

Model Reduction by Moment Matching for Linear Time-Delay Systems

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Abstract: The model reduction problem by moment matching for linear time-delay systems is addressed. A parameterized family of models achieving moment matching is characterized. The parameters can be exploited to derive a delay-free reduced order model or reduced order models with additional properties. The theory is illustrated by an example borrowed from the problem of automatic control of a platoon of vehicles.

1. INTRODUCTION

Time-delay systems are a class of infinite dimensional systems extensively studied (see *e.g.* the monographs of Hale (1977); Stépán (1989); Hale (1993); Niculescu (2001); Zhong (2006); Michiels and Niculescu (2007); Bekiaris-Liberis and Krstic (2013)). From a practical point of view every controlled system presents delays of some extent. Delays in closed-loop systems can generate unexpected behaviors (as oscillations or instability). For instance "small" delays may be destabilizing (Hale and Verduyn Lunel (2001)), while "large" delays may be stabilizing (MacDonald (1986); Beddington and May (1986)).

Herein, the model reduction technique presented in Astolfi (2010) and Ionescu et al. (2014) (see also Antoulas (2005), Ionescu and Iftime (2012) and Iftime (2012)), is extended to linear time-delay systems. It is shown that even in this case the moments of the system are fully characterized by the solution of a Sylvester-like equation. Although Sylvester equations have been widely studied (see for instance Lancaster (1969, 1970)), some care is needed to extend the classical results to the particular Sylvester-like equation that arises in the paper. A family of systems that achieve moment matching is characterized and connections with the results in Astolfi (2010) are drawn.

As noted in Halevi (1996) a reduced order model with time delays may lead to improvements in the approximation. Accordingly, the possibility to maintain the delay in the reduced order model is discussed and, in addition, it is shown that the introduction of delays can be used to improve the approximation, interpolating a larger number of points.

The rest of the paper is organized as follows. In Section 2 some preliminaries are given. In Section 3 the notion of moment is extended to linear time-delay systems and the solution of the resulting Sylvester-like equation is discussed. In Section 4 a family of systems achieving moment matching is presented and the possibility of interpolating a larger number of points maintaining the same "number of equations" is investigated. In Section 5 the platooning problem (Ioannou and Chien (1993); Huang (1999); Swa-

roop and Hedrick (1996)) is used to illustrate the theory. Finally in Section 6 some conclusions are drawn.

2. PRELIMINARIES

In this section we recall the notion of moment matching for linear systems as presented in Astolfi (2010). Consider a linear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad (1)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^n$ and $C \in \mathbb{R}^{1 \times n}$. Let

$$W(s) = C(sI - A)^{-1}B,$$

be the associated transfer function and assume that (1) is minimal, *i.e.* controllable and observable.

Definition 1. The 0-moment at $s_i \in \mathbb{C}$ of system (1) is the complex number $\eta_0(s_i) = C(s_i I - A)^{-1}B$. The k -moment of system (1) at $s_i \in \mathbb{C}$ is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!} \left[\frac{d^k}{ds^k} (C(sI - A)^{-1}B) \right]_{s=s_i},$$

with $k \geq 1$ and integer.

A characterization of the moments of system (1) can be given in terms of the solution of a Sylvester equation, as follows.

Lemma 1. (Astolfi (2010)). Consider system (1) and $s_i \in \mathbb{C}$, with $i = 1, \dots, \eta$. Suppose $s_i \notin \sigma(A)$. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \dots, \eta_{k_1}(s_1), \dots, \eta_0(s_\eta), \dots, \eta_{k_\eta}(s_\eta)$ and the matrix $C\Pi$, where Π is the unique solution of the Sylvester equation

$$A\Pi + BL = \Pi S,$$

with $S \in \mathbb{R}^{\nu \times \nu}$ any non-derogatory matrix with characteristic polynomial

$$p(s) = \prod_{i=1}^{\eta} (s - s_i)^{k_i},$$

where

$$\nu = \sum_{i=1}^{\eta} k_i,$$

and L such that the pair (L, S) is observable.

Finally, as shown in Astolfi (2010), the family of systems

$$\begin{aligned} \dot{\xi} &= (S - \Delta L)\xi + \Delta u, \\ \psi &= C\Pi\xi, \end{aligned} \quad (2)$$

with Δ any matrix such that $\sigma(S) \cap \sigma(S - \Delta L) = \emptyset$, contains all the models of dimension ν interpolating the moments of system (1) at the eigenvalues of the matrix S . Hence, we say that system (2) is a model of (1) at S .

