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Approximate Predictive Control of Polytopic Systems ⋆

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Abstract: Robust model predictive control algorithms often suffer from a high computational complexity due to the large number of variables and constraints involved in the optimiztion problems that are solved at every sampling instant. In this paper we propose an approximate control scheme for polytopic systems based on the interpolation of offline computed robust invariant tubes. The feasible set of the control scheme is the convex hull of all the sets in the invariant tubes. Online, the current control input is computed by interpolating between the control laws associated with these tubes. This interpolation requires the solution of an optimization problem. Compared with a direct solution of a robust model predictive control problem, the interpolation approach proposed in this paper requires much less computation time.

1. INTRODUCTION

The basic idea in model predictive control (MPC) is to include a simulation model in the controller that maps a sequence of predicted inputs onto a sequence of predicted states. At every sampling instant an optimization problem is solved with the predicted inputs as decision variables. Performance criteria and constraints on the inputs or states can be included explicitly in the optimization problem. This feature makes MPC very attractive for control applications, as it is one of the few control methods able to handle hard constraints on the inputs and states. For an overview of MPC, see for example Rawlings and Mayne [2009].

If uncertainty is present in the system to be controlled, the predictions of the system states become set-valued. Predicting the exact worst case evolutions of the system can result in an exponentially growing complexity in the prediction horizon, see for example Langson et al. [2004] and the references therein. In this paper, we restrict our considerations to uncertain linear time-varying systems, where the system matrices are contained in a polytope. In the past, multiple approaches have been proposed where the worst case evolution of the system is overapproximated by sets of fixed complexity, allowing a trade-off between the computational effort and the conservativeness of the predictions, thereby obtaining tractable optimization problems. A notable example is Kothare et al. [1996], where the uncertain predicted states of the system are contained in ellipsoids centered at the origin. An approach based on general ellipsoidal predictions was proposed in Brooms et al. [2001]. Robust MPC schemes based on polytopes instead of ellipsoids were for example proposed in Lee and Kouvaritakis [2000] and Langson et al. [2004].

Common to these robust control schemes is that they, implicitly or explicitly, define at every sampling instant a "tube" in the state space, which is a sequence of sets $(\mathcal{X}_0, \mathcal{X}_1, \mathcal{X}_2, \ldots)$ with an associated sequence of control laws $(\kappa_0, \kappa_1, \kappa_2, \ldots)$. For any given state x in a set $\mathcal{X}_i$ in the tube, the control law $\kappa_i(x)$ ensures that *for all realizations of the uncertainty* the state at the next sampling instant will be contained in the set $\mathcal{X}_{i+1}$. This definition of robust invariant tubes was proposed in Langson et al. [2004]. While ensuring constraint satisfaction and stability under the worst case uncertainty, these MPC schemes often still suffer from a high computational effort due to a large number of variables in the optimization problems. One way to decrease computation time is to solve optimization problems parametrically offline as proposed in Besselmann et al. [2012a], where the optimal controller is obtained by a dynamic programming iteration.

In this paper, we propose an approach to decrease the computational effort based on the ideas presented in Brunner et al. [2013] for systems without uncertainty. The main observation made in that paper was that given a trajectory satisfying convex constraints on the inputs and states and terminating in an invariant set, then the convex hull of the invariant set and the states on this trajectory is control invariant. Moreover, a stabilizing controller and a Lyapunov function can be defined on this convex hull. Extending these results to the uncertain case is the main objective of this paper. Specifically, it will be shown that the convex hull of sets $\mathcal{X}_i$ in one (or multiple) robust invariant tubes is robust control invariant. A robustly stabilizing controller on the convex hull is obtained by interpolating between the control laws κ_i online depending on the current system state. The multipliers defining the interpolation are the solution of an

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optimization problem which is far less complex than the optimization problem in a direct tube MPC approach. The main reason for this is that the uncertainty description is not included explicitly in the optimization problem yielding the multipliers. Robustness is induced by the robust invariance of the tubes which are computed offline. If the tubes between which the interpolation takes place are computed by solving a tube MPC problem, then the interpolated control law can be seen as an approximation of the control law obtained by solving the MPC problem directly. A similar approach based on the interpolation between offline computed tubes has for example been proposed in Lee and Kouvaritakis [2002] for a specific parameterization of the sets $\mathcal{X}_i$ and controllers κ_i. In Bacic et al. [2003], Pluymers et al. [2005], and Ding and Rossiter [2007] large invariant sets for MPC are constructed by taking the convex hull of multiple invariant sets centered on the origin. See also the references therein. A recent robustly stabilizing control scheme based on interpolation was proposed in Nguyen et al. [2013].

The remainder of the paper is structured as follows. The problem setup is given in Section 2. In Section 3 a tube MPC scheme based on Langson et al. [2004] is described which yields the robust invariant tubes necessary for our control method. Section 4 contains our main results, that is the definition of the optimization problem that is solved online in order to obtain the control input. In Section 5 it is shown how under certain conditions this optimization problem can be formulated as a linear program. In Section 6 several ways to simplify the optimization problem are described. Section 7 contains a discussion of the complexity of the optimization problem when compared to a direct tube MPC approach. Further, some possible extensions are described. Section 8 contains a simulation example and Section 9 concludes the paper.

Notation: Given natural numbers $a, b \in \mathbb{N}$ with $a \leq b$, the sets $\{k \in \mathbb{N} \mid a \leq k\}$ and $\{k \in \mathbb{N} \mid a \leq k \leq b\}$ are denoted by $\mathbb{N}_{\geq a}$ and $\mathbb{N}_{[a,b]}$, respectively. The set of nonnegative real numbers is denoted by $\mathbb{R}_{\geq 0}$. For any $n \in \mathbb{N}$, a compact and convex subset of $\mathbb{R}^n$ containing the origin is called a C-set. A C-set is called *proper* if it contains the origin in its (nonempty) interior. Given a set $\mathcal{Z} \subseteq \mathbb{R}^n$ and a scalar $a \in \mathbb{R}$, define $a\mathcal{Z} := \{x \in \mathbb{R}^n \mid \exists z \in \mathcal{Z}: \ x = az\}$. Given sets $\mathcal{Y}, \mathcal{Z} \subseteq \mathbb{R}^n$, define the Minkowski set addition by $\mathcal{Y} \oplus \mathcal{Z} := \{x \in \mathbb{R}^n \mid \exists y \in \mathcal{Y}, \ \exists z \in \mathcal{Z}: \ x = y + z\}$. Given a set $\mathcal{Z} \subset \mathbb{R}^n$ and a vector $v \in \mathbb{R}^n$, define further $v \oplus \mathcal{Z} := \mathcal{Z} \oplus v := \mathcal{Z} \oplus \{v\}$. Given a set $\mathcal{Z} \subseteq \mathbb{R}^n$, the convex hull of $\mathcal{Z}$ is denoted by $\operatorname{convh}(\mathcal{Z})$.

