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H_∞ Control of Discrete-Time Stochastic State-multiplicative Systems Constrained in State by Equality Constraints

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Abstract: Design conditions for the existence of the H_∞ state feedback control for discrete-time stochastic systems with state-multiplicative noise stabilizing the closed-loop in such a way that the state variables satisfy equality constraints in the mean are presented in the paper. Using an enhanced form of the bounded real lemma for discrete-time stochastic systems with state-multiplicative noise, the LMI-based procedure is provided for computation of the gain matrix of state control law and the influence of equality constraints is explained. The approach is illustrated on example demonstrating the validity of the proposed method.

Keywords: Discrete-time linear stochastic systems, multiplicative noise, state control, equality constraints, Lyapunov function, matrix formulation.

1. INTRODUCTION

In the recent decades, many significant results have spurred interest in the problem of determining the control laws for the systems with constraints. In the typical case (Benzaouia and Gurgat (1998)) where a system state reflects certain physical entities, this class of constraints rises because of physical limits and these constraints usually keep the system state in the region of technological conditions. Subsequently, this problem can be formulated using a technique dealing with the state constraints directly to be coped efficiently using modified linear techniques (Ko and Bitmead (2007)). Notably, a special form of the problems was defined while the system state variables satisfy constraints (Hahn (1992)) and are interpreted as descriptor systems. Because a system with state equality constraints generally does not satisfy the conditions under which the results of descriptor systems can be applicable, this approach is very limited. If the design task is interpreted as a singular problem, associated methods can be developed to design the controller parameters (Filasová and Krokavec (2010), Filasová and Krokavec (2012b)).

In principle, it is possible and ever easy to apply direct design methods, namely to design a controller that stabilizes the systems while the system state variables satisfy equality constraints. Following the idea of linear quadratic (LQ) control, such a technique has been introduced in Ko and Bitmead (2007) and was extensively used in the reconfigurable control design (Filasová and Krokavec (2012a), Krokavec and Filasová (2009)). Direct extension to control design for stochastic discrete-time systems with state-multiplicative noise and the state variables tied up in the mean by equality constraints (Krokavec and Filasová (2011)) is very conservative, since the design conditions given in terms of linear matrix inequalities (LMI) are ill conditioned and have to be regularized.

Considering new results in system control (He et al. (2005)) and in bounded real lemma forms for discrete-time stochastic systems (El Bouhtouri et al. (1999), Gershon and Shaked (2008), Gershon and Shaked (2013b)), new design conditions for H_∞ state-constrained control based on an enhanced form of the bounded real lemma for linear discrete-time stochastic systems with state-multiplicative noise are derived in the paper. To present this, the paper is divided in these sections. Following the introduction in Sec. 1, the control task tying up the state variables in the mean by equality constraints for discrete-time stochastic systems with state-multiplicative noise is presented in Sec. 2. The preliminary results focused on two bounded real lemma forms for such stochastic systems are presented in Sec. 3, and Sec. 4 provides the controller design conditions in the equivalent forms of LMIs. Subsequently, in Sec. 5 there are derived the design conditions for H_∞ control with the mean state equality constraints for the considered stochastic systems. Sec. 6 illustrates the constrained control design task by a numerical solution and Sec. 7 draws some conclusions.

Throughout the paper, the notations are narrowly standard in such a way that $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $\boldsymbol{X} = \boldsymbol{X}^T > 0$, (≥ 0), means that $\boldsymbol{X}$ is a symmetric positive definite (semidefinite) matrix, $rank(\,\cdot\,)$ remits the rank of a matrix, the symbol $\boldsymbol{I}_n$ indicates the n-th order identity matrix, $\mathcal{Z}_+$ is the set of all positive integers, $I\!R$ denotes the set of real numbers, $I\!R^{n\times r}$ refers to the set of all $n \times r$ real matrices and $L_2\langle 0, +\infty)$ entails the space of square summable discrete vector random sequence over $\langle 0, +\infty)$.

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2. PROBLEM FORMULATION

Throughout the paper, the task is concerned with state feedback design to control the stochastic discrete-time linear dynamic system given by the set of equations

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} + \boldsymbol{U}o(i))\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{u}(i) + \boldsymbol{V}\boldsymbol{v}(i) \tag{1}$$

$$\boldsymbol{y}(i) = \boldsymbol{C}\boldsymbol{q}(i) \tag{2}$$

where $\boldsymbol{q}(i) \in I\!R^n$, $\boldsymbol{u}(i) \in I\!R^r$, $\boldsymbol{y}(i) \in I\!R^m$ is the state, input, and output vector, respectively, $\boldsymbol{v}(i) \in I\!R^{r_v}$ is an exogenous disturbance vector, and $\boldsymbol{F} \in I\!R^{n\times n}$, $\boldsymbol{G} \in I\!R^{n\times r}$, $\boldsymbol{C} \in I\!R^{m\times n}$, $\boldsymbol{V} \in I\!R^{n\times r_v}$, $\boldsymbol{U} \in I\!R^{n\times n}$ are real matrices.

