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# An Approach to Distributed Robust Model Predictive Control of Discrete-Time Polytopic Systems

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**Abstract:** A suboptimal approach to distributed robust MPC for uncertain systems consisting of polytopic subsystems with coupled dynamics subject to both state and input constraints is proposed. The approach applies the dynamic dual decomposition method and reformulates the original centralized robust MPC problem into a distributed robust MPC problem. It is based on distributed on-line optimization by applying an accelerated gradient method. The suggested approach is illustrated on a simulation example of an uncertain system consisting of two interconnected polytopic subsystems.

*Keywords:* Predictive control, Distributed control, Decomposition methods, Interconnected systems, Uncertainty, Robust control, Constraints.

## 1. INTRODUCTION

Model predictive control (MPC) involves the solution at each sampling instant of a finite horizon optimal control problem subject to the system dynamics, and state and input constraints. Solving in a centralized way MPC problems for large-scale systems may be impractical due to the topology of the plant and data communication and the large number of decision variables. Recently, several approaches for decentralized and parallel implementation of MPC algorithms have been proposed, Constantinides (2009), Scattolini (2009), Christofides et al. (2013), Maestre and Negenborn (2014).

In Zhang and Li (2007), Venkat et al. (2008), Alessio et al. (2011), Giselsson and Rantzer (2010), Giselsson et al. (2013), approaches for distributed/decentralized MPC for systems consisting of *linear* interconnected subsystems have been developed. The approach in Giselsson and Rantzer (2010) is based on the dual decomposition methods (Dantzig and Wolfe (1961), Cohen and Miara (1990)), where large-scale optimization problems are handled by using Lagrange multipliers to relax the couplings between the sub-problems. In Giselsson et al. (2013), a distributed optimization algorithm based on accelerated gradient methods using dual decomposition is proposed and its performance is evaluated on optimization problems arising in distributed MPC. Also, approaches for distributed MPC for systems composed of several *nonlinear* subsystems have been proposed (e.g. Raimondo et al. (2007), Dunbar (2007), Heidarinejad et al. (2011), Grancharova and Johansen (2014)).

There are only a few papers considering the problem of robust distributed MPC of *polytopic* uncertain systems. Thus, in Zhang et al. (2013), a distributed MPC algorithm for polytopic systems subject to actuator saturation is proposed, where the distributed MPC controller is designed by solving a linear matrix inequality (LMI) optimization problem. In Al-Gherwi et al. (2011), an online distributed MPC algorithm that deals explicitly with model errors is proposed. The algorithm requires decomposing the entire system into subsystems, which are coupled through their inputs. Only constraints on the inputs are considered and the upper bound on the robust performance objective is minimized by using a time-varying state-feedback controller for each subsystem.

In this paper, a suboptimal approach to distributed robust MPC for uncertain systems consisting of *polytopic* subsystems with coupled dynamics subject to both state and input constraints is proposed. The approach applies the dynamic dual decomposition method (Cohen and Miara (1990), Giselsson and Rantzer (2010)) and reformulates the centralized robust MPC problem into a distributed robust MPC problem. It is based on distributed on-line optimization and can be applied to large-scale polytopic systems.

## 2. FORMULATION OF ROBUST MODEL PREDICTIVE CONTROL PROBLEM FOR POLYTOPIC SYSTEMS

Consider a system composed by the interconnection of $M$ linear uncertain subsystems described by the following polytopic discrete-time models:

$$x_i(t+1) = A_i(t)x_i(t) + B_i(t)u_i(t) + \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij}x_j(t),\ i = 1,2,...,M \quad (1)$$

$$[A_i(t),\ B_i(t)] \in \Omega_i$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t) \in \mathbb{R}^{m_i}$ are the state and control input vectors, related to the $i$-th subsystem, $A_i(t) \in \mathbb{R}^{n_i \times n_i}$ and $B_i(t) \in \mathbb{R}^{n_i \times m_i}$ are uncertain time-varying matrices, and $A_{ij} \in \mathbb{R}^{n_i \times n_i}$ $j = 1,2, \dots ,M$, $j \neq i$ are known constant martices. For a polytopic uncertainty description, $\Omega_i$ $i = 1,2, \dots ,M$ are polytopes:

$$\Omega_i = Co\{[A_i^1, B_i^1],[A_i^2, B_i^2], \dots ,[A_i^{L_i}, B_i^{L_i}]\},\ i = 1,2, \dots ,M \quad (2)$$

where $Co\{\cdot\}$ denotes convex hull and $[A_i^r, B_i^r]$, $r = 1, 2, \dots, L_i$ are its vertices.