2. PROBLEM SETUP AND PRELIMINARIES

We consider uncertain linear time-varying systems of the form

$$x^+ = \Phi(x, u, \theta) \tag{1}$$

where for all $x \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$, and all $\theta \in \mathbb{R}^r$ it holds that

$$\Phi(x, u, \theta) = \sum_{i=1}^{r} \theta_i (A_i x + B_i u) \tag{2}$$

for given matrices A_i, B_i. The parameter θ is not measurable, may change at any time step, but known to satisfy $\theta \in \Theta$, where $\Theta = \{\theta \in \mathbb{R}^r \mid \forall i \in \mathbb{N}_{[1,r]}: \ \theta_i \geq 0, \ \sum_{i=1}^{r} \theta_i = 1\}$.

Remark 1. For any $x, y \in \mathbb{R}^n$, any $u, v \in \mathbb{R}^m$, any $a, b \in \mathbb{R}$, and any $\theta \in \mathbb{R}^r$ it holds that $\Phi(ax + by, au + bv, \theta) = a\Phi(x, u, \theta) + b\Phi(y, v, \theta)$.

Further, define with slight abuse of notation

$$\Phi(x, u) := \operatorname{convh}\left(\bigcup_{i=1}^{r} \{A_i x + B_i u\} \right),$$

such that for all $x \in \mathbb{R}^n$, all $u \in \mathbb{R}^m$ and all $\theta \in \Theta$ it holds that $x^+ \in \Phi(x, u)$. The goal is to stabilize the origin of (1), while satisfying the mixed constraints $(x^\mathsf{T}, u^\mathsf{T})^\mathsf{T} \in \mathbb{Y}$ where $\mathbb{Y} \subset \mathbb{R}^n \times \mathbb{R}^m$ is a proper C-set. Additionally, given a stage cost function ℓ and an initial state x_0, we are interested in minimizing the worst case infinite horizon cost function

$$V_{\sup}(x_0, \kappa) = \sup_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \sum_{k=0}^{\infty} \ell(x_k, u_k),$$

where $\kappa : \mathbb{R}^n \to \mathbb{R}^m$ is the chosen control law which is to be optimized, $\boldsymbol{\theta} = (\theta^0, \theta^1, \ldots)$ is the sequence of future realizations of the uncertainty, further $\boldsymbol{\Theta} = \Theta \times \Theta \times \cdots$, $u_k = \kappa(x_k)$, and $x_{k+1} = \phi(x_k, u_k, \theta^k)$ for all $k \in \mathbb{N}$. The control scheme in this note requires a proper C-set $\mathcal{X}_\mathrm{f}$, called the terminal set, a terminal control law $\kappa_\mathrm{f} : \mathcal{X}_\mathrm{f} \to \mathbb{R}^m$ and a terminal cost function $V_\mathrm{f} : \mathcal{X}_\mathrm{f} \to \mathbb{R}$ satisfying the following standing assumptions.

Assumption 2. The functions ℓ and V_f are positive definite and convex.

Assumption 3. For all $x \in \mathcal{X}_\mathrm{f}$ it holds that $(x^\mathsf{T}, \kappa_\mathrm{f}(x)^\mathsf{T})^\mathsf{T} \in \mathbb{Y}$. Furthermore, for all $x \in \mathcal{X}_\mathrm{f}$ it holds that $\Phi(x, \kappa_\mathrm{f}(x)) \subseteq X_\mathrm{f}$ and $V_\mathrm{f}(x^+) \leq V_\mathrm{f}(x) - \ell(x, \kappa_\mathrm{f}(x))$ for all $x^+ \in \Phi(x, \kappa_\mathrm{f}(x))$.

Definition 4. (Compare Langson et al. [2004]). A robust invariant tube $\mathbb{T}_N$ is given by $\mathbb{T}_N = \big((\mathcal{X}_0, \ldots, \mathcal{X}_{N-1}), (\kappa_0, \ldots, \kappa_{N-1})\big)$, where for all $i \in \mathbb{N}_{[0,N-1]}$ it holds that $\mathcal{X}_i \subseteq \mathbb{R}^n$ and $\kappa_i : \mathcal{X}_i \to \mathbb{R}^m$. For all $i \in \mathbb{N}_{[0,N-1]}$ and any $x \in \mathcal{X}_i$ it holds that $(x^\mathsf{T}, \kappa_i(x)^\mathsf{T})^\mathsf{T} \in \mathbb{Y}$. Further, for all $i \in \mathbb{N}_{[0,N-2]}$ and any $x \in \mathcal{X}_i$ it holds that $\Phi(x, \kappa_i(x)) \subseteq \mathcal{X}_{i+1}$. Finally, for any $x \in \mathcal{X}_{N-1}$ it holds that $\Phi(x, \kappa_{N-1}(x)) \subseteq \mathcal{X}_\mathrm{f}$.

3. TUBE MODEL PREDICTIVE CONTROL

In this section, we give an example of how robust invariant tubes can be constructed. The parameterization and the optimization problem have been adapted from Langson et al. [2004]. The sets $\mathcal{X}_i$ are polytopes defined by $\mathcal{X}_i = \operatorname{convh}\left(\bigcup_{j=1}^{q_i} \{v_{ij}\}\right)$ for $v_{ij} \in \mathbb{R}^n$. The controllers κ_i are parameterized as vertex controllers. Given a set $\mathcal{X}_i = \operatorname{convh}\left(\bigcup_{j=1}^{q_i} \{v_{ij}\}\right)$ and associated inputs $u_{ij} \in \mathbb{R}^m$, for any $x \in \mathcal{X}_i$ the control input $\kappa_i(x)$ is defined by the optimization problem

$$(\lambda_1^\star, \ldots, \lambda_{q_i}^\star) = \underset{\lambda_1, \ldots, \lambda_{q_i}}{\operatorname{argmin}} \ \ell\Big(x, \sum_{j=1}^{q_i} \lambda_j u_{ij}\Big) \tag{3a}$$

$$\text{s.t.} \quad \forall j \in \mathbb{N}_{[1,q_i]}: \ \lambda_j \geq 0 \tag{3b}$$

$$\sum_{j=1}^{q_i} \lambda_j = 1, \qquad \sum_{j=1}^{q_i} \lambda_j v_{ij} = x \tag{3c}$$