3. MOMENT MATCHING FOR LINEAR TIME-DELAY SYSTEMS

3.1 Definition of moment

Consider a linear, single-input, single-output, continuous-time, time-delay system described by the equations

$$\begin{aligned} \dot{x}(t) &= A_0x(t) + \sum_{j=1}^{\mu} A_jx(t - \tau_j) + Bu(t - \tau_u), \\ y(t) &= Cx(t), \end{aligned} \quad (3)$$

with $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A_j \in \mathbb{R}^{n \times n}$ with $j = 0, \dots, \mu$, $B \in \mathbb{R}^n$, $C \in \mathbb{R}^{1 \times n}$, $\tau_j \in \mathbb{R}_+$ with $j = 1, \dots, \mu$, $\tau_u \in \mathbb{R}_+$ and let

$$W(s) = C(sI - A_0 - \sum_{j=1}^{\mu} e^{-s\tau_j} A_j)^{-1} e^{-s\tau_u} B, \quad (4)$$

be the associated transfer function. We begin with defining the moments of system (3) at some s_i and showing that there exists a one to one relation between the moments and the (unique) solution of a Sylvester-like equation.

Definition 2. The 0-moment at $s_i \in \mathbb{C}$ of system (3) is the complex number

$$\eta_0(s_i) = C(s_i I - A_0 - \sum_{j=1}^{\mu} e^{-s_i\tau_j} A_j)^{-1} e^{-s_i\tau_u} B.$$

The k -moment of system (3) at $s_i \in \mathbb{C}$ is the complex number

$$\begin{aligned} \eta_k(s_i) &= \\ &= \frac{(-1)^k}{k!} \left[\frac{d^k}{ds^k} (C(sI - A_0 - \sum_{j=1}^{\mu} e^{-s\tau_j} A_j)^{-1} e^{-s\tau_u} B) \right]_{s=s_i}, \end{aligned}$$

with $k \geq 1$ and integer.

Lemma 2. Consider system (3) and $s_i \in \mathbb{C}$. Let $\bar{A}(s) = A_0 + \sum_{j=1}^{\mu} e^{-s\tau_j} A_j$ and suppose $s_i \notin \sigma(\bar{A}(s_i))$. Then

$$\begin{bmatrix} \eta_0(s_i) \\ \vdots \\ \eta_k(s_i) \end{bmatrix} = (C\Pi\Psi_k),$$

where

$$\Psi_k = \text{diag}(1, -1, 1, \dots, (-1)^k) \in \mathbb{R}^{(k+1) \times (k+1)},$$

and Π is the unique solution of the Sylvester-like equation

$$A_0\Pi + \sum_{j=1}^{\mu} A_j\Pi e^{-\Sigma_k\tau_j} + BL_k e^{-\Sigma_k\tau_u} = \Pi\Sigma_k, \quad (5)$$

with $L_k = [1 \ 0 \ \dots \ 0] \in \mathbb{R}^{(k+1)}$ and

$$\Sigma_k = \begin{bmatrix} s_i & 1 & 0 & \dots & 0 \\ 0 & s_i & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & s_i & 1 \\ 0 & \dots & \dots & 0 & s_i \end{bmatrix} \in \mathbb{R}^{(k+1) \times (k+1)}.$$

To remove the disadvantage that the matrix Σ_k is complex and that Σ_k and L_k have a special structure, note that the moments are coordinates invariant. The following holds.

Lemma 3. Consider system (3) and $s_i \in \mathbb{C}$, with $i = 1, \dots, \eta$. Let $\bar{A}(s) = A_0 + \sum_{j=1}^{\mu} e^{-s\tau_j} A_j$ and suppose $s_i \notin \sigma(\bar{A}(s_i))$ for all $i = 1, \dots, \eta$. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \dots, \eta_{k_1}(s_1), \dots, \eta_0(s_\eta), \dots, \eta_{k_\eta}(s_\eta)$ and the matrix $C\Pi$, where Π is the unique solution of the Sylvester-like equation

$$A_0\Pi + \sum_{j=1}^{\mu} A_j\Pi e^{-S\tau_j} - \Pi S = -BLE^{-S\tau}, \quad (6)$$

with $S \in \mathbb{R}^{\nu \times \nu}$ any non-derogatory matrix with characteristic polynomial

$$p(s) = \prod_{i=1}^{\eta} (s - s_i)^{k_i}, \quad (7)$$

where

$$\nu = \sum_{i=1}^{\eta} k_i,$$

and L such that the pair (L, S) is observable.