The following constraints are imposed on the subsystems:

$$u_{\min,i} \le u_i(t) \le u_{\max,i}\,,\; x_{\min,i} \le x_i(t) \le x_{\max,i}\,,\; i = 1, 2, \dots, M \quad (3)$$

The following assumption is made:

**A1.** $x_{\min,i} < 0 < x_{\max,i}$, $u_{\min,i} < 0 < u_{\max,i}$, $i = 1, \dots, M$.

Let $x(t)$ and $u(t)$ denote the overall state and the overall control input, i.e.:

$$x(t) = [x_1(t), x_2(t), \dots, x_M(t)] \in \mathbb{R}^n\,,\; n = \sum_{i=1}^{M} n_i \quad (4)$$

$$u(t) = [u_1(t), u_2(t), \dots, u_M(t)] \in \mathbb{R}^m\,,\; m = \sum_{i=1}^{M} m_i \quad (5)$$

Another assumption will be made with respect to the rate of variation of parameters, mainly with respect to the prediction horizon as it will be shown next in the MPC design.

**A2.** The uncertain pairs $[A_i(t), B_i(t)] \in \Omega_i$, $i = 1, \dots, M$ have infrequent changes in the sense that $[A_i(t), B_i(t)] = \text{const}$, $i = 1, \dots, M$ for periods of time, which are not less than $\tilde{N}$ ($\tilde{N} \in \mathbb{N}$ is supposed to be sufficiently large).

Before formulating the robust MPC problem, a set $\tilde{\Omega}_i$ is introduced, which is a finite subset of $\Omega_i$. First, let $\Omega_i^{\text{vert}} = \{[A_i^r, B_i^r], r = 1, 2, \dots, L_i\}$ be the set of vertices of $\Omega_i$ and $\Omega_i^{\text{int}} = \{[A_i^{L_i+j}, B_i^{L_i+j}] \in \text{int}(\Omega_i), j = 1, 2, \dots, K_i\}$ be a finite set which includes interior points of the set $\Omega_i$. Then, the finite set $\tilde{\Omega}_i \subset \Omega_i$ is defined as $\tilde{\Omega}_i = \Omega_i^{\text{vert}} \bigcup \Omega_i^{\text{int}}$. The reason for this definition of the set $\tilde{\Omega}_i$ is explained in Remark 1.

It is supposed that a full measurement $x = [x_1, x_2, \dots, x_M]$ of the overall state is available at the current time $t$. The robust regulation problem is considered where the goal is to steer the overall state of the system (1) to the origin. Let $N$ be a finite horizon such that $N < \tilde{N}$. By Assumption A2 it can be accepted that $[A_i(t+k), B_i(t+k)] = \text{const} = [A_i, B_i]$, $k = 0, 1, \dots, N$, $i = 1, \dots, M$. Then, for the current $x$, the robust regulation MPC solves the optimization problem:

**Problem P1 (Centralized robust MPC):**

$$V^*(x) = \min_{U} \max_{\substack{[A_i, B_i] \in \tilde{\Omega}_i \\ i=1,2,\dots,M}} J(U, x, [A_1, B_1], \dots, [A_M, B_M]) \quad (6)$$

subject to $x_{t|t} = x$ and:

$$x_{i,t+k|t} \in \mathcal{X}_i\,,\; \forall [A_i, B_i] \in \tilde{\Omega}_i\,,\; i = 1, \dots, M\,,\; k = 1, \dots, N \quad (7)$$

$$u_{i,t+k} \in \mathcal{U}_i\,,\; i = 1, \dots, M\,,\; k = 0, 1, \dots, N-1 \quad (8)$$

$$\begin{aligned} & x_{i,t+k+1|t} = A_i x_{i,t+k|t} + B_i u_{i,t+k} + \sum_{j=1, j\ne i}^{M} A_{ij} x_{j,t+k|t} \\ & [A_i, B_i] \in \tilde{\Omega}_i\,,\; i = 1, \dots, M\,,\; k = 0, 1, \dots, N-1 \end{aligned} \quad (9)$$

$$x_{t+k|t} = [x_{1,t+k|t}, x_{2,t+k|t}, \dots, x_{M,t+k|t}],\; k = 0, 1, \dots, N \quad (10)$$

$$u_{t+k} = [u_{1,t+k}, u_{2,t+k}, \dots, u_{M,t+k}],\; k = 0, 1, \dots, N-1 \quad (11)$$

with $U = [u_t, u_{t+1}, \dots, u_{t+N}]$ and the cost function given by:

$$J(U, x, [A_1, B_1], \dots, [A_M, B_M]) = \sum_{k=0}^{N} l(x_{t+k|t}, u_{t+k}) \quad (12)$$

$$\text{with}\quad l(x_{t+k|t}, u_{t+k}) = \sum_{i=1}^{M} l_i(x_{i,t+k|t}, u_{i,t+k}) \quad (13)$$