$$\kappa_i(x) = \sum_{j=1}^{q_i} \lambda_j^\star u_{ij}. \tag{3d}$$

A particular parameterization of the sets $\mathcal{X}_i$ is $\mathcal{X}_i = z_i \oplus a_i\mathcal{X}$ for a fixed polytope $\mathcal{X} = \operatorname{convh}\left(\bigcup_{j=1}^{q}\{v_j\}\right)$ where $z_i, v_j \in \mathbb{R}^n$ and $a_i \in \mathbb{R}_{\geq 0}$. With this parameterization, the vertices v_{ij} are given by $v_{ij} = z_i + a_i v_j$. Furthermore, a polytopic terminal set $\mathcal{X}_\mathrm{f}$ is assumed to be given. Summarizing, for fixed $\mathcal{X}, \mathcal{X}_\mathrm{f}$ the parameters of the tube are a_i, z_i, and u_{ij}. In order to satisfy Definition 4, these parameters are defined as the solution of an optimization problem for a given $x \in \mathbb{R}^n$. Let $d_N = (z_1, \ldots, z_{N-1}, a_1, \ldots, a_{N-1}, u_{01}, \ldots, u_{0q}, \ldots, u_{N-11}, \ldots, u_{N-1q})$ denote the optimization variables. The constraints are given by

$$x \in z_0 \oplus a_0\mathcal{X} \tag{4a}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}\colon\ a_i \geq 0 \tag{4b}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}\colon\ \forall j \in \mathbb{N}_{[1,q]}\colon\quad (z_i + a_i v_{ij}, u_{ij}) \in \mathbb{Y} \tag{4c}$$

$$\forall l \in \mathbb{N}_{[1,r]}\colon\ \forall i \in \mathbb{N}_{[0,N-2]}\colon\ \forall j \in \mathbb{N}_{[1,q]}\colon\quad A_l(z_i + a_i v_{ij}) + B_l u_{ij} \in z_{i+1} \oplus a_{i+1}\mathcal{X} \tag{4d}$$

$$\forall l \in \mathbb{N}_{[1,r]}\colon\ \forall j \in \mathbb{N}_{[1,q]}\colon\quad A_l(z_{N-1} + a_{N-1}v_{ij}) + B_l u_{N-1j} \in \mathcal{X}_\mathrm{f}. \tag{4e}$$

The cost function of the optimization problem is defined by

$$J_N^{\mathrm{TMPC}}(d_N) = \sum_{i=0}^{N-1} \max_{j \in \mathbb{N}_{[1,q]}} \ell(z_i + a_i v_{ij}, u_{ij}) + \max_{\substack{j \in \mathbb{N}_{[1,q]} \\ l \in \mathbb{N}_{[1,r]}}} V_\mathrm{f}(A_l(z_{N-1j} + a_{N-1}v_{N-1j}) + B_l u_{N-1j})$$

Finally, the optimization problem is given by

$$\begin{aligned} d_N^\star(x) = \operatorname*{argmin}_{d_N} J_N^{\mathrm{TMPC}}(d_N) \\ \text{s.t. (4a) to (4e).} \end{aligned} \tag{5}$$

4. APPROXIMATE PREDICTIVE CONTROL

This section contains our main results. That is, it is shown how a stabilizing controller can be defined on the convex hull of a robust invariant tube. For the sake of simplicity, only a single tube is used in the construction of the controller in this section. The extension to multiple tubes is obvious and is briefly discussed in Section 7. Note that the results in this section are not dependent on the parameterization of the sets $\mathcal{X}_i$ and the controllers κ_i. In particular, it is not necessary that the sets and controllers are parameterized as in Section 3.

Given an invariant tube $\mathbb{T}_N$ where all $\mathcal{X}_i$ are convex and compact sets, define the set

$$\mathcal{X}_\mathrm{E} := \operatorname{convh}\left(\bigcup_{i=0}^{N-1} \mathcal{X}_i \cup \mathcal{X}_\mathrm{f}\right).$$

By the definition of the convex hull of a set as the union of all convex combinations in the set it holds that for any $x \in \mathcal{X}_\mathrm{E}$ there exists $\hat{x}_i \in \mathcal{X}_i$, $\hat{x}_\mathrm{f} \in \mathcal{X}_\mathrm{f}$, and scalars $\rho_i, \rho_\mathrm{f} \geq 0$ with $\sum_{i=1}^{N-1}\rho_i + \rho_\mathrm{f} = 1$, such that $x = \sum_{i=0}^{N-1}\rho_i\hat{x}_i + \rho_\mathrm{f}\hat{x}_\mathrm{f}$. Hence it holds that $\mathcal{X}_\mathrm{E} = \{x \in \mathbb{R}^n \mid \Gamma(x) \neq \emptyset\}$, where

$$\Gamma(x) := \Bigg\{(\rho_0, \ldots, \rho_{N-1}, \rho_\mathrm{f}, \hat{x}_0, \ldots, \hat{x}_{N-1}, \hat{x}_\mathrm{f}) \in \mathbb{R} \times \ldots \times \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \;\Bigg|\; \forall i \in \mathbb{N}_{[0,N-1]}\colon\ \rho_i \geq 0,\ \rho_\mathrm{f} \geq 0,\ \sum_{i=0}^{N-1}\rho_i + \rho_\mathrm{f} = 1, \forall i \in \mathbb{N}_{[0,N-1]}\colon\ \hat{x}_i \in \mathcal{X}_i,\ \hat{x}_\mathrm{f} \in \mathcal{X}_\mathrm{f},\ x = \sum_{i=0}^{N-1}\rho_i\hat{x}_i + \rho_\mathrm{f}\hat{x}_\mathrm{f}\Bigg\}.$$

For any $x \in \mathcal{X}_\mathrm{E}$ and any $p \in \Gamma(x)$ define the control law

$$\kappa_\mathrm{E}^\mathrm{p}(p) := \sum_{i=0}^{N-1}\rho_i\kappa_i(\hat{x}_i) + \rho_\mathrm{f}\kappa_\mathrm{f}(\hat{x}_\mathrm{f}), \tag{6}$$

where $p = (\rho_0, \ldots, \rho_{N-1}, \rho_\mathrm{f}, \hat{x}_0, \ldots, \hat{x}_{N-1}, \hat{x}_\mathrm{f})$. Further, for all $i \in \mathbb{N}_{[0,N-1]}$ define functions $J_i : \mathcal{X}_i \to \mathbb{R}$. The following assumption holds throughout the remainder of the section.

Assumption 5. For all $i \in \mathbb{N}_{[0,N-2]}$, all $x \in \mathcal{X}_i$ and all $x^+ \in \Phi(x, \kappa_i(x))$ it holds that

$$J_{i+1}(x^+) - J_i(x) \leq -\ell(x, \kappa_i(x)).$$

For all $x \in \mathcal{X}_{N-1}$ and all $x^+ \in \Phi(x, \kappa_{N-1}(x))$ it holds that

$$V_\mathrm{f}(x^+) - J_{N-1}(x) \leq -\ell(x, \kappa_{N-1}(x)).$$

Remark 6. Multiple ways of defining functions J_i satisfying Assumption 5 will be discussed in Section 5 and Section 6.

