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Synchronizing Linear Heterogeneous Networks by Output Homogenization

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Abstract: This article deals with the synchronization of heterogeneous multi-agent systems. A simple distributed control law is provided to solve the problem. The presented approach is based on the idea to ascribe the heterogeneous case to a homogeneous synchronization task, for which the solution is well known. The presented method is applicable to networks of completely different agents, even with different dimensions. Additionally, the communication effort is minimized because only the states which are to be synchronized need to be exchanged through the network. An example illustrates the efficiency of the concept.

Keywords: Synchronization; Multi-agent systems; Heterogeneous agents; Distributed control.

1. INTRODUCTION

In recent years, the problem of distributed control for networked dynamic systems – so-called *multi-agent systems* – which are connected through a communication topology has been receiving increasing attention in many research areas (see for instance Olfati-Saber et al. [2007], Ren et al. [2007] and the references therein). Especially, the problem of *synchronization*, meaning the agreement of all agents to a common trajectory, is a widely studied problem.

The first research contributions led to conditions and algorithms to achieve *consensus* for single and double integrator dynamics (Ren et al. [2007], Ren [2008]) and based on these results, the consensus problem for high order integrator dynamics has been investigated (e.g. in Jiang et al. [2009]). Topics like formation (Fax and Murray [2004]), swarming and flocking (Tanner et al. [2007]), rendezvous problems (Lin et al. [2007]) and many more have been investigated. Also the influence of communication delays and time-varying communication networks was considered (Moreau [2004], Olfati-Saber et al. [2007]).

However, an increasing interest has turned to more sophisticated systems with general linear dynamics. While in systems with only integrator dynamics the goal is to find a state consensus for all agents, in systems with general linear dynamics the task is to track a common time-varying trajectory. In the last decade, many results have been achieved for *homogeneous* agents, i.e. systems with identical dynamics (e.g. Tuna [2008], Scardovi and Sepulchre [2009], Ma and Zhang [2010]). The article of Ma and Zhang [2010] gives a detailed overview of necessary and sufficient conditions for the synchronization of identical linear time-invariant systems that are connected by a fixed communication topology. A fundamental result is given by Tuna [2008], where a simple LQR-based feedback design is presented, which guarantees synchronization for homogeneous agents.

In this paper, we focus on synchronization for *heterogeneous* multi-agent systems, that is agents with non-identical dynamics. The consideration of heterogeneous agents is attractive for two reasons. First, in real world applications, no two systems are exactly identical and second, synchronization of completely different systems is a very desirable result, because this offers more application possibilities. While in the homogeneous case the goal is to synchronize all internal states of the agents, in heterogeneous networks this is not possible in general. Due to different system dynamics and dimensions, synchronization of all internal states is not possible and meaningful. Here, the goal is to achieve *output synchronization* in the sense of synchronizing physically comparable states.

Interestingly, there is only little literature about the synchronization of heterogeneous multi-agent systems. Most of the results are based on the output regulation theory and the *internal model principle* of control theory (Francis and Wonham [1976]). The main idea of these contributions is to track a common trajectory which is generated by an *exosystem* and was first published by Wieland and Allgöwer [2009]. A similar approach is used in Kim et al. [2011] where the output synchronization problem for uncertain systems was studied. Describing the heterogeneous synchronization problem as an output tracking problem is a common method that is used in different approaches (e.g. Wieland et al. [2011], Listmann et al. [2011b], Grip et al. [2012]). However, a major drawback of this method lies in the high design complexity and its increased communication load. Moreover, the solution is based on the regulator equations, which lead to large computational efforts.

In contrast to previous works, this article provides a simple and more transparent control law which allows to ascribe the heterogeneous output synchronization task to the homogeneous case and use known methods to solve the problem. We show that under certain conditions, a straightforward feedback structure is sufficient to synchronize heterogeneous agents, also with different dimensions.

The idea is based on a two-step procedure consisting of an inner feedback law, which homogenizes the output dynamics of each agent, and an outer feedback law, which depends on the communication topology of the network and ensures output synchronization. Fig. 1 illustrates the control strategy for each agent.

The remainder of this paper is organized as follows: We start in Section 2 with some notations and graph-theoretical basics. In Section 3, the problem under study is set up and a brief review of the results that have been achieved so far for the synchronization of homogeneous agents is given. Our main result is presented in Section 4, where we show how the output synchronization problem for heterogeneous agents can be traced back to the homogeneous case. In Section 5, a numerical example is given to illustrate the efficiency of the approach. A conclusion is provided in Section 6.

2. PRELIMINARIES

2.1 Notation and matrix theory

The identity matrix of order k is denoted by $\boldsymbol{I}_k$ and a matrix with only zero elements of appropriate dimension is written as $\mathbf{0}$. The column vector whose elements are all one is abbreviated by $\mathbf{1}$. Given the vectors $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$ with $i \in \{1,\ldots,N\}$, for notational convenience we define the stacked vector $\boldsymbol{x}^\top = \begin{bmatrix}\boldsymbol{x}_1^\top & \ldots & \boldsymbol{x}_N^\top\end{bmatrix} \in \mathbb{R}^n$, with $n = \sum_{i=1}^N n_i$.