Here, $l_i(x_{i,t+k|t}, u_{i,t+k}) = \| x_{i,t+k|t} \|_{W_{x_i}}^2 + \| u_{i,t+k} \|_{W_{u_i}}^2$ is the stage cost for the $i$-th subsystem with weighting matrices $W_{x_i}, W_{u_i} \succ 0$. The sets $\mathcal{X}_i$ and $\mathcal{U}_i$ are defined by:

$$\mathcal{X}_i = \{\lambda_i \in \mathbb{R}^{n_i} \mid x_{\min,i} \le \lambda_i \le x_{\max,i}\} \quad (14)$$

$$\mathcal{U}_i = \{\eta_i \in \mathbb{R}^{m_i} \mid u_{\min,i} \le \eta_i \le u_{\max,i}\} \quad (15)$$

It follows from (14)–(15) that $\mathcal{X}_i$ and $\mathcal{U}_i$ are convex (polyhedral) sets, which include the origin in their interior (due to Assumption A1). It should be noted that the state constraints (7) guarantee the robust feasibility of the solution in sense that the state constraints in (3) will be satisfied for the worst-case uncertainty realizations in $\tilde{\Omega}_i$, $i = 1, \dots, M$.

**Remark 1:**
The description of the overall system dynamics, corresponding to (1) is:

$$x(t+1) = (A(t) + \tilde{A})x(t) + B(t)u(t),\; [A(t), B(t)] \in \Omega \quad (16)$$

where $A(t)$, $B(t)$, $\tilde{A}$ are block-matrices:

$$\begin{aligned} & A(t) = \text{diag}\{A_1(t), \dots, A_M(t)\}, B(t) = \text{diag}\{B_1(t), \dots, B_M(t)\} \\ & \tilde{A} = \begin{Bmatrix} A_{ij}\,, & i \ne j \\ 0\,, & i = j \end{Bmatrix},\; i, j = 1, \dots, M \end{aligned} \quad (17)$$

and $\Omega = \Omega_1 \times \Omega_2 \times \dots \times \Omega_M$. Further, by assuming that $[A(t+k), B(t+k)] = \text{const} = [A, B]$, $k = 0, 1, \dots, N$, the predicted overall state is:

$$x_{t+k|t} = (A + \tilde{A})^k x + \sum_{j=0}^{k-1} (A + \tilde{A})^j B u_{t+k-1-j} \quad (18)$$

Then, the cost function (12) will be in general non-convex with respect to the uncertain matrix $A$, because the predicted state includes the powers of $(A + \tilde{A})$. Therefore, considering only the vertices of the sets $\Omega_i$, $i = 1, \dots, M$ when computing the worst-case cost may not be sufficient. For this reason, the finite uncertainty sets $\tilde{\Omega}_i$, $i = 1, \dots, M$ should include some interior elements in addition to the vertices.

## 3. DISTRIBUTED ROBUST MPC APPROACH BY DUAL DECOMPOSITION

### *3.1 Distributed robust MPC by dual decomposition*

Problem P1 can be decomposed by using the dynamic dual decomposition approach (Cohen and Miara (1990), Giselsson and Rantzer (2010)). The following *decoupled* state equations can be formulated:

$$\begin{aligned} & x_i(t+1) = A_i(t)x_i(t) + B_i(t)u_i(t) + v_i(t),\; i = 1, 2, \dots, M \\ & [A_i(t), B_i(t)] \in \Omega_i \end{aligned} \quad (19)$$

with the additional constraints that (Giselsson and Rantzer (2010)):

$$v_i(t) = \sum_{j=1, j\ne i}^{M} A_{ij} x_j(t)\,,\; i = 1, \dots, M \quad \text{for all } t \quad (20)$$

Similar to (Giselsson and Rantzer (2010)), the constraints (20) are relaxed by introducing the corresponding Lagrange multipliers $p_i \in \mathbb{R}^{n_i}$ (also referred to as prices) in the cost function (12) and the original problem P1 is reformulated as a dual problem in view of *distributed* robust MPC:

**Problem P2 (Distributed robust MPC):**

$$\max_P \min_{U,V} \max_{\substack{[A_i,\, B_i] \in \tilde{\Omega}_i \\ i=1,2,\dots,M}} \sum_{k=0}^{N} \sum_{i=1}^{M} [l_i(x_{i,t+k|t}, u_{i,t+k}) + p_{i,t+k}^T (v_{i,t+k} - \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ij} x_{j,t+k|t})] =$$
$$\max_P \sum_{i=1}^{M} (\min_{U_i,V_i} \max_{[A_i,\, B_i]\in\tilde{\Omega}_i} \underbrace{\sum_{k=0}^{N} [l_i(x_{i,t+k|t}, u_{i,t+k}) + p_{i,t+k}^T v_{i,t+k} - x_{i,t+k|t}^T \sum_{\substack{j=1 \\ j\neq i}}^{M} A_{ji}^T p_{j,t+k}]}_{l_i^P(u_{i,t+k}, v_{i,t+k}, P, A_i, B_i)}) \tag{21}$$