3.2 Solution of the Sylvester-like equation

Equation (6) is a Sylvester equation only if $\mu = 0$. Nevertheless, it is a linear equation in Π and it can be solved with the use of the vectorization operator and the Kronecker product. To this end, it is necessary to determine when the equation admits a unique solution.

¹ Let $x \in \mathbb{C}$ and $A(x) \in \mathbb{C}^{n \times n}$. Then $x \notin \sigma(A(x))$ means that $\det(xI - A(x)) \neq 0$.

In this subsection we give the solution of this problem in the general case and for two special cases.

Lemma 4. Equation (6) has a unique solution if and only if

$$s_l \notin \sigma(\bar{A}(s_l)),$$

for all $l = 1, \dots, \eta$, with $\bar{A}(s) = A_0 + \sum_{j=1}^{\mu} e^{-s\tau_j} A_j$.

Remark 1. Note that the condition of Lemma 4 can be verified in $O(\eta n^p)$, where $O(n^p)$ is the computational complexity to compute the determinant of a n by n matrix, while the condition resulting by the vectorization of equation (6) can be verified in $O(n^{2p})$. Then the condition of Lemma 4 is particularly advantageous for “small” reductions of “large” systems.

Lemma 5. Equation (6) has a unique solution if the following holds.

- $A_0 = 0, A_1 \neq 0, \mu = 1$, and $\sigma(A_1) \cap \sigma(Se^{S\tau}) = \emptyset$.
- The matrices A_j for $j = 0, 1, \dots, \mu$ commute and $\lambda_{0i} + \sum_{j=1}^{\mu} e^{-s_l \tau_j} \lambda_{ji} \neq s_l$ for $i = 1, \dots, n$ and $l = 1, \dots, \eta$, with $\lambda_{0i}, \lambda_{ji}$ and s_l eigenvalues of A_0, A_j and S , respectively.

4. MODEL REDUCTION BY MOMENT MATCHING

In this section a family of systems achieving moment matching is presented and the possibility of interpolating a larger number of points maintaining the same “number of equations” is investigated.

Theorem 1. Consider system (3) and let $S \in \mathbb{R}^{\nu \times \nu}$ be any non-derogatory matrix with characteristic polynomial (7).

Let $\bar{A}(s) = A_0 + \sum_{j=1}^{\mu} e^{-s\tau_j} A_j$, assume that $s_l \notin \sigma(\bar{A}(s_l))$

for all $l = 1, \dots, \eta$, and that L is such that the pair (L, S) is observable.

Then the system

$$\begin{aligned} \dot{\xi}(t) &= F_0 \xi(t) + \sum_{j=1}^{\rho} F_j \xi(t - \chi_j) + Gu(t - \chi_u), \\ \phi(t) &= H \xi(t), \end{aligned} \quad (8)$$

with $\xi(t) \in \mathbb{R}^{\nu}, \psi(t) \in \mathbb{R}, \phi(t) \in \mathbb{R}, F_j \in \mathbb{R}^{\nu \times \nu}$ for $j = 0, \dots, \rho \geq 0, \chi_j \in \mathbb{R}_+$ for $j = 1, \dots, \rho, \chi_u \in \mathbb{R}_+, G \in \mathbb{R}^{\nu}$ and $H \in \mathbb{R}^{1 \times \nu}$, is a model of system (3) at S , if

$$s_l \notin \sigma(F_0 - \sum_{j=1}^{\rho} F_j e^{-s_l \chi_j}), \quad (9)$$

for all $l = 1, \dots, \eta$, and there exists a unique solution P of the equation

$$F_0 P + \sum_{j=1}^{\rho} F_j P e^{-S \chi_j} - PS = -GLE^{-S \chi_u}, \quad (10)$$

such that

$$C\Pi = HP, \quad (11)$$

where Π is the unique solution of (6).

System (8) is a reduced order model of system (3) if $\nu < n$, or if $\rho < \mu$.