Given now any $x \in \mathcal{X}_\mathrm{E}$ and any $p \in \Gamma(x)$, define the cost function

$$V_\mathrm{E}^\mathrm{p}(p) := \sum_{i=0}^{N-1}\rho_i J_i(\hat{x}_i) + \rho_\mathrm{f}V_\mathrm{f}(\hat{x}_\mathrm{f}).$$

The next lemma establishes invariance of the set $\mathcal{X}_\mathrm{E}$ for any parameter p in the control law $\kappa_\mathrm{E}^\mathrm{p}(p)$. Further, it is shown that there always exists a choice of the parameter p, such that the cost function $V_\mathrm{E}^\mathrm{p}(p)$ decreases along the trajectories of the closed-loop system.

Lemma 7. Given any $x \in \mathcal{X}_\mathrm{E}$ and any $p \in \Gamma(x)$, then for any $x^+ \in \Phi(x, \kappa^\mathrm{p}(p))$ there exists a $p^+ \in \Gamma(x^+)$, such that $V_\mathrm{E}^\mathrm{p}(p^+) - V_\mathrm{E}^\mathrm{p}(p) \leq -\ell(x, \kappa_\mathrm{E}^\mathrm{p}(p))$.

Proof. Let $x^+ = \Phi(x, \kappa^\mathrm{p}(p), \theta)$ for a given θ. Let $p \in \Gamma(x)$ with $p = (\rho_0, \ldots, \rho_{N-1}, \rho_\mathrm{f}, \hat{x}_0, \ldots, \hat{x}_{N-1}, \hat{x}_\mathrm{f})$ and define $p^+ := (\rho_0^+, \ldots, \rho_{N-1}^+, \rho_\mathrm{f}^+, \hat{x}_0^+, \ldots, \hat{x}_{N-1}^+, \hat{x}_\mathrm{f}^+)$. In particular, let $\rho_0^+ = 0$, $\rho_i^+ = \rho_{i-1}$ for all $i \in \mathbb{N}_{[1,N-1]}$ and $\rho_\mathrm{f}^+ = \rho_{N-1} + \rho_\mathrm{f}$. It obviously holds that $\rho_i^+ \geq 0$ for all $i \in \mathbb{N}_{[0,N-1]}$, $\rho_\mathrm{f}^+ \geq 0$, and $\sum_{i=0}^{N-1}\rho_i^+ + \rho_\mathrm{f}^+ = 1$. Let further $\hat{x}_0^+$ be any state in $\mathcal{X}_0$ and for all $i \in \mathbb{N}_{[1,N-1]}$ let $\hat{x}_i^+ = \Phi(\hat{x}_{i-1}, \kappa_{i-1}(\hat{x}_{i-1}), \theta)$. By the definition of $\mathbb{T}_N$ it holds that $\hat{x}_i^+ \in \mathcal{X}_i$ for all $i \in \mathbb{N}_{[0,N-1]}$. Further, if $\rho_{N-1} + \rho_\mathrm{f} = 0$ define $\hat{x}_\mathrm{f}^+ = 0$, which implies $\hat{x}_\mathrm{f}^+ \in \mathcal{X}_\mathrm{f}$. Otherwise, define

$$\hat{x}_\mathrm{f}^+ = \frac{\rho_{N-1}}{\rho_{N-1} + \rho_\mathrm{f}}\Phi(\hat{x}_{N-1}, \kappa_{N-1}(\hat{x}_{N-1}), \theta) + \frac{\rho_\mathrm{f}}{\rho_{N-1} + \rho_\mathrm{f}}\Phi(\hat{x}_\mathrm{f}, \kappa_\mathrm{f}(\hat{x}_\mathrm{f}), \theta). \tag{7}$$

By the definition of $\mathbb{T}_N$ and by Assumption 3 it holds that $\Phi(\hat{x}_{N-1}, \kappa_{N-1}(\hat{x}_{N-1}), \theta) \in \mathcal{X}_\mathrm{f}$ and $\Phi(\hat{x}_\mathrm{f}, \kappa_\mathrm{f}(\hat{x}_\mathrm{f}), \theta) \in$

$\mathcal{X}_\mathrm{f}$, such that $\hat{x}_\mathrm{f}^+$ in (7) is a convex combination of states in $\mathcal{X}_\mathrm{f}$ and hence satisfies $\hat{x}_\mathrm{f}^+ \in \mathcal{X}_\mathrm{f}$. Further, it holds that

$$\begin{aligned} x^+ &= \Phi\left(\sum_{i=0}^{N-1} \rho_i \hat{x}_i + \rho_\mathrm{f}\hat{x}_\mathrm{f}, \sum_{i=0}^{N-1} \rho_i \kappa_i(\hat{x}_i) + \rho_\mathrm{f}\kappa_\mathrm{f}(\hat{x}_\mathrm{f}), \theta\right) \\ &= \sum_{i=0}^{N-1} \rho_i \Phi(\hat{x}_i, \kappa_i(\hat{x}_i), \theta) + \rho_\mathrm{f}\Phi(\hat{x}_\mathrm{f}, \kappa_\mathrm{f}(\hat{x}_\mathrm{f}), \theta) \\ &= \sum_{i=0}^{N-2} \rho_i \hat{x}_{i+1}^+ + \rho_{N-1}\Phi(\hat{x}_{N-1}, \kappa_{N-1}(\hat{x}_{N-1}), \theta) \\ &\qquad\qquad + \rho_\mathrm{f}\Phi(\hat{x}_\mathrm{f}, \kappa_\mathrm{f}(\hat{x}_\mathrm{f}), \theta) \\ &= \sum_{i=1}^{N-1} \rho_{i-1}\hat{x}_i^+ + \rho_\mathrm{f}^+\hat{x}_\mathrm{f}^+ = \sum_{i=0}^{N-1} \rho_i^+ \hat{x}_i^+ + \rho_\mathrm{f}^+ \hat{x}_\mathrm{f}^+ . \end{aligned}$$

Considering all of the above, it holds that $p^+ \in \Gamma(x^+)$ and hence $x^+ \in \mathcal{X}_\mathrm{f}$. Due to lack of space, only a sketch of the proof for the decrease of the cost function is given here. It holds that

$$\begin{aligned} V_\mathrm{E}^\mathrm{p}(p^+) - V_\mathrm{E}^\mathrm{p}(p) = \sum_{i=0}^{N-1} \rho_i^+ J_i(\hat{x}_i^+) - \sum_{i=0}^{N-1} \rho_i J_i(\hat{x}_i) \\ + \rho_\mathrm{f}^+ V_\mathrm{f}(\hat{x}_\mathrm{f}^+) - \rho_\mathrm{f} V_\mathrm{f}(\hat{x}_\mathrm{f}). \end{aligned}$$