subject to $x_{t|t} = x$, constraints (7)–(8) and:

$$x_{i,t+k+1|t} = A_i x_{i,t+k|t} + B_i u_{i,t+k} + v_{i,t+k}, \ [A_i,\, B_i] \in \tilde{\Omega}_i \quad i = 1,\dots,M,\ k = 0,1,\dots,N-1 \tag{22}$$

$$p_{t+N} = 0 \tag{23}$$

Here:

$$\begin{aligned} &P = [p_t, p_{t+1}, \dots, p_{t+N}] \\ &\text{with } p_{t+k} = [p_{1,t+k}, p_{2,t+k}, \dots, p_{M,t+k}],\ k = 0,1,\dots,N \\ &U = [u_t, u_{t+1}, \dots, u_{t+N}], U_i = [u_{i,t}, u_{i,t+1}, \dots, u_{i,t+N}] \\ &V = [v_t, v_{t+1}, \dots, v_{t+N}], V_i = [v_{i,t}, v_{i,t+1}, \dots, v_{i,t+N}] \\ &\text{with } v_{t+k} = [v_{1,t+k}, v_{2,t+k}, \dots, v_{M,t+k}],\ k = 0,1,\dots,N \end{aligned} \tag{24}$$

The inner *decoupled* optimization problems in problem P2 represent Quadratic Programming (QP) sub-problems. Each QP sub-problem is presented as follows:

**Problem P3$^i$ (*i*-th QP sub-problem):**

$$V_i^*(P, x_i) = \min_{U_i,V_i} \max_{[A_i,\, B_i]\in\tilde{\Omega}_i} \sum_{k=0}^{N} l_i^P(u_{i,t+k}, v_{i,t+k}, P, A_i, B_i) \tag{25}$$

subject to $x_{i,t|t} = x_i$ and:

$$x_{i,t+k|t} \in \mathcal{X}_i,\ \forall [A_i,\, B_i] \in \tilde{\Omega}_i,\ k = 1,\dots,N \tag{26}$$

$$u_{i,t+k} \in \mathcal{U}_i,\ k = 0,1,\dots,N-1 \tag{27}$$

$$x_{i,t+k+1|t} = A_i x_{i,t+k|t} + B_i u_{i,t+k} + v_{i,t+k},\ [A_i,\, B_i] \in \tilde{\Omega}_i \quad k = 0,1,\dots,N-1 \tag{28}$$

Let $U_i^* = [u_{i,t}^*, u_{i,t+1}^*, \dots, u_{i,t+N}^*]$ and $V_i^* = [v_{i,t}^*, v_{i,t+1}^*, \dots, v_{i,t+N}^*]$ be the optimal solution of P3$^i$, and $X_i^* = [x_{i,t|t}^*, x_{i,t+1|t}^*, \dots, x_{i,t+N|t}^*]$ denote the worst-case state trajectory corresponding to the optimal trajectories $U_i^*$ and $V_i^*$, i.e.:

$$x_{i,t+k+1|t}^* = A_i^* x_{i,t+k|t}^* + B_i^* u_{i,t+k}^* + v_{i,t+k}^*,\ k = 0,1,\dots,N-1 \tag{29}$$

where:

$$[A_i^*,\, B_i^*] = \arg \max_{[A_i,\, B_i]\in\tilde{\Omega}_i} \sum_{k=0}^{N} l_i^P(u_{i,t+k}^*, v_{i,t+k}^*, P, A_i, B_i) \tag{30}$$

The decomposition of the optimization problem P1 is given by the following proposition:

**Proposition 1:**
Suppose that $x = [x_1, x_2, \dots, x_M]$ is a feasible point for problem P1. Then:

$$V^*(x) = \max_P \sum_{i=1}^{M} V_i^*(P, x_i) \tag{31}$$

where maximization is subject to $p_{t+N} = 0$.

