix} u_{i+2}. \quad (2.15)$$

The array

$$\begin{matrix} E_2 & A'_{\alpha 2} & -E_2 c_2 & \ldots & -E_2 c_{i+1} & B'_2 & B_4 & 0 \\ -A_{\alpha 3} & 0 & 0 & \ldots & 0 & 0 & B_3 & B_5 \end{matrix} \quad (2.16)$$

can be obtained from (2.13) by performing a shuffle.

If $\det \begin{bmatrix} E_2 \\ -A_{\alpha 3} \end{bmatrix} \neq 0$, we can find x_{i+1} from (2.15) if not we repeat the procedure for (2.16). If the pencil is regular then after q steps we obtain a nonsingular matrix $\begin{bmatrix} E_q \\ -A_{\alpha q+1} \end{bmatrix}$ and the desired equation (2.5). □

3. SOLUTION TO THE STANDARD EQUIVALENT SYSTEMS AND POSITIVITY OF THE FRACTIONAL DESCRIPTOR SYSTEMS

To find the solution x_i of the standard discrete-time linear system (2.5) we shall apply the Z-transform method.

Let $X(z)$ be the Z-transform ($\mathcal{Z}$) of x_i defined by

$$Z[x_i] = \sum_{i=0}^{\infty} x_i z^{-i} . \tag{3.1}$$

Taking into account that

$$Z[x_{i-p}] = z^{-p} X(z) + z^{-p} \sum_{j=-1}^{-p} x_j z^{-j} \text{ , } p = 1,2,\dots \tag{3.2a}$$

$$Z[x_{i+p}] = z^{p} X(z) - \sum_{l=0}^{p-1} x_l z^{p-l} \text{ , } p = 1,2,\dots \tag{3.2b}$$

and applying the Z-transform to the equation (2.5) we obtain

$$Z[x_{i+1}] = \sum_{k=0}^{i} \overline{A}_{i-k} Z[x_{i-k}] + \sum_{j=0}^{q} \overline{B}_j Z[u_{i+j}] \tag{3.3a}$$

and

$$zX(z) - zx_0 = \sum_{k=0}^{i} \overline{A}_{i-k} z^{-k} X(z) + \sum_{j=0}^{q} \overline{B}_j z^q \left[U(z) - \sum_{l=0}^{j-1} u_l z^{-l} \right] \tag{3.3b}$$

where $U(z) = Z[u_i]$ and $x_j = 0, j = -1,\dots, -k$.

Multiplying (3.3b) by z^{-1} and solving with respect to $X(z)$ we obtain

$$X(z) = \left[I_n - \sum_{k=0}^{i} \overline{A}_{i-k} z^{-(k+1)} \right]^{-1} \left[x_0 + \sum_{j=0}^{q} \overline{B}_j z^{q-1} \left[U(z) - \sum_{l=0}^{j-1} u_l z^{-l} \right] \right]. \tag{3.4}$$

Substitution of the expansion

$$\left[I_n - \sum_{k=0}^{i} \overline{A}_{i-k} z^{-(k+1)} \right]^{-1} = \sum_{j=0}^{\infty} \Phi_j z^{-j} \tag{3.5}$$

into (3.4) yields

$$X(z) = \sum_{j=0}^{\infty} \Phi_j z^{-j} \left[x_0 + \sum_{j=0}^{q} \overline{B}_j z^{q-1} \left[U(z) - \sum_{l=0}^{j-1} u_l z^{-l} \right] \right]. \tag{3.6}$$

From definition of the inverse matrix we have

$$\begin{aligned} &\left[I_n - \sum_{k=0}^{i} \overline{A}_{i-k} z^{-(k+1)} \right] \left[\sum_{j=0}^{\infty} \Phi_j z^{-j} \right] \\ &= \left[\sum_{j=0}^{\infty} \Phi_j z^{-j} \right] \left[I_n - \sum_{k=0}^{i} \overline{A}_{i-k} z^{-(k+1)} \right] = I_n. \end{aligned} \tag{3.7}$$