By plugging in the definition of ρ_i^+, ρ_f^+, $\hat{x}_i^+$, and $\hat{x}_\mathrm{f}^+$, using Assumption 3 and Assumption 5 and the convexity of the functions ℓ and V_f, it follows that

$$\begin{aligned} &V_\mathrm{E}^\mathrm{p}(p^+) - V_\mathrm{E}^\mathrm{p}(p) \\ &\quad\le -\sum_{i=0}^{N-1} \rho_i \ell(\hat{x}_i, \kappa_i(\hat{x}_i)) - \rho_\mathrm{f}\ell(\hat{x}_\mathrm{f}, \kappa_\mathrm{f}(\hat{x}_\mathrm{f})) \\ &\quad\le -\ell(x, \kappa_\mathrm{E}^\mathrm{p}(p)), \end{aligned}$$

completing the proof. □

Lemma 7 suggests a control algorithm based on the following optimization problem. Given an invariant tube $\mathbb{T}_N$, associated functions J_i, and any state $x \in \mathcal{X}_\mathrm{E}$, problem $\mathbb{P}_\mathrm{E}(x)$ is defined by

$$V_\mathrm{E}(x) := \min_{p\in\Gamma(x)} V_\mathrm{E}^\mathrm{p}(p)$$
$$p^0(x) := \operatorname*{argmin}_{p\in\Gamma(x)} V_\mathrm{E}^\mathrm{p}(p)$$

The controller resulting from the solution of $\mathbb{P}_\mathrm{E}(x)$ is defined as $\kappa_\mathrm{E}(x) := \kappa_\mathrm{E}^\mathrm{p}(p^0(x))$. The closed-loop system for this control law is

$$x^+ = \Phi(x, \kappa_\mathrm{E}(x), \theta). \tag{9}$$

Theorem 8. The origin of (9) is asymptotically stable with a region of attraction $\mathcal{X}_\mathrm{E}$. Furthermore, for all $x \in \mathcal{X}_\mathrm{E}$ it holds that $(x^\mathsf{T}, \kappa_\mathrm{E}(x)^\mathsf{T})^\mathsf{T} \in \mathbb{Y}$. Finally, if $x_0 \in \mathcal{X}_E$ is an arbitrary initial state of the system, it holds that

$$V_\mathrm{sup}(x_0, \kappa_\mathrm{E}) \le V_\mathrm{E}(x_0). \tag{10}$$

Proof. By Lemma 7, for all $x \in \mathcal{X}_\mathrm{f}$ and all $x^+ \in \Phi(x, \kappa_\mathrm{E}(x))$ it holds that

$$V_\mathrm{E}(x^+) \le V_\mathrm{E}(x) - \ell(x, \kappa_\mathrm{E}(x)). \tag{11}$$

Positive definiteness of ℓ and V_f imply positive definiteness of V_E. Together with (11) this implies asymptotic stability of the origin. As the region where $\mathbb{P}_\mathrm{E}(x)$ is feasible is exactly $\mathcal{X}_\mathrm{E}$, this set is at the same time the region of attraction of the origin. Further, the definition of the controller in (6) implies by the definition of $\mathbb{T}_N$ that for all $x \in \mathcal{X}_\mathrm{E}$ the vector $(x^\mathsf{T}, \kappa_\mathrm{E}(x)^\mathsf{T})^\mathsf{T}$ is a convex combination of points in $\mathbb{Y}$ and is hence contained in $\mathbb{Y}$. Finally, summing up inequality (11) from 0 to ∞ yields the performance bound (10), as (11) holds for any $x^+ \in \Phi(x, \kappa_\mathrm{E}(x))$. This completes the proof. □

5. LINEAR PROGRAMMING SOLUTION

In this section we describe how problem $\mathbb{P}_\mathrm{E}(x)$ can be solved by linear programming for a certain choice of the sets $\mathcal{X}_i$, the controllers κ_i and the functions ℓ and V_f. Let the following additional assumption hold throughout this section.

Assumption 9. The sets $\mathcal{X}_i$ and the set $\mathcal{X}_\mathrm{f}$ are polytopes in $\mathbb{R}^n$. For all $i \in \mathbb{N}_{[0,N-1]}$ the functions $\ell(x, \kappa_i(x))$ are piecewise affine and convex functions. Likewise, V_f is piecewise affine.

The functions J_i are defined in the following iterative way in order to satisfy Assumption 5. By Assumption 9, these functions are convex.

$$\begin{aligned} &\forall x \in \mathcal{X}_{N-1}: \\ &J_{N-1}(x) := \ell(x, \kappa_{N-1}(x)) + \max_{\substack{z\in\mathcal{X}_{N-1} \\ z^+\in\Phi(z,\kappa_{N-1}(z))}} V_\mathrm{f}(z^+) \end{aligned} \tag{12a}$$

$$\begin{aligned} &\forall i \in \mathbb{N}_{[0,N-2]}:\ \forall x \in \mathcal{X}_i: \\ &J_i(x) := \ell(x, \kappa_i(x)) + \max_{\substack{z\in\mathcal{X}_i \\ z^+\in\Phi(z,\kappa_i(z))}} J_{i+1}(x^+). \end{aligned} \tag{12b}$$

Remark 10. A less conservative way to define the functions would be

$$\begin{aligned} &\forall x \in \mathcal{X}_{N-1}: \\ &J_{N-1}(x) := \ell(x, \kappa_{N-1}(x)) + \max_{x^+\in\Phi(x,\kappa_{N-1}(x))} V_\mathrm{f}(x^+) \\ &\forall i \in \mathbb{N}_{[0,N-2]}: \forall x \in \mathcal{X}_i: \\ &J_i(x) := \ell(x, \kappa_i(x)) + \max_{x^+\in\Phi(x,\kappa_i(x))} J_{i+1}(x^+). \end{aligned}$$

This definition is also used in Besselmann et al. [2012a] where the controllers κ_i are obtained by dynamic programming. However, we require the functions J_i to be convex and hence restrict ourselves to the more conservative definition in (12).

Remark 11. If the controllers κ_i are defined as in (3) and the stage cost ℓ is piecewise affine, then the functions $\ell(x, \kappa_i(x))$ are convex and piecewise affine by definition. Furthermore, the functions κ_i are piecewise affine in that case.