For a matrix $\boldsymbol{X} \in \mathbb{R}^{q\times r}$, with $q \leqslant r$, its pseudoinverse reads $\boldsymbol{X}^+ \in \mathbb{R}^{r\times q}$, where $\boldsymbol{X}\boldsymbol{X}^+ = \boldsymbol{I}_q$. We write the rank of $\boldsymbol{X}$ as $\operatorname{rank}(\boldsymbol{X})$. A block diagonal matrix formed from N matrices $\boldsymbol{X}_i$ is written as

$$\breve{\boldsymbol{X}}_N = \begin{bmatrix}\boldsymbol{X}_1 & & \\ & \ddots & \\ & & \boldsymbol{X}_N\end{bmatrix}.$$

If the matrices $\boldsymbol{X}_i$ are identical, i.e. $\boldsymbol{X}_i = \boldsymbol{X}$, instead of $\breve{\boldsymbol{X}}_N$ we write $\breve{\boldsymbol{X}}_N = \boldsymbol{I}_N \otimes \boldsymbol{X}$, where $\otimes$ denotes *Kronecker* product. Further, we define $\mathbb{R}^{kq\times kr} \ni \widehat{\boldsymbol{X}}_k = \boldsymbol{X} \otimes \boldsymbol{I}_k$.

A square matrix $\boldsymbol{A} \in \mathbb{R}^{k\times k}$ is called *Hurwitz* if all of its eigenvalues $\lambda_i(\boldsymbol{A})$ have strictly negative real parts and it is called *anti-stable* if all of its eigenvalues have non-negative real parts. We denote by $\boldsymbol{A} \succ (\prec)\ \mathbf{0}$ that $\boldsymbol{A}$ is positive (negative) definite.

2.2 Graph theory

In this work, we model the communication network of a multi-agent system with $N < \infty$ agents by a *directed graph* (or *digraph*) $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$, which is assumed to be time-invariant. Each agent is represented by a vertex (or node) in the set of vertices $\mathcal{V}_\mathcal{G}$, with $|\mathcal{V}_\mathcal{G}| = N$ and the information flow from a vertex i to a vertex j is described by the edge $(i,j) \in \mathcal{E}_\mathcal{G}$. The in-degree $d_{in,i}$ of a vertex i is the number of vertices which send information to vertex i and the in-degree matrix of the corresponding graph $\mathcal{G}$ is denoted by $\boldsymbol{D}_{\mathcal{G}_{in}} = \operatorname{diag}(d_{in,1},\ldots,d_{in,N})$. The *adjacency matrix* of the graph is given as $\boldsymbol{A}_\mathcal{G} = [a_{\mathcal{G}_{ij}}] \in \mathbb{R}^{N\times N}$, with

$$a_{\mathcal{G}_{ij}} = \begin{cases}1, & \text{if } (i,j) \in \mathcal{E}_\mathcal{G},\\ 0, & \text{if } (i,j) \notin \mathcal{E}_\mathcal{G}.\end{cases}$$

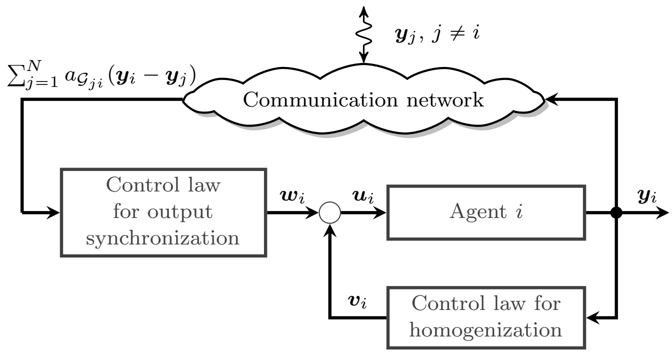

Fig. 1. Two-step control strategy for achieving output synchronization.

Furthermore, in consensus and synchronization problems the *Graph Laplacian* is an important matrix (Olfati-Saber et al. [2007]). It is defined as $\boldsymbol{L}_\mathcal{G} = \boldsymbol{D}_{\mathcal{G}_{in}} - \boldsymbol{A}_\mathcal{G}^\top = [l_{\mathcal{G}_{ij}}] \in \mathbb{R}^{N\times N}$ and its elements are

$$l_{\mathcal{G}_{ij}} = \begin{cases}\sum_{k=1}^N a_{\mathcal{G}_{ki}}, & \text{if } i = j,\\ -a_{\mathcal{G}_{ji}}, & \text{if } i \neq j.\end{cases}$$

Definition 1. A digraph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ is called *connected* if there exists at least one vertex (also called root vertex) that can reach every other vertex in the graph, using only the edges given in $\mathcal{E}_\mathcal{G}$.