*Proof:*
The cost function $J(U, x, [A_1, B_1], \dots, [A_M, B_M])$ (cf. (12)-(13)) is convex with respect to $U$ since the stage cost functions $l_i(x_{i,t+k|t}, u_{i,t+k})$, $i = 1,2,\dots,M$ are convex with respect to the optimization variables $U_i = [u_{i,t}, u_{i,t+1}, \dots, u_{i,t+N}]$, $i = 1,2,\dots,M$. Therefore, the worst-case cost function:

$$\max_{\substack{[A_i,\, B_i]\in\tilde{\Omega}_i \\ i=1,2,\dots,M}} J(U, x, [A_1, B_1], \dots, [A_M, B_M]) \tag{32}$$

is a convex function too. Also, the Slater's condition always holds for a feasible convex QP (Boyd and Vandenberghe (2004)). Then, from the duality theory (Boyd and Vandenberghe (2004)) it follows that there is no duality gap between the dual problem P2 and the problem P1. The requirement $p_{t+N} = 0$ follows from the fact that there are only $N$ equality constraints of the type (20). Therefore, (31) holds. The maximum in (31) is attained when all elements of the gradient of $\sum_{i=1}^{M} V_i^*(P, x_i)$ with respect to $P$ is zero, i.e. $v_{i,t+k}^* - \sum_{j=1, j\neq i}^{M} A_{ij} x_{j,t+k|t}^* = 0$, $i = 1,\dots,M$, $k = 0,1,\dots,N-1$ (here $x_{j,t+k|t}^*$, $k = 0,1,\dots,N-1$ are the elements of the worst-case state trajectory as defined in (29)–(30)). □

According to Proposition 1, the computation of $U_i^*$ and $V_i^*$ for given prices $P$ can be done in a decentralized way, but finding the optimal prices requires coordination. The prices $P$ are found by applying the accelerated proximal gradient method to solve the dual problem to a convex primal optimization problem (Giselsson et al. (2013) and the references therein). Given a price prediction sequence $P^r = [p_t^r, \dots, p_{t+N}^r]$ for the *r*-th iteration, the corresponding sequences $U_i^{*r} = [u_{i,t}^{*r}, \dots, u_{i,t+N}^{*r}]$, $V_i^{*r} = [v_{i,t}^{*r}, \dots, v_{i,t+N}^{*r}]$ and $X_i^{*r} = [x_{i,t|t}^{*r}, \dots, x_{i,t+N|t}^{*r}]$ are computed locally by solving problem P3$^i$ and (29)–(30). Then, the prices can be updated distributedly with the following iteration for the *i*-th subsystem:

$$\begin{aligned} &q_{i,t+k}^r = p_{i,t+k}^r + \frac{r-1}{r+2}(p_{i,t+k}^r - p_{i,t+k}^{r-1}), k = 0,1,\dots,N-1 \\ &p_{i,t+k}^{r+1} = q_{i,t+k}^r + \frac{1}{L} \nabla_{P_i} S(P, x)|_{P=Q^r}, k = 0,1,\dots,N-1 \\ &\text{with } p_{i,t+N}^{r+1} = p_{i,t+N}^r = 0 \end{aligned} \tag{33}$$

where $Q^r = [Q_1^r, Q_2^r, \dots, Q_M^r]$, $Q_i^r = [q_{i,t}^r, \dots, q_{i,t+N}^r]$, $i = 1,2,\dots,M$ (with $q_{i,t+N}^r = p_{i,t+N}^r = 0$), and $S(P, x)$ is the dual function (cf. (21)):

$$S(P,x)=\min_{U,V}\max_{\substack{[A_i,B_i]\in\tilde{\Omega}_i\\ i=1,2,\dots,M}}$$

$$\sum_{k=0}^{N}\sum_{i=1}^{M}[l_i(x_{i,t+k|t},u_{i,t+k})+p_{i,t+k}^{T}(v_{i,t+k}-\sum_{\substack{j=1\\ j\neq i}}^{M}A_{ij}x_{j,t+k|t})] \tag{34}$$

The gradient of the dual function $S(P,x)$ with respect to the prices $P_i$ at $P=Q^r$ is:

$$\nabla_{P_i}S(P,x)|_{P=Q^r}=v_{i,t+k}^{*}-\sum_{\substack{j=1\\ j\neq i}}^{M}A_{ij}x_{j,t+k|t}^{*}\ ,\ k=0,1,\dots,N-1 \tag{35}$$

From (35), it can be seen that in order to compute the gradient $\nabla_{P_i}S(P,x)|_{P=Q^r}$ in (33) it is necessary to have the worst-case state trajectories $X_j^*=[x_{j,t|t}^*,x_{j,t+1|t}^*,\dots,x_{j,t+N|t}^*]$, $j=1,\dots,M$, $j\neq i$ of the interconnected subsystems, which on their hand depend on the values $P_i$ of prices.

In (33), $L$ is the Lipschitz constant to the gradient function $\nabla_P S(P,x)$. In section 3.2, an off-line algorithm to obtain an estimate of $L$ is provided.

*3.2 A suboptimal approach to distributed robust MPC based on on-line optimization*

An estimate $\hat{L}$ of the Lipschitz constant $L$ to the dual function gradient $\nabla_P S(P,x)$ is determined with the following off-line algorithm.