4.1 Model reduction with free F_j

To construct a family of models that achieves moment matching at ν points select

$$\begin{aligned} F_0 &= S - \Delta L e^{-S \chi_u} - \sum_{j=1}^{\rho} \Gamma_j e^{-S \chi_j}, & G &= \Delta, \\ F_j &= \Gamma_j, & H &= C\Pi, \end{aligned} \quad (12)$$

and note that this selection solves equations (10), (11) for $P = I$. This yields the family of reduced order models

$$\begin{aligned} \dot{\xi}(t) &= (S - \Delta L e^{-S \chi_u} - \sum_{j=1}^{\rho} \Gamma_j e^{-S \chi_j}) \xi(t) + \\ &+ \sum_{j=1}^{\rho} \Gamma_j \xi(t - \chi_j) + \Delta u(t - \chi_u), \\ \phi(t) &= C\Pi \xi(t), \end{aligned} \quad (13)$$

with Δ and Γ_j any matrices such that condition (9) holds.

The proposed model has several free design parameters, namely $\Delta, \Gamma_j, \chi_j, \rho$. We note that selecting $\Gamma_j = 0$ for all $j = 1, \dots, \rho$, yields a reduced order model with no delays. In other words, we reduce an infinite dimensional system to a finite dimensional one of dimension ν . This reduced order model coincides with the one in Astolfi (2010) and all results therein are directly applicable: the parameter Δ can be selected to achieve matching with prescribed eigenvalues, matching with prescribed relative degree, etc.

However, the choice of eliminating the delays is likely to destroy some underlying dynamics of the model and, as shown in Halevi (1996), delays are not always negative to stability. With this in mind, a possible choice is to keep Γ_j free with $\rho = \mu$. In this case we can use the matrices Γ_j to maintain some important physical properties of the delay structure of the system.

Example 1. To illustrate the above idea consider the example in Section 2.5 of Niculescu (2001). Therein a model of a LC transmission line is discussed. The system is such that if $R_0 \sqrt{C/L} = 1$ the delay part of the system disappears (a phenomenon called line-matching) and the model can be described by a system of ordinary differential equations. In the reduced model it is desirable to maintain this property to preserve the physical structure of the system.

In the next subsection, the case in which the matrices Γ_j are non-zero is presented and it is shown how to exploit them to obtain some properties on the reduced order system.

4.2 Interpolation at $(\rho + 1)\nu$ points

The matrices Γ_j in (13) are design parameters. In this subsection we show how to exploit them to achieve moment matching at more than ν points, still maintaining the same dimension ν of the matrix F_0 . We analyze the case in which $\rho = 1$, for ease of notation. The general case can be analyzed in a similar way.

Proposition 1. Let $S_a \in \mathbb{R}^{\nu \times \nu}$ and $S_b \in \mathbb{R}^{\nu \times \nu}$ be two non-derogatory matrices such that $\sigma(S_a) \cap \sigma(S_b) = \emptyset$ and let L_a and L_b be such that the pairs (L_a, S_a) and (L_b, S_b) are observable. Let $\Pi = \Pi_a$ be the unique solution of (6) with $L = L_a$ and let $\Pi = \Pi_b$ be the unique solution of (6) with $L = L_b$. Consider $F_0, F_1, G,$ and H as in (12) with $S = S_a$ and $L = L_a$ and assume Δ is any matrix such that condition (9) holds for S_a and S_b .

Then there exists a matrix P_b such that

$$F_0 P_b + F_1 P_b e^{-S_b \tau} - P_b S_b = -\Delta L_b,$$

$$C \Pi_a P_b = C \Pi_b.$$

In addition the selection

$$\Gamma_1 = (P_b S_b - S_a P_b + \Delta L_a P_b - \Delta L_b) (P_b e^{-S_b \tau} - e^{-S_a \tau} P_b)^{-1}, \quad (14)$$

is such that system (13) is a reduced order model of system (3) achieving moment matching at S_a and S_b .

The family of systems characterized in Proposition 1 achieves moment matching at 2ν interpolation points. Note that the matrix Δ remains a free parameter and it can be used to achieve the properties discussed in Astolfi (2010). Note, however, that Δ has only ν free parameters. Hence, for instance, one can assign the eigenvalues of F_0 but not of F_1 at the same time.