Comparison of the coefficients at the same powers of z^{-k}, $k = 0,1,\dots$ from (3.7) yields $\Phi_0 = I_n$, $\Phi_1 = \overline{A}_i$, $\Phi_2 = \overline{A}_i \Phi_1 + \overline{A}_{i-1}$, $\Phi_3 = \overline{A}_i \Phi_2 + \overline{A}_{i-1} \Phi_1 + \overline{A}_{i-2}$, ... and in general

$$\begin{aligned} \Phi_k &= \overline{A}_i \Phi_{k-1} + \overline{A}_{i-1} \Phi_{k-2} + \dots + \overline{A}_{i-k+1} \Phi_0 \\ &= \Phi_{k-1} \overline{A}_i + \Phi_{k-2} \overline{A}_{i-1} + \dots + \Phi_0 \overline{A}_{i-k+1} \end{aligned} \text{ , } k = 1,2,\dots \tag{3.8}$$

Applying the inverse Z-transform and the convolution theorem (Kaczorek, 2012) to (3.6) we obtain the desired solution

$$x_i = \Phi_i x_0 + \sum_{k=0}^{i-1} \Phi_{i-k-1} \left(\sum_{j=0}^{q} \overline{B}_j u_{j+k} \right), \ i \in Z_+ . \tag{3.9}$$

Therefore, the following theorem has been proved.

Theorem 3.1. The solution of the equation (2.5) has the form (3.9) where the matrices Φ_k are given by (3.8).

Definition 3.1. The fractional discrete system (2.1) is called (internally) positive if $x_i \in \Re_+^n$ for every consistent $x_0 \in \Re_+^n$ and all inputs $u_i \in \Re_+^m$, $i \in Z_+$.

Theorem 3.2. The fractional descriptor system (2.1) is positive if and only if the matrices of the equivalent standard system (2.5) satisfy the conditions

$$\overline{A}_k \in \Re_+^{n \times n}, \ \ k = 0,1,\dots,i \text{ and } \overline{B}_j \in \Re_+^{n \times m}, \ \ j = 0,1,\dots,q . \tag{3.10}$$

Proof. It is well-known that the state vector of the descriptor system (2.1) and the equivalent standard system (2.5) is the same. By Definition 3.1 the descriptor system (2.1) is positive if and only if the standard system (2.5) is positive. From (3.9) it follows that $x_i \in \Re_+^n$, $i \in Z_+$ if the conditions (3.10) are met and $x_0 \in \Re_+^n$, $u_i \in \Re_+^m$. Necessity of the condition (3.10) follow the fact that $x_i \in \Re_+^n$ for every consistent $x_0 \in \Re_+^n$ and arbitrary $u_i \in \Re_+^m$, $i \in Z_+$ (Kaczorek, 2012). □

4. REACHABILITY OF THE POSITIVE FRACTIONAL DESCRIPTOR SYSTEMS

Consider the positive fractional descriptor discrete-time system (2.1). The positive descriptor system (2.1) is called reachable in n steps if and only if the equivalent standard system (2.5) is reachable in n steps.

Definition 4.1. The positive system (2.5) is called reachable in n steps if for any given $x_f \in \Re_+^n$ there exists an input sequence $u_k \in \Re_+^m$ for $k = 0,1,\dots,n + q - 1$ that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$, i.e. $x_n = x_f$.

Theorem 4.1. The positive system (2.5) is reachable in n steps if and only if the reachability matrix

$$R_h = [\Phi_{n-1} \overline{B}_0 \quad \Phi_{n-2} \overline{B}_1 \quad \dots \quad \Phi_{n-q-2} \overline{B}_{q-1} \quad \Phi_{n-q-1} \overline{B}_q] \in \Re^{n \times h} \tag{4.1}$$

$h = (i+2)m$, contains n linearly independent monomial columns.