Assumption 9 implies that there exists matrices $F_{ij}, F_{\mathrm{f}j} \in \mathbb{R}^{1\times n}$ and scalars $g_{ij}, g_{\mathrm{f}j} \in \mathbb{R}$, where $i \in \mathbb{N}_{[0,N-1]}$ and $j \in \mathbb{N}_{[1,\hat{q}_i]}$ and $j \in \mathbb{N}_{[1,\hat{q}_\mathrm{f}]}$, respectively, for some $\hat{q}_i, \hat{q}_\mathrm{f} \in \mathbb{N}$, such that for all $i \in \mathbb{N}_{[0,N-1]}$, all $\hat{x}_i \in \mathcal{X}_i$, and all $\hat{x}_\mathrm{f} \in \mathcal{X}_\mathrm{f}$ it holds that

$$J_i(\hat{x}_i) = \min\{t_i \in \mathbb{R} \mid \forall j \in \mathbb{N}_{[1,\hat{q}_i]}:\ F_{ij}\hat{x}_i + g_{ij} \le t_i\}$$

and

$$V_\mathrm{f}(\hat{x}_\mathrm{f}) = \min\{t_\mathrm{f} \mid \forall j \in \mathbb{N}_{[1,\hat{q}_i]}:\ F_{\mathrm{f}j}\hat{x}_\mathrm{f} + g_{\mathrm{f}j} \le t_\mathrm{f}\}.$$

With these parameterizations problem $\mathbb{P}_\mathrm{E}(x)$ takes the form

$$V_{\mathrm{E}}(x) = \min_{\substack{(t_0,\ldots,t_{N-1},t_{\mathrm{f}},\\ \rho_0,\ldots,\rho_{N-1},\rho_{\mathrm{f}}\\ \hat{x}_0,\ldots,\hat{x}_{N-1},\hat{x}_{\mathrm{f}})}} \sum_{i=0}^{N-1} \rho_i t_i + \rho_{\mathrm{f}} t_{\mathrm{f}} \tag{14a}$$

s. t.

$$\forall i \in \mathbb{N}_{[0,N-1]}: \ \rho_i \geq 0, \ \rho_{\mathrm{f}} \geq 0 \tag{14b}$$

$$\sum_{i=0}^{N-1} \rho_i + \rho_{\mathrm{f}} = 1, \qquad x = \sum_{i=0}^{N-1} \rho_i \hat{x}_i + \rho_{\mathrm{f}} \hat{x}_{\mathrm{f}} \tag{14c}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}: \hat{x}_i \in \mathcal{X}_i, \qquad \hat{x}_{\mathrm{f}} \in \mathcal{X}_{\mathrm{f}} \tag{14d}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}: \forall j \in \mathbb{N}_{[1,\hat{q}_i]}: F_{ij}\hat{x}_i + g_{ij} \leq t_i \tag{14e}$$

$$\forall j \in \mathbb{N}_{[1,\hat{q}_{\mathrm{f}}]}: F_{\mathrm{f}j}\hat{x}_{\mathrm{f}} + g_{\mathrm{f}j} \leq t_{\mathrm{f}}. \tag{14f}$$

Due to the multiplication of optimization variables (that is, for example, ρ_i and t_i) this problem is not a linear program. By multiplying the constraints in the lines (14d) to (14f) with ρ_i and ρ_{f}, respectively, and by substituting $\tilde{x}_i = \rho_i \hat{x}_i$, $\tilde{x}_{\mathrm{f}} = \rho_{\mathrm{f}} \hat{x}_{\mathrm{f}}$, $s_i = \rho_i t_i$ and $s_{\mathrm{f}} = \rho_{\mathrm{f}} t_{\mathrm{f}}$, we obtain the equivalent problem $\mathbb{P}_{\mathrm{E}}^{\mathrm{LP}}(x)$, defined by

$$V_{\mathrm{E}}(x) = \min_{\substack{(s_0,\ldots,s_{N-1},s_{\mathrm{f}},\\ \rho_0,\ldots,\rho_{N-1},\rho_{\mathrm{f}}\\ \tilde{x}_0,\ldots,\tilde{x}_{N-1},\tilde{x}_{\mathrm{f}})}} \sum_{i=0}^{N-1} s_i + s_{\mathrm{f}} \tag{15a}$$

s. t.

$$\forall i \in \mathbb{N}_{[0,N-1]}: \ \rho_i \geq 0, \ \rho_{\mathrm{f}} \geq 0 \tag{15b}$$

$$\sum_{i=0}^{N-1} \rho_i + \rho_{\mathrm{f}} = 1, \qquad x = \sum_{i=0}^{N-1} \tilde{x}_i + \tilde{x}_{\mathrm{f}} \tag{15c}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}: \tilde{x}_i \in \rho_i \mathcal{X}_i, \qquad \tilde{x}_{\mathrm{f}} \in \rho_{\mathrm{f}} \mathcal{X}_{\mathrm{f}} \tag{15d}$$

$$\forall i \in \mathbb{N}_{[0,N-1]}: \forall j \in \mathbb{N}_{[1,\hat{q}_i]}: F_{ij}\tilde{x}_i + \rho_i g_{ij} \leq s_i \tag{15e}$$

$$\forall j \in \mathbb{N}_{[1,\hat{q}_{\mathrm{f}}]}: F_{\mathrm{f}j}\tilde{x}_{\mathrm{f}} + \rho_{\mathrm{f}} g_{\mathrm{f}j} \leq s_{\mathrm{f}} \tag{15f}$$

which is in fact a linear program.

Lemma 12. The problems $\mathbb{P}_{\mathrm{E}}(x)$ and $\mathbb{P}_{\mathrm{E}}^{\mathrm{LP}}(x)$ are equivalent in the following sense. If a solution is feasible for any of the two problems, this implies the existence of a solution to the other problem with the same values of the objective function and the same resulting control input for both problems.

The evaluation of the control law $\kappa_{\mathrm{E}}(x)$ requires values for the variables $\hat{x}_i$. However, problem $\mathbb{P}_{\mathrm{E}}^{\mathrm{LP}}(x)$ only yields the variables $\tilde{x}_i$ as a solution. Consider that if $\rho_i = 0$ for a specific $i \in \mathbb{N}_{[0,N-1]}$ it also holds that $\rho_i \kappa_i(\hat{x}_i) = 0$ for any value of $\hat{x}_i$. Hence, a re-substitution is only necessary for those $i \in \mathbb{N}_{[0,N-1]}$ for which $\rho_i > 0$. In this case, a feasible choice is $\hat{x}_i = \tilde{x}_i / \rho_i$. The same holds true for ρ_{f} and $\tilde{x}_{\mathrm{f}}$.

6. SIMPLIFICATIONS

In this section we describe several modifications to the control algorithm which reduce the complexity of problem $\mathbb{P}_{\mathrm{E}}^{\mathrm{LP}}(x)$ while leading to a more conservative performance bound.

In a first step, the cost function is simplified. That is, all functions J_i are replaced by *constants* $\tilde{J}_i \in \mathbb{R}$ defined by

$$\tilde{J}_{N-1}(x) := \max_{x \in \mathcal{X}_{N-1}} \ell(x, \kappa_{N-1}(x)) + \max_{\substack{z \in \mathcal{X}_{N-1}\\ z^+ \in \Phi(z, \kappa_{N-1}(z))}} V_{\mathrm{f}}(z^+) \tag{16a}$$

$$\forall i \in \mathbb{N}_{[0,N-2]}: \ J_i(x) := \max_{x \in \mathcal{X}_i} \ell(x, \kappa_i(x)) + \tilde{J}_{i+1}. \tag{16b}$$

For all $i \in \mathbb{N}_{[0,N-2]}$ it holds that

$$\tilde{J}_{i+1} - \tilde{J}_i = -\max_{x \in \mathcal{X}_i} \ell(x, \kappa_i(x)).$$

Hence, Assumption 5 still holds, but the bound in (10) will be more conservative. On the other hand, if the linear programming formulation is used, the inequalities in (15e) can be dropped completely and the optimization variables s_i replaced with $\rho_i \tilde{J}_i$.