If the digraph is connected, it is also said that it contains a directed spanning tree (Wu [2005]). Based on Geršgorin's Circle Theorem the following Lemma is elementary (Lafferriere et al. [2005]):

Lemma 1. Given the digraph $\mathcal{G} = (\mathcal{V}_\mathcal{G}, \mathcal{E}_\mathcal{G})$ is connected, then $\lambda_1(\boldsymbol{L}_\mathcal{G}) = 0$ is a simple eigenvalue and $\sigma_i > 0, \forall i \in \{2,\ldots,N\}$, with $\sigma_i = \mathfrak{Re}\{\lambda_i(\boldsymbol{L}_\mathcal{G})\}$.

Remark 1. The Laplacian matrix has always at least one eigenvalue in zero with corresponding right eigenvector $\mathbf{1}$, since all rows sum up to zero.

3. PROBLEM SETUP AND SYNCHRONIZATION OF HOMOGENEOUS AGENTS

In this section, we define the systems under study and the assumptions which are necessary to solve the problem. Since the idea of our approach is to reduce the output synchronization problem for heterogeneous agents to a synchronization task for homogeneous agents, some results of the homogeneous case are also summarized.

3.1 System description and problem definition

We consider a multi-agent system consisting of N linear agents

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}_i\boldsymbol{x}_i + \boldsymbol{B}_i\boldsymbol{u}_i, \tag{1a}$$
$$\boldsymbol{y}_i = \boldsymbol{C}_i\boldsymbol{x}_i, \qquad i \in \{1,\ldots,N\}, \tag{1b}$$

where $\boldsymbol{x}_i \in \mathbb{R}^{n_i}$, $\boldsymbol{u}_i \in \mathbb{R}^{m_i}$ and $\boldsymbol{y}_i \in \mathbb{R}^p$ are the state, control and output vectors. The matrices $\boldsymbol{A}_i \in \mathbb{R}^{n_i\times n_i}$, $\boldsymbol{B}_i \in \mathbb{R}^{n_i\times m_i}$ and $\boldsymbol{C}_i \in \mathbb{R}^{p\times n_i}$ are assumed to be constant and known. Note that all agents can have different state and input dimensions n_i and m_i, while the output dimension p is the same for all agents.

In some literature it is assumed that only relative information is available to every agent, meaning that they do not have any information about their own states (Listmann et al. [2011b], Grip et al. [2012]). However, we consider in this paper that the agents know their own outputs. Since in general the agents have access to internal state measurements, we refer to Fax and Murray [2004] and Olfati-Saber et al. [2007] and assume that every agent can measure its internal states or estimate non-measurable states by a (reduced order) Luenberger observer. For heterogeneous multi-agent systems it is also possible to estimate the absolute states of the agents based on the agent heterogeneity and the relative information as shown in Listmann et al. [2011b]. Hence, we assume in the following that every agent has access to its output vector $\boldsymbol{y}_i$.

The goal is to find a distributed control law so that output synchronization is achieved, meaning

$$\lim_{t\to\infty} \|\boldsymbol{y}_i(t) - \boldsymbol{\eta}(t)\| = \boldsymbol{0}, \quad \forall i \in \{1, \dots, N\}, \tag{2}$$

where $\boldsymbol{\eta}(t)$ is the synchronization trajectory.

In order to solve this problem, we make the following assumptions, which have to be fulfilled for every agent:

Assumption 1. The pair $(\boldsymbol{A}_i, \boldsymbol{C}_i)$ is detectable.

Assumption 2. The system $(\boldsymbol{A}_i, \boldsymbol{B}_i, \boldsymbol{C}_i)$ has no invariant zeros in the closed right half of the s-plane.

Assumption 3. $\mathrm{rank}(\boldsymbol{C}_i\boldsymbol{B}_i) = p$.

Assumption 1 is a common assumption in control theory, which is necessary for the construction of a Luenberger observer, and the necessity of Assumption 2 will be shown in the next section. Assumption 3 was also assumed in the work of Rodrigues de Campos et al. [2012] and means that the input vector influences the dynamics of the output vector directly. Furthermore, this implies that only as many states as available actuators can be synchronized. Since in the heterogeneous case we are interested in output synchronization, this is not a major restriction due to the fact that often $m_i \geqslant p$ is satisfied.

Next, we give a brief summary of the results for the synchronization of homogeneous agents, before we present the design procedure to achieve the goal defined in (2).

3.2 Synchronization of homogeneous multi-agent systems

In a homogeneous multi-agent system the dynamics of the agents are described by

$$\dot{\boldsymbol{x}}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{B}\boldsymbol{u}_i, \tag{3a}$$

$$\boldsymbol{u}_i = -\boldsymbol{K}\sum_{j=1}^{N} a_{\mathcal{G}_{ji}}(\boldsymbol{x}_i - \boldsymbol{x}_j), \quad i, j \in \{1, \dots, N\}, \tag{3b}$$

with identical matrices $\boldsymbol{A} \in \mathbb{R}^{n\times n}$, $\boldsymbol{B} \in \mathbb{R}^{m\times m}$ and $\boldsymbol{K} \in \mathbb{R}^{m\times n}$ for every agent.