Proof. Using (3.9) for $i = n$ and $x_0 = 0$ we obtain

$$x_f = x_n = R_n \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n+q} \end{bmatrix}$$
$$= [\Phi_{n-q-1}\overline{B}_q \quad \Phi_{n-q-2}\overline{B}_{q-1} \quad \ldots \quad \Phi_{n-2}\overline{B}_1 \quad \Phi_{n-1}\overline{B}_0] \begin{bmatrix} u_{n+q} \\ u_{n+q-1} \\ \vdots \\ u_0 \end{bmatrix} \tag{4.2}$$

where R_n is defined by (4.1).

From (4.2) it follows that there exists an input sequence $u_k \in \Re_+^m$ for $k = 0,1,\ldots,i + q$ if and only if the matrix (4.1) contains n linearly independent monomial columns. □

Remark 4.1. Assuming zero the components of the input sequence that do not correspond to the chosen linear independent monomial columns we obtain different input sequence which steers the state vector from $x_0 = 0$ to $x_n = x_f$.

5. MINIMUM ENERGY CONTROL PROBLEM

Consider the fractional descriptor positive system (2.1) reduced to the form (2.5). In section 4 it was shown that if the positive system is reachable then there exist many input sequences that steer the state of the system from $x_0 = 0$ to the given final state $x_f \in \Re_+^n$. Among these input sequences we are looking for sequence $u_k \in \Re_+^m$ for $k = 0,1,\ldots,n + q - 1$ that minimizes the performance index

$$I(u) = \sum_{i=0}^{h-1} u_i^T Q u_i \tag{5.1}$$

where $Q \in \Re_+^{m\times m}$ is a symmetric positive defined matrix such that

$$Q^{-1} \in \Re_+^{m\times m} \tag{5.2}$$

and $h = n + q$ is the number of steps in which the state of the system is transferred from $x_0 = 0$ to the given final state $x_f \in \Re_+^n$.

The minimum energy control problem for the fractional descriptor positive discrete-time linear systems (2.1) with bounded inputs can be stated as follows.

Given the matrices E, A, B of the descriptor positive system (2.1), α, the final state $x_f \in \Re_+^n$ and the matrix $Q \in \Re_+^{n\times n}$ of the performance index (5.1) satisfying the condition (5.2), find a sequence of inputs $u_k \in \Re_+^m$ for $k = 0,1,\ldots,h - 1$ satisfying

$$u_k \le U \;\; (U \in \Re_+^m \text{ is given) for } k = 0,1,\ldots,h-1 \tag{5.3}$$

that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (5.1).

To solve the problem we define the matrix

$$W_h = R_h Q_h^{-1} R_h^T \in \Re_+^{n\times n} \tag{5.4}$$

where $R_h \in \Re_+^{n\times hm}$ is defined by (4.1) and

$$Q_h^{-1} = \text{blockdiag}\,[Q^{-1},\ldots,Q^{-1}] \in \Re_+^{hm\times hm}. \tag{5.5}$$

The matrix (5.4) is non-singular if the positive system is reachable in h steps.

For a given $x_f \in \Re_+^n$ we may define the input sequence

$$\hat{u}_h = \begin{bmatrix} \hat{u}_{h-1} \\ \hat{u}_{h-2} \\ \vdots \\ \hat{u}_0 \end{bmatrix} = Q_h^{-1} R_h^T W_h^{-1} x_f \tag{5.6}$$

where Q_h^{-1}, W_h and R_h are defined by (5.5), (5.4) and (4.1), respectively.

Lemma 5.1. If the system (2.5) is reachable and all columns of the matrix (4.1) are monomial and the matrix Q is diagonal then

$$W_h^{-1} x_f \in \Re_+^n \tag{5.7}$$

for any $x_f \in \Re_+^n$.