In order to further simplify the optimization problem, we assume a particular parameterization for the sets $\mathcal{X}_i$. That is, the sets are parameterized by $\mathcal{X}_i = z_i \oplus a_i \mathcal{X}$, where $z_i \in \mathbb{R}^n$ and $a_i \in \mathbb{R}_{\geq 0}$. This kind of parameterization is common in robust MPC, see for example Langson et al. [2004] and Raković et al. [2012].

With these two simplifications, it is possible to aggregate some of the variables in the optimization problem. Specifically, with the substitution

$$\tilde{x} = \sum_{i=0}^{N-1} (\tilde{x}_i - \rho_i z_i)$$

the optimization problem becomes

$$V_{\mathrm{E}}(x) = \min_{\substack{(s_{\mathrm{f}},\rho_0,\ldots,\rho_{N-1},\rho_{\mathrm{f}}\\ \tilde{x},\tilde{x}_{\mathrm{f}})}} \sum_{i=0}^{N-1} \rho_i \tilde{J}_i + s_{\mathrm{f}} \tag{17a}$$

s. t.

$$\forall i \in \mathbb{N}_{[0,N-1]}: \ \rho_i \geq 0, \ \rho_{\mathrm{f}} \geq 0 \tag{17b}$$

$$\sum_{i=0}^{N-1} \rho_i + \rho_{\mathrm{f}} = 1, \qquad x = \sum_{i=0}^{N-1} \rho_i z_i + \tilde{x} + \tilde{x}_{\mathrm{f}} \tag{17c}$$

$$\tilde{x} \in \sum_{i=0}^{N-1} \rho_i a_i \mathcal{X}, \qquad \tilde{x}_{\mathrm{f}} \in \rho_{\mathrm{f}} \mathcal{X}_{\mathrm{f}} \tag{17d}$$

$$\forall j \in \mathbb{N}_{[1,\hat{q}_{\mathrm{f}}]}: F_{\mathrm{f}j}\tilde{x}_{\mathrm{f}} + \rho_{\mathrm{f}} g_{\mathrm{f}j} \leq s_{\mathrm{f}}. \tag{17e}$$

After the optimization problem has been solved, the variables $\hat{x}_i$ can be obtained from $\tilde{x}$ by setting

$$\hat{x}_i = \begin{cases} \dfrac{a_i}{\sum_{j=0}^{N-1} \rho_j a_j} \tilde{x} + z_i & \text{if} \quad \displaystyle\sum_{j=0}^{N-1} \rho_j a_j \neq 0 \\ z_i & \text{else.} \end{cases}$$

Remark 13. This resubstitution formula reveals why it is necessary to simplify the cost function. By making the cost function constant on every set $\mathcal{X}_i$, it becomes irrelevant where $\hat{x}_i$ is chosen in $\mathcal{X}_i$. In fact, by aggregating the variables $\hat{x}_i$ into $\tilde{x}$, any information about the individual sets $\mathcal{X}_i$ is lost.

Alternatively, the variables $\hat{x}_i$ may also be obtained by solving an optimization problem.

7. DISCUSSION AND EXTENSIONS

In this section we discuss the complexity of our approach and consider possible extensions.

7.1 Complexity

We compare our control approach based on solving optimization problem (17) to a direct robust MPC approach based on solving (5) at every sampling instant. We neglect the computational effort involved with evaluating the control laws κ_i.

The complexity of both optimization problems depends on the length N of the tube, on the number of vertices q of the set $\mathcal{X}$, and on the dimension n of the state space. However, the optimization problem (5) additionally depends on the dimension m of the input variables, the complexity of the constraint set $\mathbb{Y}$, and, most importantly, on the number of vertices r in the uncertainty description (2). Neither of the three latter system properties has an influence on the complexity of (17). Hence, the approach in this paper allows the use of a very nonconservative description of the uncertainty, that is, a large r in (2), without increasing the (online) computational effort. Of course, in order to obtain an invariant tube, it is still necessary to solve an optimization problem of type (5) offline.

7.2 Multiple Tubes

The approach in this paper is easily extended to a setup where $\mathcal{X}_\mathrm{E}$ is defined by multiple tubes. Let M robust invariant tubes be given, where the jth tube is of length N_j. In particular, let the tubes be defined by

$$\forall j \in \mathbb{N}_{[1,M]}: \mathbb{T}^j_{N_j} = \big((\mathcal{X}_{0j}, \ldots, \mathcal{X}_{N_j-1j}), (\kappa_{0j}, \ldots, \kappa_{N_j-1j})\big).$$

Then the set $\mathcal{X}_\mathrm{E}$ is defined by

$$\mathcal{X}_\mathrm{E} := \operatorname{convh}\left(\bigcup_{j=1}^{M} \bigcup_{i=0}^{N_j-1} \mathcal{X}_{ij} \cup \mathcal{X}_\mathrm{f}\right)$$

and the set $\Gamma(x)$ takes the form

$$\Gamma(x) := \Bigg\{ (\rho_{01}, \ldots, \rho_{N_1-11}, \ldots, \rho_{0M}, \ldots, \rho_{N_M-1M}, \rho_\mathrm{f}, \hat{x}_{01}, \ldots, \hat{x}_{N_1-11}, \ldots, \hat{x}_{0M}, \ldots, \hat{x}_{N_M-1M}, \hat{x}_\mathrm{f}) \in \mathbb{R} \times \ldots \times \mathbb{R} \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \;\Bigg|\; \forall j \in \mathbb{N}_{[1,M]}: \ \forall i \in \mathbb{N}_{[0,N_j-1]}: \ \rho_{ij} \geq 0, \ \rho_\mathrm{f} \geq 0, \ \forall j \in \mathbb{N}_{[1,M]}: \ \forall i \in \mathbb{N}_{[0,N_j-1]}: \ \hat{x}_{ij} \in \mathcal{X}_{ij}, \ \hat{x}_\mathrm{f} \in \mathcal{X}_\mathrm{f}, \ x = \sum_{j=1}^{M} \sum_{i=0}^{N_j-1} \rho_{ij}\hat{x}_{ij} + \rho_\mathrm{f}\hat{x}_\mathrm{f}, \ \sum_{j=1}^{M} \sum_{i=0}^{N_j-1} \rho_{ij} + \rho_\mathrm{f} = 1 \Bigg\}.$$

The cost function V_E and the controller κ_E can also be defined analogously to the case of one robust invariant tube.