The synchronization task is given as

$$\lim_{t\to\infty} \|\boldsymbol{x}_i(t) - \boldsymbol{x}_j(t)\| = \boldsymbol{0}, \quad \forall i, j \in \{1, \dots, N\}.$$

It is well known in literature (Tuna [2008], Ma and Zhang [2010]) that stabilizing the matrices

$$\boldsymbol{A} - \lambda_i(\boldsymbol{L}_{\mathcal{G}})\boldsymbol{B}\boldsymbol{K}, \quad \forall i \in \{2, \dots, N\}, \tag{4}$$

leads to synchronization. That is, the synchronization problem of homogeneous agents can be reduced to a stabilization problem rendering the matrices (4) Hurwitz. The synchronization trajectory for the multi-agent system is then determined by

$$\begin{aligned}\lim_{t\to\infty} \boldsymbol{x}_{\mathrm{Sync}}(t) &= \lim_{t\to\infty} \big(\boldsymbol{r}^\top \otimes e^{\boldsymbol{A}(t-t_0)}\big)\boldsymbol{x}(t_0) \\ &= \lim_{t\to\infty} e^{\boldsymbol{A}(t-t_0)}\big(r_1\boldsymbol{x}_1(t_0) + \ldots + r_N\boldsymbol{x}_N(t_0)\big),\end{aligned}$$

where $\boldsymbol{r}^\top = [r_1, \dots, r_N] \in \mathbb{R}^N$ is the left eigenvector to the corresponding eigenvalue $\lambda_1(\boldsymbol{L}_{\mathcal{G}}) = 0$. Further, we know from Wu [2005] that the elements of $\boldsymbol{r}$ are all non-negative and according to Remark 1, we get $\boldsymbol{r}^\top\boldsymbol{1} = \sum_{j=1}^N r_j = 1$. That is, the agents synchronize to a trajectory which is determined by the weighted average of their initial values $\boldsymbol{x}_i(t_0)$, where the weights are given by $\boldsymbol{r}^\top$.

Listmann et al. [2011a] have shown that it is sufficient to consider only the real parts of the eigenvalues of the Laplacian matrix and they have derived a linear matrix inequality (LMI) approach, which will also be used in the following Lemma.

Lemma 2. Consider a homogeneous multi-agent system with dynamics (3a) and control law (3b). Assume the digraph which describes the underlying communication topology of the network is connected and let the pair $(\boldsymbol{A}, \boldsymbol{B})$ be stabilizable. Then, the feedback matrix $\boldsymbol{K} = \boldsymbol{V}\boldsymbol{X}^{-1}$, with $\boldsymbol{X} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{V} \in \mathbb{R}^{m\times n}$ achieves synchronization if and only if there is a matrix

$$\boldsymbol{X} = \boldsymbol{X}^\top \succ \boldsymbol{0},$$

so that the following two LMIs hold:

$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A}^\top - \underline{\sigma}\big(\boldsymbol{B}\boldsymbol{V} + \boldsymbol{V}^\top\boldsymbol{B}^\top\big) \prec \boldsymbol{0},$$
$$\boldsymbol{A}\boldsymbol{X} + \boldsymbol{X}\boldsymbol{A}^\top - \overline{\sigma}\big(\boldsymbol{B}\boldsymbol{V} + \boldsymbol{V}^\top\boldsymbol{B}^\top\big) \prec \boldsymbol{0},$$

with $\underline{\sigma} = \min_i(\sigma_i)$ and $\overline{\sigma} = \max_i(\sigma_i)$, $i \in \{2, \dots, N\}$. See Listmann et al. [2011a] for the proof.

Remark 2. In Tuna [2008] an LQR-based method is presented to calculate an appropriate feedback matrix $\boldsymbol{K}$, which can be used alternatively.

4. MAIN RESULT

The synchronization task defined in (2) will be solved in two steps (cf. Fig. 1). Given that Assumptions 1, 2 and 3 hold, we first introduce a local control law which homogenizes the dynamics of each agent and decouples the states of interest, i.e. the output states, from the rest of the system. After doing so, it is easy to show that, based on the communicated variables, an appropriate control law solves the output synchronization problem for the heterogeneous agents.

4.1 Output synchronization for heterogeneous agents

Consider again the heterogeneous multi-agent system (1). Without loss of generality, we assume that the output vector describes the states which we wish to synchronize and that the output matrix is given by $\boldsymbol{C}_i = [\boldsymbol{I}_p\ \boldsymbol{0}]$, meaning we want to synchronize the first p states of every agent. Note that it is always possible to transform and re-arrange the states of a system so that the above condition for $\boldsymbol{C}_i$ holds. Hence, we have $\boldsymbol{x}_i^\top = [\boldsymbol{y}_i^\top\ \bar{\boldsymbol{x}}_i^\top]$ with $\bar{\boldsymbol{x}}_i^\top = [x_{i,p+1} \dots x_{i,n_i}]$.