Proof. If the assumptions are satisfied then the matrix $W_h = R_h Q_h^{-1} R_h^T$ is diagonal and $W_h^{-1} \in \Re_+^{n\times n}$ since the diagonal entries of W_q are positive. Therefore, the condition (5.7) is met for any $x_f \in \Re_+^n$. □

Lemma 5.2. If the system (2.5) is reachable and all columns of the matrix (4.1) are monomial and the matrix Q is diagonal then the input sequence (5.6) steers the positive system from $x_0 = 0$ to $x_f \in \Re_+^n$.

Proof. Using the solution of the equation (2.5) for $x_0 = 0$ and $i = h$ and (5.6) we obtain

$$x_h = R_h \hat{u}_h = R_h Q_h^{-1} R_h^T W_h^{-1} x_f = x_f \tag{5.8}$$

since $W_h = R_h Q_h^{-1} R_h^T$ holds. □

Lemma 5.3. If the diagonal matrix Q is the scalar matrix

$$Q = \text{diag}[q_1,\ldots,q_1] \in \Re_+^{m\times m} \tag{5.9}$$

then the input sequence (5.6) is independent of Q and is given by

$$\hat{u}_h = R_h^T [R_h R_h^T]^{-1} x_f \in \Re_+^{qm} \tag{5.10}$$

for any $x_f \in \Re_+^n$.

Proof. If (5.9) holds then from (5.4) we have

$$W_h = \frac{1}{q_1} R_h R_h^T \in \Re_+^{n\times n} \tag{5.11}$$

and

$$W_h^{-1} = q_1 [R_h R_h^T]^{-1} \in \Re_+^{n\times n}. \tag{5.12}$$

In this case the input sequence (5.6) is given by

$$\hat{u}_h = Q_h^{-1} R_h^T W_h^{-1} x_f = \frac{1}{q_1} R_h^T q_1 [R_h R_h^T]^{-1} x_f = R_h^T [R_h R_h^T]^{-1} x_f \tag{5.13}$$

for any $x_f \in \Re_+^n$. □

Theorem 5.1. Let the fractional descriptor positive system (2.1) be reachable in h steps and the conditions (5.2) and (5.7) be satisfied. Moreover, let

$$\bar{u} = \begin{bmatrix} \bar{u}_{h-1} \\ \bar{u}_{h-2} \\ \vdots \\ \bar{u}_0 \end{bmatrix} \in \Re_+^{hm} \tag{5.14}$$

be an input sequence satisfying (5.3) that steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Then the input sequence (5.6) satisfying (5.3) also steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$ and minimizes the performance index (5.1), i.e.

$$I(\hat{u}) \le I(\bar{u}). \tag{5.15}$$

The minimal value of the performance index (5.1) is given by

$$I(\hat{u}) = x_f^T W_h^{-1} x_f. \tag{5.16}$$

Proof. If the conditions (5.2) and (5.7) are met and the system is reachable in h steps then the input sequence (5.6) is well defined and $\hat{u} \in \Re_+^{hm}$. We shall show that the input sequence (5.6) satisfying (5.3) steers the state of the system from $x_0 = 0$ to $x_f \in \Re_+^n$. Using (4.2) and (5.6) we obtain

$$x_h = R_h \hat{u} = R_h Q_h^{-1} R_h^T W_h^{-1} x_f = x_f \tag{5.17}$$

since by (5.4) $R_h Q_h^{-1} R_h^T = W_h$. Hence $x_f = R_h \hat{u} = R_h \bar{u}$ or

$$R_h [\hat{u} - \bar{u}] = 0. \tag{5.18}$$

The transposition of (5.18) yields

$$[\hat{u} - \bar{u}]^T R_h^T = 0. \tag{5.19}$$

Postmultiplying the equality (5.19) by $W_h^{-1} x_f$ we obtain

$$[\hat{u} - \bar{u}]^T R_h^T W_h^{-1} x_f = 0. \tag{5.20}$$

From (5.6) we have $Q_h \hat{u} = R_h^T W_h^{-1} x_f$. Substitution of this equality into (5.20) yields