Remark 2. The result can be generalized to $\rho > 1$ delays, obtaining a reduced order model that interpolates at $(\rho + 1)\nu$ points. This result can be used also when the system to be reduced is not a time-delay system. In other words, a system described by ordinary differential equations or differential time-delay equations can be reduced to a system described by differential time-delay equations with an arbitrary number of delays ρ achieving moment matching at $(\rho + 1)\nu$ points. This property of interpolating an arbitrary large number of points comes to the cost that the reduced order model becomes an infinite dimensional system. However, as noted in Halevi (1996), a reduced model with time delays may have better properties than one without delays.

Remark 3. Although it is possible to interpolate at several different points s_i maintaining the same dimension ν , the maximum k -moment at s_i cannot be more than ν because it is limited by definition to the dimension of the matrix S_i .

5. EXAMPLE

To illustrate the results in the paper we consider a controlled platoon of vehicles as presented in Ioannou and Chien (1993); Huang (1999). The platooning problem consists in controlling a group of vehicles tightly spaced following a leader, all moving in longitudinal direction. The advantages of the automatic cruise control is twofold. First, the use of automatic control to replace human drivers and their low-predictable reaction time with respect to traffic problems (spacing of around 30 m at 60 km/h) can reduce the spacing distance between vehicles, consequently decreasing the traffic congestion. Second, the automatic control reduces the human error factor and then increases safety. In recent years successful experiments involving autonomous vehicles have been carried out (e.g. the Google

driver-less cars), and the use of this technology may be possible in the immediate future. However, when a large number of vehicles is considered, to study the dynamics of the whole platoon to guarantee individual vehicle stability and avoid slinky-type effect (i.e., the amplification of the spacing errors between subsequent vehicles as the vehicle "index" increases) can be computationally demanding (see Middleton and Braslavsky (2010)).

In what follows we use a model well-studied (see Niculescu (2001); Ioannou and Chien (1993); Huang (1999)) for which the solution of the platooning problem is known. In particular, we are interested in reducing the number of vehicles to only a leader and a following car.

Let $x_i(t)$ be the position of the i -th vehicle with respect to some well-defined reference, $v_i(t)$ its speed, $a_i(t)$ its acceleration and denote with $\delta_i = x_{i+1} - x_i - H_i$ the spacing error, with $H_i > 0$ the minimum separation distance. The resulting model is described by the equations

$$\dot{\delta}_i(t) = v_{i+1}(t) - v_i(t),$$

$$\dot{v}_i(t) = a_i(t),$$

$$\dot{a}_i(t) = -\frac{a_i(t)}{c} + \frac{1}{c} [k_s \delta_i(t - \tau) + k_v (v_{i+1}(t - \tau) - v_i(t - \tau))], \quad (15)$$

where $c > 0$ is the engine time constant, $\tau > 0$ is the total delay (including fueling and transport, etc.) for each vehicle, and k_s and k_v are the transmission gains between the vehicles. To this platoon we add a leader car with dynamics described by the equations

$$\dot{v}_n(t) = a_n(t),$$

$$\dot{a}_n(t) = -\frac{a_n(t)}{c} + \frac{1}{c} k_v (u(t) - v_n(t)), \quad (16)$$

where $u(t)$ is a desired velocity imposed on the leader with no delay. We select as output of the system the sum of all the spacing errors, namely the distance between the first and the last vehicle. We rewrite the system in compact form as

$$\dot{x}(t) = A_0 x(t) + A_1 x(t - \tau) + B u(t),$$

$$y(t) = C x(t), \quad (17)$$

with

$$A_0 = \frac{1}{c} \begin{bmatrix} A_0^1 & A_0^2 & 0 & \dots & 0 & 0 \\ 0 & A_0^1 & A_0^2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & A_0^1 & A_0^2 & 0 \\ 0 & \dots & \dots & 0 & A_0^1 & A_0^3 \\ 0 & \dots & \dots & \dots & 0 & A_0^4 \end{bmatrix},$$

$$A_1 = \frac{1}{c} \begin{bmatrix} A_1^1 & A_1^2 & 0 & \dots & 0 & 0 \\ 0 & A_1^1 & A_1^2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 & \vdots \\ \vdots & \ddots & 0 & A_1^1 & A_1^2 & 0 \\ 0 & \dots & \dots & 0 & A_1^1 & A_1^3 \\ 0 & \dots & \dots & \dots & \dots & 0 \end{bmatrix},$$