We propose a distributed control law consisting of two components

$$\boldsymbol{u}_i = \boldsymbol{v}_i + \boldsymbol{w}_i, \tag{5a}$$

with

$$\boldsymbol{v}_i = -\boldsymbol{F}_i \hat{\boldsymbol{x}}_i, \tag{5b}$$

$$\boldsymbol{w}_i = -\boldsymbol{K}_i \sum_{j=1}^{N} a_{\mathcal{G}_{ji}} (\boldsymbol{y}_i - \boldsymbol{y}_j). \tag{5c}$$

In (5b) the vector $\hat{\boldsymbol{x}}_i$ describes state estimations given by a Luenberger observer

$$\dot{\hat{\boldsymbol{x}}}_i = \boldsymbol{A}_i \hat{\boldsymbol{x}}_i + \boldsymbol{B}_i \boldsymbol{u}_i + \boldsymbol{H}_i \boldsymbol{C}_i (\boldsymbol{x}_i - \hat{\boldsymbol{x}}_i),$$

with the observer matrix $\boldsymbol{H}_i \in \mathbb{R}^{n_i \times p}$ and the error dynamics

$$\dot{\boldsymbol{e}}_i = \dot{\boldsymbol{x}}_i - \dot{\hat{\boldsymbol{x}}}_i = \big(\boldsymbol{A}_i - \boldsymbol{H}_i \boldsymbol{C}_i\big) \boldsymbol{e}_i.$$

The goal is that each agent tracks a common trajectory described by

$$\dot{\boldsymbol{\eta}} = \boldsymbol{S}\boldsymbol{\eta}, \tag{6}$$

with $\boldsymbol{S} \in \mathbb{R}^{p \times p}$. Referring to Wieland and Allgöwer [2009], system (6) can be interpreted as an exosystem whose trajectory should be tracked by each agent.

Remark 3. We assume that $\boldsymbol{S}$ is anti-stable. If $\boldsymbol{S}$ is Hurwitz a trivial solution for the synchronization problem is given by stabilizing the dynamics of every agent.

Theorem 1. Given the agent dynamics (1) with control law (5) and let Assumptions 1 and 3 hold. Then, in the asymptotic case the output dynamics of every agent read

$$\dot{\boldsymbol{y}}_i = \boldsymbol{S}\boldsymbol{y}_i + \boldsymbol{C}_i \boldsymbol{B}_i \boldsymbol{w}_i, \tag{7}$$

if $\boldsymbol{A}_i - \boldsymbol{H}_i \boldsymbol{C}_i$ is Hurwitz and $\boldsymbol{F}_i$ is chosen as

$$\boldsymbol{F}_i = (\boldsymbol{C}_i \boldsymbol{B}_i)^{+} \big(\boldsymbol{C}_i \boldsymbol{A}_i - [\boldsymbol{S}\ \boldsymbol{0}]\big). \tag{8}$$

Proof. Combining (1a) and (1b) leads to

$$\dot{\boldsymbol{y}}_i = \boldsymbol{C}_i \boldsymbol{A}_i \boldsymbol{x}_i + \boldsymbol{C}_i \boldsymbol{B}_i \boldsymbol{u}_i, \tag{9}$$

and applying (5) to (9) we get

$$\dot{\boldsymbol{y}}_i = \boldsymbol{C}_i \boldsymbol{A}_i \boldsymbol{x}_i - \boldsymbol{C}_i \boldsymbol{B}_i \boldsymbol{F}_i \hat{\boldsymbol{x}}_i + \boldsymbol{C}_i \boldsymbol{B}_i \boldsymbol{w}_i. \tag{10}$$

Now, if $\boldsymbol{F}_i$ is determined as in (8), the output dynamics can be written as

$$\dot{\boldsymbol{y}}_i = [\boldsymbol{S}\ \boldsymbol{0}]\, \boldsymbol{x}_i + \big(\boldsymbol{C}_i \boldsymbol{A}_i - [\boldsymbol{S}\ \boldsymbol{0}]\big) \boldsymbol{e}_i + \boldsymbol{C}_i \boldsymbol{B}_i \boldsymbol{w}_i. \tag{11}$$

Since Assumption 1 holds, the observer matrix $\boldsymbol{H}_i$ can be calculated such that the matrix of the error dynamics $\boldsymbol{A}_i - \boldsymbol{H}_i \boldsymbol{C}_i$ is Hurwitz, leading to

$$\lim_{t \to \infty} \boldsymbol{e}_i(t) = 0.$$

Hence, in the asymptotic case (11) reads

$$\dot{\boldsymbol{y}}_i = [\boldsymbol{S}\ \boldsymbol{0}]\, \boldsymbol{x}_i + \boldsymbol{C}_i \boldsymbol{B}_i \boldsymbol{w}_i.$$

By assumption, the state vector is partitioned as $\boldsymbol{x}_i^\top = \big[\boldsymbol{y}_i^\top\ \bar{\boldsymbol{x}}_i^\top\big]$. Therefore, the output dynamics are determined by (7) finally. ■

Because the observer is asymptotically converging, we assume in the following that $\boldsymbol{x}_i$ is available for the control law (5b).