$$[\hat{u} - \bar{u}]^T Q_h \hat{u} = 0 \tag{5.21}$$

where $Q_h = \text{blockdiag}[Q,...,Q] \in \Re_+^{hm\times hm}$.
From (5.21) it follows that

$$\bar{u}^T Q_h \bar{u} = \hat{u}^T Q_h \hat{u} + [\bar{u} - \hat{u}]^T Q_h [\bar{u} - \hat{u}] \tag{5.22}$$

since by (5.21) $\bar{u}^T Q_h \hat{u} = \hat{u}^T Q_h \hat{u} = \hat{u}^T Q_h \bar{u}$.
From (5.22) it follows that (5.15) holds since $[\bar{u} - \hat{u}]^T Q_h [\bar{u} - \hat{u}] \ge 0$. To find the minimal value of the performance index (5.1) we substitute (5.6) into (5.1) and we obtain

$$\begin{aligned} I(\hat{u}) &= \sum_{i=0}^{h-1} \hat{u}_i^T Q \hat{u}_i = \hat{u}^T Q_h \hat{u} = [Q_h^{-1} R_h^T W_h^{-1} x_f]^T Q_h [Q_h^{-1} R_h^T W_h^{-1} x_f] \\ &= x_f^T W_h^{-1} R_h Q_h^{-1} R_h^T W_h^{-1} x_f = x_f^T W_h^{-1} x_f \end{aligned} \tag{5.23}$$

since by (5.4) $W_h^{-1} R_h Q_h^{-1} R_h^T = I_n$. □

Theorem 5.2. Let the matrix Q have the form (5.9) and the assumptions of Theorem 5.1 be satisfied. Then the minimum energy control problem with bounded inputs has a solution if the last m columns of the input sequence vector (5.6) satisfy the condition

$$\hat{u}'_m = \bar{B}_q^T [BB^T]^{-1} x_f < U \in \Re_+^m \tag{5.24a}$$

where

$$B = [\bar{B}_0 \quad \bar{B}_1 \quad \dots \quad \bar{B}_q] \in \Re_+^{n\times(q+1)m}. \tag{5.24b}$$

Proof. If the matrix Q has the form (3.8) then by Lemma 5.3 the input sequence is given by (5.10) for any $x_f \in \Re_+^n$. From the structure of R_q and (5.10) it follows that the last m components of the input sequence vector $\hat{u}_q$ is given by

$$\hat{u}'_m = \bar{B}_q^T [BB^T]^{-1} x_f. \tag{5.25}$$

Therefore, the minimum energy control problem with bounded inputs has a solution if the condition (5.3) is satisfied. □

The optimal input sequence (5.6) and the minimal value of the performance index (5.16) can be computed by the use of the following procedure.

Procedure 5.1.

Step 1. Knowing the matrices A, B, Q and using (3.8) and (4.1) compute the matrices R_h and W_h for a chosen h such that the matrix R_h contains at least n linearly independent monomial columns and check the condition (5.24). If this condition is satisfied then go to step 2.

Step 2. Using (5.6) find the input sequence $u_k \in \Re_+^m$ $k = 0,1,\ldots,h-1$ satisfying the condition (5.3). If the condition (5.3) is not satisfied increase h by one and repeat the computation for $i + 1$. If the matrix W_h is diagonal after some number of steps we obtain the desired input sequence satisfying the condition (5.3).

Step 3. Using (5.16) compute the minimal value of the performance index $I(\hat{u})$.

6. CONCLUDING REMARKS

Necessary and sufficient conditions for the positivity and reachability of fractional descriptor positive discrete-time linear systems have been established (Theorem 2.1and Theorem 4.1). The transformation of the fractional descriptor system to equivalent standard system by the use of the shuffle algorithm has been addressed. The minimum energy control problem for the fractional descriptor positive systems has been formulated and solved (Theorem 5.1). A procedure for computation of optimal input sequences and minimal value of the performance index has been proposed (Procedure 5.1). An open problem is an extension of these considerations to fractional positive descriptor 2D continuous-discrete linear systems.

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