7.3 Iterative Construction

A set $\mathcal{X}_\mathrm{E}$, controller κ_E and cost function V_E defined as in Section 4 satisfy the requirements on the terminal set, controller, and cost function in Assumption 3. This suggests the following iterative procedure.

Algorithm 1 Iterative Controller Construction

1: Obtain an invariant tube for a given terminal set $\mathcal{X}_\mathrm{f}$, terminal cost V_f, and terminal controller κ_f, for example by solving (5).
2: Define the set $\mathcal{X}_\mathrm{E}$, the cost function V_E, and controller κ_E.
3: Set $\mathcal{X}_\mathrm{f} := \mathcal{X}_\mathrm{E}$, $V_\mathrm{f} := V_\mathrm{E}$, and $\kappa_\mathrm{f} := \kappa_\mathrm{E}$.
4: Go to 1.

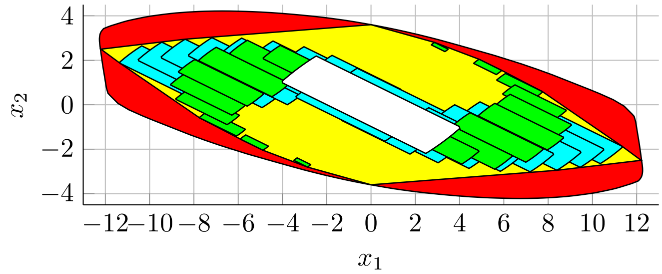

Fig. 1. Initial robust invariant tubes (cyan, green), convex hull (yellow), and approximation of the feasible set of the tube MPC scheme (red). The terminal set X_f (white) is also shown.

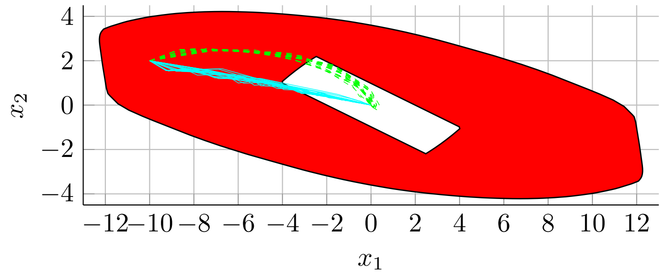

Fig. 2. closed-loop trajectories for the initial condition $x_0 = [-10, 2]^\mathsf{T}$ for the tube MPC scheme based on (5) (green, dashed) and the approximate control scheme based on (15) (cyan, solid).

8. ILLUSTRATIVE EXAMPLE

Let the matrices of the system be given by

$$A_i = \begin{bmatrix} 1 + 0.1\sin(2\pi i/50) & 1 + 0.1\cos(2\pi i/50) \\ 0 & 1 \end{bmatrix}, \quad B_i = \begin{bmatrix} 0.5 + 0.1\sin(2\pi i/50) \\ 1 + 0.1\cos(2\pi i/50) \end{bmatrix}, \tag{18}$$

where $i \in \mathbb{N}_{[1,50]}$. The constraints on the state and input are given by $x \in [-20, 20] \times [-20, 20]$ and $u \in [-1, 1]$. A locally stabilizing linear controller $u = K_\mathrm{f}x$ has been obtained using an algorithm in Pluymers et al. [2006]. It is given by $K_\mathrm{f} = [-0.5016 \;\; -1.0227]$. A robust invariant set $\mathcal{X}_\mathrm{f}$ for the closed-loop system with this controller satisfying the constraints has been found using Algorithm 2.4 in Pluymers [2006]. It is shown in Figure 1. The stage cost is given by $\ell(x, u) = \|[\begin{smallmatrix} x \\ u \end{smallmatrix}]\|_\infty$. A terminal cost $V_\mathrm{f}(x) = \|Px\|_\infty$ for a matrix $P \in \mathbb{R}^{34\times 2}$ satisfying Assumption 3 has been obtained using the results in Raković and Lazar [2012] based on a contractive set for the closed-loop system with the controller K_f. We chose a homothetic parameterization of the sets $\mathcal{X}_i$ as described in Section 3, where $\mathcal{X} = \mathcal{X}_\mathrm{f}$. Four robust invariant tubes were computed by solving the optimization problem in (5) for a horizon length of $N = 10$ and initial conditions $x_0^1 = [-12.2 \;\; 2.5]^\mathsf{T}$, $x_0^2 = [12.2 \;\; -2.5]^\mathsf{T}$, $x_0^3 = [0 \;\; 3.6]^\mathsf{T}$, $x_0^4 = [0 \;\; -3.6]^\mathsf{T}$. These tubes, together with the resulting convex hull and an approximation of the set where the problem in (5) is feasible, are shown in Figure 1. The region where the approximate control scheme is feasible (the convex hull of the tubes) is smaller than the feasible region of the original tube MPC scheme. However, by using more initial tubes, the feasible region of the original MPC scheme can be approximated to arbitrary precision.

Closed-loop simulations were performed for the initial condition $x_0 = [-10, 2]^\mathsf{T}$ and 20 random realizations of the uncertainty, were at every time step the matrices (A, B) were chosen as a random vertices of the uncertainty set in (18). The runtime of the simulations was $T = 30$. We used the simplified cost function in (16), but did not aggregate the variables. Because of numerical difficulties the inequality in (15d) was tightened to $\tilde{x}_i \in 0.99\rho_i\mathcal{X}_i$. Furthermore, when evaluating (6), all $\rho_i < 0.001$ were set to zero. For comparison, the closed-loop with a tube MPC controller based on the solution of (5) was also simulated. The resulting trajectories are shown in Figure 2. There is an obvious deviation between the trajectories. One reason for this is that the original tube MPC scheme yields an optimal solution with respect to the current state whereas the approximate scheme might interpolate between solutions based on points in the state space which are actually far away from the current system state. Another factor is the conservatism in the definition of the cost function of the approximate scheme.

A comparison of the average computation times and the average performance indices $V_{\text{perf}}(x_0) = \sum_{i=0}^{T-1} \ell(x_i, u_i)$ for the simulations associated with Figure 2 is shown in Table 1. The computations were performed on an Intel Core i3-3110M 2.40 GHz CPU.

Table 1. Comparison of Computation Time and Performance of Tube MPC and Approximate Predictive Control

	Tube MPC based on (5)	Approximate Control based on (15)
average comp. time [ms]	2226.5	16.6
average performance index	31.64	55.48

9. CONCLUSION

In this paper it was shown how a robustly stabilizing controller can be defined on the convex hull of robust invariant tubes. As demonstrated in the example, this method considerably reduces the computation time needed for the evaluation of the control law. On the other hand, this reduction in computational complexity comes with a degradation in performance. We expect that the performance can be improved by including additional robust invariant tubes with initial conditions at arbitrary points in the feasible set in the construction. This would imply a trade-off between computational complexity, which increases with the number of tubes, and the closed-loop performance. An investigation of the exact relations is a topic for future research.

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