Now, for every agent the dynamics of the output vectors $\boldsymbol{y}_i$ are described by homogeneous system matrices $\boldsymbol{S}$ and are completely decoupled from the remaining states $\bar{\boldsymbol{x}}_i$. Hence, there are internal dynamics, which do not influence the output vector, but it is reasonable to ensure that these dynamics will not be unstable. So the question is, how will the internal dynamics be influenced by controller (5b).

Proposition 1. Since Assumption 2 is satisfied, controller (5b) leads to stable internal dynamics.

Proof. To prove Proposition 1, we consider the transfer matrix of the closed loop system (1) with (5b):

$$\boldsymbol{G}_{\boldsymbol{y}_i \boldsymbol{w}_i}(s) = \boldsymbol{C}_i \big(s\boldsymbol{I}_{n_i} - \boldsymbol{A}_i + \boldsymbol{B}_i \boldsymbol{F}_i\big)^{-1} \boldsymbol{B}_i. \tag{12}$$

From Theorem 1, we know that the input-output relationship can also be described by (7), with transfer matrix

$$\boldsymbol{G}_{\boldsymbol{y}_i \boldsymbol{w}_i}(s) = \big(s\boldsymbol{I}_p - \boldsymbol{S}\big)^{-1} \boldsymbol{C}_i \boldsymbol{B}_i. \tag{13}$$

While (12) describes a system with n_i poles, in (13) we have a system with p poles. This is only possible if $n_i - p$ poles in (12) are canceled by corresponding invariant zeros of the system. Hence, the controller (5b) shifts $n_i - p$ poles under the invariant zeros of the system. Since we have assumed that the agents do not have any invariant zeros in the closed right half of the s-plane, the internal dynamics will be stable. ■

Remark 4. Assumption 2 ensures that the internal dynamics of the agents will be stable. However, it is sufficient to have no unstable modes. Hence, we may have invariant zeros on the imaginary axis, if the marginal stability of the internal dynamics is still guaranteed.

After homogenizing the output dynamics of the agents, the next step is to find a suitable matrix $\boldsymbol{K}_i$ in (5c), leading to output synchronization. It should be noted that the controller (5c) will not change the stability properties of the internal dynamics.

Theorem 2. Given the system dynamics (1) together with control law (5) and the digraph describing the underlying communication topology of the network is connected. Then, all agents synchronize to a common trajectory

$$\lim_{t \to \infty} \boldsymbol{\eta}(t) = \lim_{t \to \infty} \big(\boldsymbol{r}^\top \otimes e^{\boldsymbol{S}(t-t_0)}\big) \boldsymbol{y}(t_0), \tag{14}$$

if we choose $\boldsymbol{F}_i$ as described in (8) and

$$\boldsymbol{K}_i = \big(\boldsymbol{C}_i \boldsymbol{B}_i\big)^{+} \boldsymbol{P}, \tag{15}$$

where $\boldsymbol{P}$ is determined such that $\boldsymbol{S} - \sigma_i \boldsymbol{P}$ is Hurwitz for all $i \in \{2, \ldots, N\}$.

Proof. As we have shown in Theorem 1, the dynamics of the output vectors are determined by homogeneous system matrices $\boldsymbol{S}$ and heterogeneous input matrices $\boldsymbol{C}_i \boldsymbol{B}_i$, if we calculate $\boldsymbol{F}_i$ as given in (8). The overall multi-agent system (regarding only the output dynamics) is then described by

$$\dot{\boldsymbol{y}} = \big(\widetilde{\boldsymbol{S}}_N - \breve{\boldsymbol{C}}_N \breve{\boldsymbol{B}}_N \breve{\boldsymbol{K}}_N \widehat{\boldsymbol{L}}_{\mathcal{G},p}\big) \boldsymbol{y}. \tag{16}$$

Since Assumption 3 holds, we choose $\boldsymbol{K}_i = \big(\boldsymbol{C}_i \boldsymbol{B}_i\big)^{+} \boldsymbol{P}$ for every agent and the expression in (16) changes to

$$\dot{\boldsymbol{y}} = \big(\widetilde{\boldsymbol{S}}_N - \widetilde{\boldsymbol{P}}_N \widehat{\boldsymbol{L}}_{\mathcal{G},p}\big) \boldsymbol{y}. \tag{17}$$

Equation (17) shows that the output vectors are described by a completely homogeneous multi-agent system if we calculate the feedback matrices $\boldsymbol{F}_i$ and $\boldsymbol{K}_i$ as given in Theorem 1 and 2. From Section 3.2, we know that synchronization is achieved if the matrices $\boldsymbol{S} - \sigma_i \boldsymbol{P}$ are Hurwitz for all $i \in \{2, \ldots, N\}$, which proves the claim. ■

Remark 5. For the calculation of the matrix $\boldsymbol{P}$, we can use Lemma 2 by setting $\boldsymbol{A} = \boldsymbol{S}$ and $\boldsymbol{B} = \boldsymbol{I}_p$. Note that the pair $(\boldsymbol{S}, \boldsymbol{I}_p)$ is stabilizable, independently of $\boldsymbol{S}$. Hence, a solution for $\boldsymbol{P}$ can always be calculated, provided that the graph is connected.