**Algorithm 1 (Off-line estimation of the Lipschitz constant):**

**1.** Given $x_{\min,i}$, $x_{\max,i}$, $P_{\min,i}$, $P_{\max,i}$, $i=1,2,\dots,M$, numbers $N_x$ and $N_P$.

**2. for** $j=1,\dots,N_x$ **do**

**3.** Generate a random initial state $x^j=[x_1^j,\dots,x_M^j]$ of the overall system, where $x_i^j\in[x_{\min,i},x_{\max,i}]$, $i=1,\dots,M$.

**4. for** $l=1,\dots,N_P$ **do**

**5.** Generate a random price sequence $P^l=[P_1^l,\dots,P_M^l]$, where $P_i^l\in[P_{\min,i},P_{\max,i}]$, $i=1,\dots,M$.

**6.** Compute the gradient elements $\nabla_{P_i}S(P,x)|_{x=x^j,P=P^l}$, $i=1,2,\dots,M$ according to (35). For this purpose, the QP sub-problems P3$^i$, $i=1,2,\dots,M$ are solved for initial state $x_i=x_i^j$ and price sequence $P=P^l$, and the worst-case state trajectories $X_j^{*r}=[x_{j,t|t}^{*r},x_{j,t+1|t}^{*r},\dots,x_{j,t+N|t}^{*r}]$, $j=1,\dots,M$ are determined.

**7. end**

**8. end**

**9.** Let:

$$\hat{L}=\max_{j=1,2,\dots,N_x}\ \max_{\substack{l=1,2,\dots,N_P\\ k=1,2,\dots,N_P\\ l\neq k}}\frac{\|\nabla_P S(P,x)|_{x=x^j,P=P^l}-\nabla_P S(P,x)|_{x=x^j,P=P^k}\|}{\|P^l-P^k\|}$$

The following suboptimal algorithm to distributed robust MPC of uncertain polytopic systems is proposed.

**Algorithm 2 (Distributed robust MPC by on-line optimization):**

**1.** Given an estimate $\hat{L}$ of the Lipschitz constant to the gradient $\nabla_P S(P,x)$, number $R$ of iterations, and arbitrary guesses $P_i^0$, $i=1,2,\dots,M$ for the price sequences. Let $t=0$.

**2.** Let the state at time $t$ be $x(t)=x=[x_1,\dots,x_M]$.

**3. for** $r=0,1,\dots,R-1$ **do**

**4.** For $i$-th subsystem, $i=1,2,\dots,M$, communicate the price sequences $P_j^r=[p_{j,t}^r,\dots,p_{j,t+N}^r]$, $j=1,\dots,M$, $j\neq i$ of the interconnected subsystems.

**5.** Compute the sequences $U_i^{*r}=[u_{i,t}^{*r},\dots,u_{i,t+N}^{*r}]$ and $V_i^{*r}=[v_{i,t}^{*r},\dots,v_{i,t+N}^{*r}]$ corresponding to the price sequence $P^r=[p_t^r,\dots,p_{t+N}^r]$ by solving *distributedly* the QP sub-problems P3$^i$, $i=1,2,\dots,M$. Compute the worst-case state trajectories $X_i^{*r}=[x_{i,t|t}^{*r},x_{i,t+1|t}^{*r},\dots,x_{i,t+N|t}^{*r}]$, $i=1,2,\dots,M$ from (29)–(30).

**6.** For $i$-th subsystem, $i=1,2,\dots,M$, communicate the worst-case state trajectories $X_j^{*r}=[x_{j,t|t}^{*r},\dots,x_{j,t+N|t}^{*r}]$, $j=1,\dots,M$, $j\neq i$ of the interconnected subsystems.

**7.** Compute *distributedly* the updates $P_i^{r+1}=[p_{i,t}^{r+1},\dots,p_{i,t+N}^{r+1}]$, $i=1,2,\dots,M$ of the price sequences by applying (33) with $L=\hat{L}$ and using (35).

**8. end**

**9.** Let $P_i^0=P_i^R$, $i=1,2,\dots,M$.

**10.** Apply to the overall system the control inputs $u_i(t)=u_{i,t}^{*R-1}$, $i=1,2,\dots,M$.

**11.** Let $t=t+1$ and go to step 2.

*3.3 Remarks on the robust stability and performance*

Let $U_i^{*r}=[u_{i,t}^{*r},\dots,u_{i,t+N}^{*r}]$, $i=1,2,\dots,M$ be the solutions to the problems P3$^i$, $i=1,2,\dots,M$, corresponding to the price prediction sequence $P^r=[p_t^r,\dots,p_{t+N}^r]$ for the $r$-th iteration and $U^{*r}=[u_t^{*r},u_{t+1}^{*r}\dots,u_{t+N}^{*r}]$ be the overall control input sequence. According to the receding horizon strategy, the MPC law is $u_{\mathrm{MPC}}(t)=u_t^{*r}$. Then the state evolution of the uncertain closed-loop system is described by:

$$x(t+1)=(A(t)+\tilde{A})x(t)+B(t)u_{\mathrm{MPC}}(t),\ [A(t),\ B(t)]\in\Omega \tag{36}$$

with $\tilde{A}$ defined in (17). Given the initial state $x(0)=\bar{x}$ of the system (36), we define the following infinite horizon worst-case cost for the feedback law $u_{\mathrm{MPC}}(t)$:

$$V_{\mathrm{MPC}}^{\infty}(\bar{x})=\max_{[A(t),\ B(t)]\in\Omega}\sum_{t=0}^{\infty}l(x(t),u_{\mathrm{MPC}}(t)) \tag{37}$$

where $l(\cdot,\cdot)$ is defined by (13). Also, let:

$$V^{\infty}(\bar{x})=\min_{u}\max_{[A(t),\ B(t)]\in\Omega}\sum_{t=0}^{\infty}l(x(t),u(t)) \tag{38}$$

where the evolution of the state is obtained according to (16). Based on the results of the relaxed dynamic programming approach (Grüne and Rantzer (2008)), it can be claimed that if for a given performance parameter $\alpha \in (0, 1)$ the optimal value of the worst-case cost (as defined in (6)) decreases in the following way:

$$V^*(x(t)) \geq V^*(x(t+1)) + \alpha l(x(t), u_{\text{MPC}}(t)) \quad \forall [A(t),\ B(t)] \in \Omega \tag{39}$$

for every $t \geq 0$, then the closed-loop system (36) will be robustly asymptotically stable with worst-case performance that satisfies:

$$\alpha V_{\text{MPC}}^{\infty}(\bar{x}) \leq V^{\infty}(\bar{x}) \tag{40}$$

For the purpose of the robust MPC design, it can be assumed that the uncertain pair $[A(t),\ B(t)]$ does not change at time $t+1$ and it can be required for the inequality (39) to hold only for a finite set of uncertain pairs, i.e. for $[A(t),\ B(t)] \in \tilde{\Omega} = \tilde{\Omega}_1 \times \tilde{\Omega}_2 \times \ldots \times \tilde{\Omega}_M$. Then in Algorithm 2, instead of performing the steps 4 to 7 for the preliminary specified number of iterations, a modified version of the stopping criterion in Giselsson and Rantzer (2010) can be incorporated so as to ensure the satisfaction of (39) with the mentioned assumptions.

## 4. EXAMPLE

Consider the following system composed of two interconnected polytopic subsystems $S_1$ and $S_2$:

$$\begin{aligned} S_1:\ & x_1(t+1) = A_1(t)x_1(t) + B_1u_1(t) + A_{12}x_2(t),\ A_1(t) \in \Omega_1 \\ S_2:\ & x_2(t+1) = A_2(t)x_2(t) + B_2u_2(t) + A_{21}x_1(t),\ A_2(t) \in \Omega_2 \end{aligned} \tag{41}$$

where:

$$\begin{aligned} A_1(t) &= \begin{bmatrix} \alpha_1(t) & -0.09 \\ 0.17 & 0.79 \end{bmatrix},\ \alpha_1(t) \in [0.43,\ 0.83] \\ A_2(t) &= \begin{bmatrix} \alpha_2(t) & -0.09 \\ 0.17 & 0.69 \end{bmatrix},\ \alpha_2(t) \in [0.53,\ 0.93] \\ B_1 &= \begin{bmatrix} 0.06 \\ 0.01 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.07 \\ 0.01 \end{bmatrix};\ A_{12} = A_{21} = \begin{bmatrix} 0 & 0 \\ 0 & 0.1 \end{bmatrix} \end{aligned} \tag{42}$$

Here, $\alpha_1$ and $\alpha_2$ are uncertain parameters. The sets $\Omega_1$ and $\Omega_2$ have two vertices corresponding to $\alpha_1 = 0.43$, $\alpha_1 = 0.83$ and $\alpha_2 = 0.53$, $\alpha_2 = 0.93$, respectively. The finite sets $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ are defined as:

$$\begin{aligned} \tilde{\Omega}_1 &= \{[A_1(\alpha_1), B_1],\ \alpha_1 \in \{0.43,\ 0.53,\ 0.63,\ 0.73,\ 0.83\}\} \\ \tilde{\Omega}_2 &= \{[A_2(\alpha_2), B_2],\ \alpha_2 \in \{0.53,\ 0.63,\ 0.73,\ 0.83,\ 0.93\}\} \end{aligned} \tag{43}$$

The following state and input constraints are imposed on the system (41):

$$\begin{bmatrix} -0.1 \\ -0.1 \end{bmatrix} \leq x_i(t),\ -2 \leq u_i(t) \leq 2,\ i = 1,2 \tag{44}$$