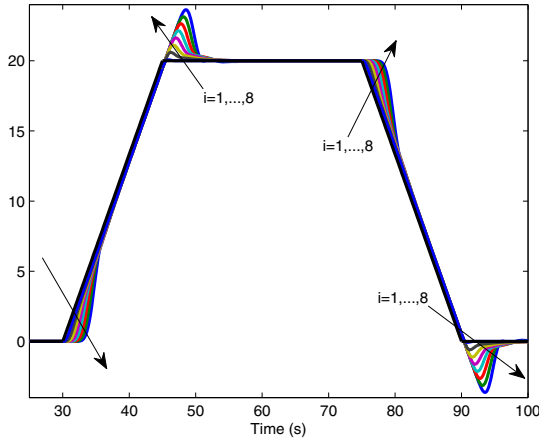


Fig. 1. Speed of the eight vehicles.

$$B = \left[0 \ 0 \ 0 \mid 0 \ 0 \ 0 \mid \dots \mid 0 \ 0 \ 0 \mid 0 \ \frac{k_v}{c} \right]^T,$$

$$C = \left[1 \ 0 \ 0 \mid 1 \ 0 \ 0 \mid \dots \mid 1 \ 0 \ 0 \mid 0 \ 0 \right],$$

where

$$A_0^1 = \begin{bmatrix} 0 & -c & 0 \\ 0 & 0 & c \\ 0 & 0 & -1 \end{bmatrix}, \quad A_0^2 = \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_0^3 = \begin{bmatrix} c & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_0^4 = \begin{bmatrix} 0 & c \\ -k_v & -1 \end{bmatrix},$$

$$A_1^1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k_s & -k_v & 0 \end{bmatrix}, \quad A_1^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k_v & 0 \end{bmatrix}, \quad A_1^3 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ k_v & 0 \end{bmatrix}.$$

5.1 Simulations

We consider $n = 8$ identical vehicles with $c = 0.25 \text{ s}$, $k_s = 0.875 \text{ s}^{-2}$, $k_v = 2.5 \text{ s}^{-1}$ and $\tau = 0.005 \text{ s}$.

We propose two reduced order models that match the 0-moments at $s_1 = 0$, $s_{2,3} = \pm\pi/5$, $s_{4,5} = \pm\pi/30$, with $u(t) = L\omega(t)$, $\dot{\omega}(t) = S\omega(t)$, $L = [1 \ 0 \ 1 \ 0 \ 1]^T$, and described by the equations

$$\dot{\xi}(t) = F_0\xi(t) + F_1\xi(t - \tau) + \Delta u(t), \quad (18)$$

$$\psi(t) = C\Pi\xi(t),$$

with F_0 and F_1 defined as in (12). Note that the number of equations decreases from $3n - 1$ to ν . We denote with ψ_I the output of the system (18) when F_1 is defined as

$$F_1 = \frac{1}{c} \begin{bmatrix} A_1^1 & A_1^3 \\ 0 & 0 \end{bmatrix}. \quad (19)$$

Note that F_1 has the same structure of A_1 . We denote with ψ_0 the output of the system (18) when $F_1 = 0$. In the latter case all the eigenvalues of the matrix F_0 have been placed at $-\frac{1}{2}$. The input given to the system consists of a speed increase from 0 to $20 \text{ m/s} = 72 \text{ km/h}$ in 15 s, a constant speed of 20 m/s for 30 s and a deceleration to 0 m/s in 15 s. The speed of the vehicles are shown in Fig. 1. Fig. 2 shows the output signals $y(t)$ (solid line), $\psi_I(t)$ (dashed line), $\psi_0(t)$ (dotted line). Fig. 3 shows the absolute errors between $y(t)$ and $\psi_I(t)$ (dashed line), and between $y(t)$

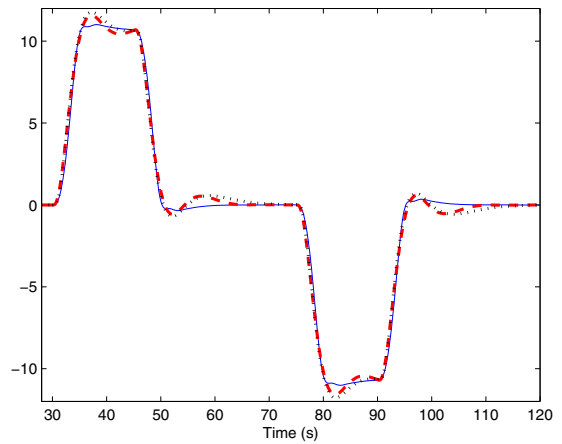


Fig. 2. Output signals $y(t)$ (solid line), $\psi_I(t)$ (dashed line), and $\psi_0(t)$ (dotted line).