Summarizing the results, the following steps determine the design procedure for achieving output synchronization.

Step 1. Given the heterogeneous multi-agent system (1) with control law (5) and Assumptions 1, 2 and 3 are satisfied, use observers to estimate non-measurable states for all $i \in \{1, \ldots, N\}$.

Step 2. Calculate the feedback matrices $\boldsymbol{F}_i$ as given in Theorem 2 to adjust the output dynamics of the agents to the dynamics of the synchronization trajectory (6), leading to homogeneous system matrices.

Step 3. Calculate the feedback matrices $\boldsymbol{K}_i$ as shown in Theorem 2 and output synchronization is achieved, with synchronization trajectory (14).

4.2 Discussion and further notes

The presented approach provides several advantages. The solution is applicable to heterogeneous systems with potentially different dimensions. Moreover, the presented solution is based on static feedback laws, i.e. no controller dynamics are necessary, unless we need estimations of internal states. Furthermore, the communication load is minimized compared to previous works. No controller or exosystem states need to be exchanged, as it is the case e.g. in Wieland et al. [2011]. Only the output vectors need to be exchanged through the communication network.

The major restriction of the above design procedure is based on the rank conditions given in Assumption 3, but it should be noted that this is only a sufficient condition. Assumption 3 must hold since we need an inverse of $\boldsymbol{C}_i\boldsymbol{B}_i$ to calculate the feedback matrices $\boldsymbol{F}_i$ and $\boldsymbol{K}_i$. Under certain conditions it is still possible to generate a solution, even if $\boldsymbol{C}_i\boldsymbol{B}_i$ is not right-invertible. Regarding again (8) and (15), we know that the following two equations must be fulfilled:

$$\begin{aligned} \boldsymbol{C}_i\boldsymbol{B}_i\boldsymbol{F}_i &= \boldsymbol{C}_i\boldsymbol{A}_i - [\boldsymbol{S}\ \boldsymbol{0}], \\ \boldsymbol{C}_i\boldsymbol{B}_i\boldsymbol{K}_i &= \boldsymbol{P}. \end{aligned} \tag{18}$$

If $\operatorname{rank}(\boldsymbol{C}_i\boldsymbol{B}_i) \neq p$ then $\boldsymbol{C}_i\boldsymbol{B}_i$ is not invertible. However, a solution is still available if the following more general conditions hold. First, note that $\boldsymbol{C}_i\boldsymbol{B}_i = \boldsymbol{B}_{i,1} \in \mathbb{R}^{p\times m_i}$ and $\boldsymbol{C}_i\boldsymbol{A}_i = [\boldsymbol{A}_{i,11}\ \ \boldsymbol{A}_{i,12}]$, with $\boldsymbol{A}_{i,11} \in \mathbb{R}^{p\times p}$ and $\boldsymbol{A}_{i,12} \in \mathbb{R}^{p\times n_i - p}$, since we have assumed that $\boldsymbol{C}_i = [\boldsymbol{I}_p\ \ \boldsymbol{0}]$. Then, it is clearly evident that the equations (18) are solvable for $\boldsymbol{F}_i$ and $\boldsymbol{K}_i$ if

$$\begin{aligned} \operatorname{rank}(\boldsymbol{B}_{i,1}) &= \operatorname{rank}\big([\boldsymbol{B}_{i,1}\ \ \boldsymbol{A}_{i,11} - \boldsymbol{S}\ \ \boldsymbol{A}_{i,12}]\big), \\ \operatorname{rank}(\boldsymbol{B}_{i,1}) &= \operatorname{rank}\big([\boldsymbol{B}_{i,1}\ \boldsymbol{P}]\big). \end{aligned} \tag{19}$$

Since we have the opportunity to influence the matrices $\boldsymbol{S}$ and $\boldsymbol{P}$, a solution can be found if (19) is met. Suppose for example a multi-agent system with a leader follower structure, where for the leader agent k Assumption 3 is not fulfilled. Then we can choose $\boldsymbol{S} = \boldsymbol{A}_{k,11}$ resulting in the reduced rank condition

$$\operatorname{rank}(\boldsymbol{B}_{k,1}) = \operatorname{rank}\big([\boldsymbol{B}_{k,1}\ \ \boldsymbol{A}_{k,12}]\big), \tag{20}$$

because for the leader we can use $\boldsymbol{K}_k = \boldsymbol{0}$. So, synchronization to agent k is still possible if condition (20) holds.