The prediction horizon is $N = 5$ and the weighting matrices are $W_{x_i} = I$, $W_{u_i} = 0.01$, $i = 1,2$. The centralized robust MPC problem (problem P1) is represented as a distributed robust MPC problem (problem P2) by applying the dual decomposition approach. The estimate of the Lipschitz constant to the dual function gradient, obtained with Algorithm 1, is $\hat{L} = 2.117$. Then, Algorithm 2 with number of iterations $R = 10$ is used to generate the two control inputs for an initial state of the overall system $x(0) = [2\ \ 2\ \ 2\ \ 2]$. The simulations are performed for the variations of the uncertain parameters, shown in Fig. 1. The computed trajectories of the control inputs $u_1, u_2$ and the states $x_1^1, x_1^2$ and $x_2^1, x_2^2$, associated to the subsystems $S_1$ (i.e. $x_1 = [x_1^1,\ x_1^2]$) and $S_2$ (i.e. $x_2 = [x_2^1,\ x_2^2]$) are depicted in Fig. 2 to Fig. 4. They are compared with the trajectories corresponding to the centralized robust MPC. The results show that the suboptimal trajectories obtained with the distributed robust MPC keep both the state and input constraints and differ only slightly from the centralized MPC trajectories.

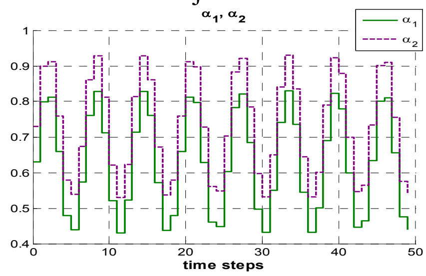


Fig. 1. The variation of parameters $\alpha_1, \alpha_2$.

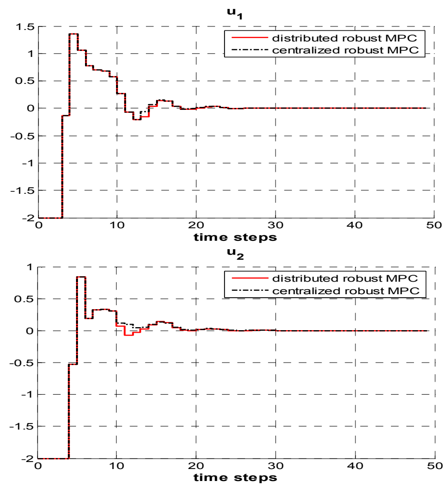


Fig. 2. The control inputs $u_1, u_2$.

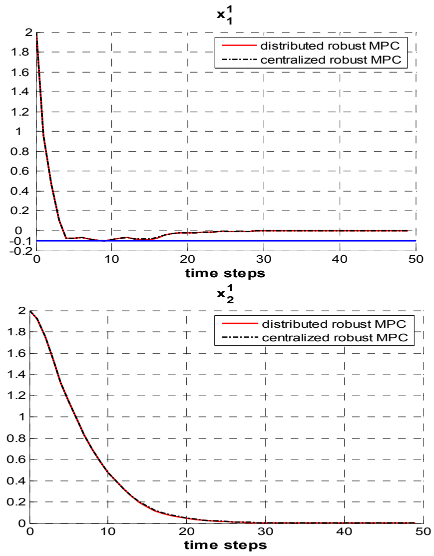


Fig. 3. The states $x_2^1, x_2^2$ of subsystem $S_1$.

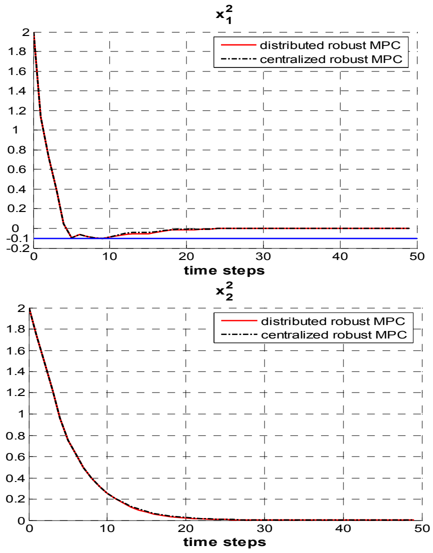


Fig. 4. The states $x_2^1$, $x_2^2$ of subsystem $S_2$.

## 5. CONCLUSIONS

In this paper, a suboptimal approach to distributed robust MPC for uncertain systems consisting of polytopic constrained subsystems is proposed and its performance is illustrated with a numerical example. A further extension of the approach would include incorporation of a stopping criterion that will ensure the robust stability and performance of the closed-loop system, as well as consideration of the network-induced constraints (cf. Hespanha et al. (2007), Zhang et al. (2013)), associated to a networked control system structure.

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