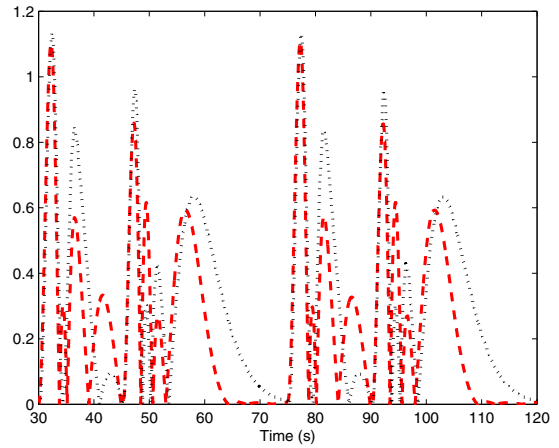


Fig. 3. Absolute errors between $y(t)$ and $\psi_I(t)$ (dashed line), and between $y(t)$ and $\psi_0(t)$ (dotted line).

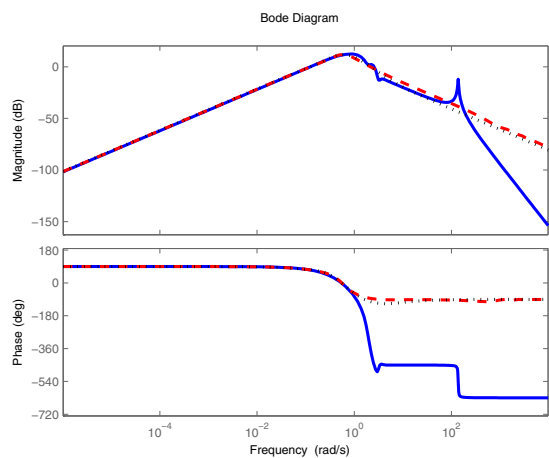


Fig. 4. Bode plots of the system (solid line), of the reduced order model with delays (dashed line), and of the reduced order model with no delays (dotted line).

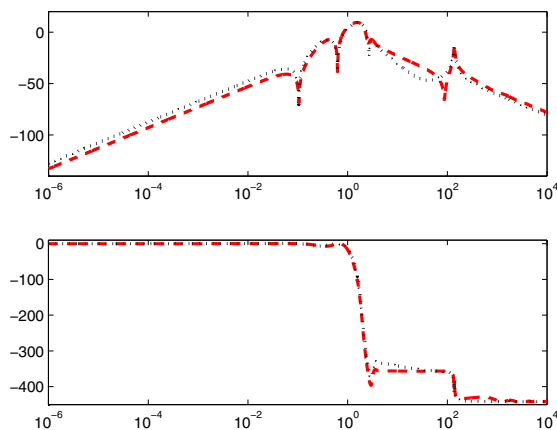


Fig. 5. Absolute errors of the Bode plots: with the reduced order model with delays (dashed line), and with the reduced order model with no delays (dotted line).

and $\psi_0(t)$ (dotted line). We see that the output is similar in the three cases and that the reduced order model with delays is tighter to the system, *i.e.* the ratio between the area under the error curve of the model with delays and the area under the error curve of the model with no delays is 0.799. Fig. 4 shows the Bode plots of the system (solid line), of the reduced order model with delays (dashed line), and of the reduced order model with no delays (dotted line). The three lines are superposed for frequencies below 1 rad/s. Note that this is the region where we expect the input to be “concentrated”, to avoid sudden accelerations.

6. CONCLUSION

The problem of model reduction by moment matching for linear time-delay systems has been solved. The notion of moment in terms of a solution of a Sylvester-like equation has been given and its solvability has been discussed. A family of systems achieving moment matching has been proposed. The possibility of interpolating a larger number of points maintaining the same number of equations has been studied. The theory has been illustrated by an example in the automotive domain.

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