5. EXAMPLE

To illustrate the efficiency of the given approach we consider a multi-agent system consisting of $N = 3$ different kinds of aircraft. The communication topology is given as in Fig. 2, which describes a connected digraph.

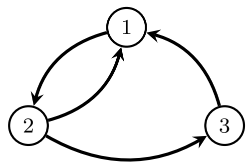

Fig. 2. Communication network for the three aircraft.

Agent 1 is a Grumman X-29A forward swept wing (FSW), Agent 2 is a McDonnell Douglas F/A-18/HARV and Agent 3 is a PUMA XW 241 helicopter, whose linear dynamics are taken from Bosworth [1992], Shewchun and Feron [1999] and Padfield [2007]. The linearized system matrices describe the lateral dynamics of the agents in cruise flight:

$$\boldsymbol{A}_1 = \begin{bmatrix} -2.590 & 0.9970 & -16.55 & 0 \\ -0.1023 & -0.0673 & 6.779 & 0 \\ -0.0603 & -0.9928 & -0.1645 & 0.04413 \\ 1 & 0.07168 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B}_1 = \begin{bmatrix} 1.347 & 0.2365 \\ 0.09194 & -0.07056 \\ -0.0006141 & 0.0006866 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{A}_2 = \begin{bmatrix} -2.3142 & 0.5305 & -15.5763 & 0 \\ -0.016 & -0.1287 & 3.0081 & 0 \\ 0.049 & 0.998 & -0.1703 & 0.044 \\ 1 & 0.0491 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B}_2 = \begin{bmatrix} 23.3987 & 21.4133 & 3.2993 \\ -0.1644 & 0.3313 & -1.9836 \\ -0.0069 & -0.0153 & 0.038 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{A}_3 = \begin{bmatrix} -1.6119 & 0.0713 & -0.0491 \\ -0.1361 & -0.2850 & 0.0249 \\ -0.6983 & 0.1415 & -0.0374 \end{bmatrix},$$

$$\boldsymbol{B}_3 = \begin{bmatrix} 23.1286 & 1.9123 \\ 2.7198 & -7.6343 \\ 9.7042 & 3.8463 \end{bmatrix}.$$

Note the different state and input dimensions of the agents. Because Agents 1 and 3 have only two actuators, at most two states can be synchronized. We have re-arranged the states of the systems such that the first two states of every agent describe the roll (p_i) and yaw rate (r_i). Our aim is to synchronize these two states leading to output matrices

$$\boldsymbol{C}_i = [\boldsymbol{I}_2\ \ \boldsymbol{0}], \qquad \forall i \in \{1, 2, 3\}.$$

Assumptions 1 and 3 are fulfilled for every agent, but Assumption 2 is not satisfied for Agent 1 and 2, due to the fact that they have an invariant zero in $s = 0$. However, as discussed in Remark 4 the internal dynamics will be marginally stable, so synchronization of the roll and yaw rates is still possible.

In order to synchronize the states to a feasible trajectory, we use a harmonic oscillator model

$$\boldsymbol{S} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}.$$

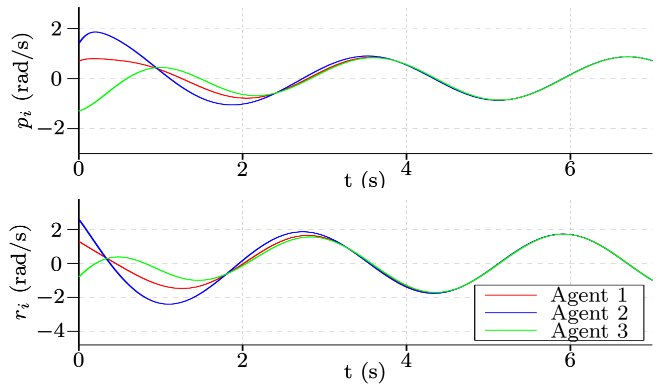

Fig. 3. Synchronization of the roll and yaw rates for the three aircraft.

Applying the described control structure results in the behavior shown in Fig. 3, which demonstrates the efficiency of the approach. Obviously, synchronization of the roll and yaw rates is achieved successfully. Due to observer errors at the beginning, the outputs diverge firstly, but synchronization is achieved after approximately 3 seconds.

6. CONCLUSION

In this paper, a distributed control scheme for synchronizing heterogeneous linear multi-agent systems is proposed. The main idea is to set up a control law which consists of two parts. First, absolute information is used to homogenize the output dynamics of every agent to a specifiable system, which describes the synchronization trajectory. Based on relative information, the second part of the control law is determined such that synchronization for these homogenized systems is achieved. Necessary and sufficient conditions are deduced to synchronize heterogeneous networks of coupled linear systems, with potentially different dimensions, and the approach was successfully tested for a multi-agent system consisting of various types of aircraft. The presented method appears to be very efficient compared to existing results on literature, since the complexity of the control design and the communication load between the agents are reduced to a minimum. Future work includes the relaxation of the assumptions, especially with respect to the rank conditions given in Assumption 3.

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