It is assumed that the multiplicative noise $o(i)$, $0 \le i \le j$ satisfies the condition

$$E_o\{o(i)\} = 0, \quad E_o\{o(i)o(j)\} = \delta_{ij} \tag{3}$$

where $E_o\{\cdot\}$ denotes expectation with respect to $o(i)$ and δ_{ij} is the Kronecker delta function. Disturbance vector is a non-anticipative process, where $\{\boldsymbol{v}(i)\} \in L_2(\langle 0,\infty); I\!R^{r_v})$.

The problem of the interest is to design in the mean square sense stable and in the state variables constrained a closed-loop system by the state feedback controller of the form

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) \tag{4}$$

where $\boldsymbol{K} \in I\!R^{r\times n}$ is the feedback controller gain matrix, the design constraint takes the equality form

$$E_o\{\boldsymbol{q}(i+1)\} \in \mathcal{N}_{\boldsymbol{L}} = \{\boldsymbol{q} : \boldsymbol{L}\boldsymbol{q} = \boldsymbol{0}\} \text{ for all } i \in \mathcal{Z}_+ \tag{5}$$

and $\text{rank}(\boldsymbol{L}) = p < r$ (Krokavec and Filasová (2011)).

3. BASIC PRELIMINARIES

Proposition 1. (Matrix pseudoinverse) If $\boldsymbol{\Theta}$ is a matrix variable and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{\Lambda}$ are known non-square matrices of the appropriate dimensions such the equality

$$\boldsymbol{A}\boldsymbol{\Theta}\boldsymbol{B} = \boldsymbol{\Lambda} \tag{6}$$

can be set, then all solution to $\boldsymbol{\Theta}$ means

$$\boldsymbol{\Theta} = \boldsymbol{A}^{\ominus 1}\boldsymbol{\Lambda}\boldsymbol{B}^{\ominus 1} + \boldsymbol{\Theta}^{\circ} - \boldsymbol{A}^{\ominus 1}\boldsymbol{A}\boldsymbol{\Theta}^{\circ}\boldsymbol{B}\boldsymbol{B}^{\ominus 1} \tag{7}$$

where $\boldsymbol{A}^{\ominus 1}$ and $\boldsymbol{B}^{\ominus 1}$ is Moore-Penrose pseudoinverse of $\boldsymbol{A}$, $\boldsymbol{B}$, respectively, and $\boldsymbol{\Theta}^{\circ}$ is an arbitrary matrix of appropriate dimension (see, e.g., Boyd et al. (1994)).

Proposition 2. If $\boldsymbol{H} \in I\!R^{n\times n}$ is a real square matrix with non-repeated eigenvalues, satisfying the constraint

$$\boldsymbol{e}^T\boldsymbol{H} = \boldsymbol{0} \tag{8}$$

then one from its eigenvalues is zero, and (normalized) $\boldsymbol{e}^T$ is the left raw eigenvector of $\boldsymbol{H}$ associated with the zero eigenvalue (see, e.g., Filasová and Krokavec (2012b)).

In order to create more convenient space for new design task conditions, two modifications of the stability solution for stochastic systems (Gershon and Shaked (2008) and El Ghaoui (1995), Hinrichsen and Pritchard (1998), Gao and Wang (2004), respectively) are presented first. Since these modifications constitute the first stage in the solution of the considered problem, to the best of authors' belief, inclusion of the complete proofs was an easier way than to interpret in detail how to adapt the original bounded real lemma (BRL) formulation for the discrete-time stochastic systems with multiplicative noise.

Lemma 1. (BRL for discrete-time stochastic systems with multiplicative noise) The unforced system (1), (2) is stable in the mean square sense and with the quadratic performance γ if there exists a positive definite matrix $\boldsymbol{P} \in I\!R^{n\times n}$ and a positive scalar $\gamma \in I\!R$ such that

$$\boldsymbol{P} = \boldsymbol{P}^T > 0, \qquad \gamma > 0 \tag{9}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * & * \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} & * & * & * \\ \boldsymbol{P}\boldsymbol{F} & \boldsymbol{P}\boldsymbol{V} & -\boldsymbol{P} & * & * \\ \boldsymbol{P}\boldsymbol{U} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{P} & * \\ \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{10}$$

Here, and hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof. Let the Lyapunov function candidate take the form

$$p(\boldsymbol{q}(i)) = \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \sum_{j=0}^{i-1}\big(\boldsymbol{y}^T(j)\boldsymbol{y}(j) - \gamma\boldsymbol{v}^T(j)\boldsymbol{v}(j)\big) \tag{11}$$

where $\gamma \in I\!R$ is square of the H_∞ norm of the transfer function matrix defined for the disturbance input $\boldsymbol{v}$ and the system output $\boldsymbol{y}$. Taking expectation with respect to $o(i)$ it yields

$$\begin{aligned} &E_o\{p(\boldsymbol{q}(i+1))\} - p(\boldsymbol{q}(i)) = \\ &= E_o\{\boldsymbol{q}^T(i+1)\boldsymbol{P}\boldsymbol{q}(i+1)\} - \boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i) + \\ &+\boldsymbol{y}^T(i)\boldsymbol{y}(i) - \gamma\boldsymbol{v}^T(i)\boldsymbol{v}(i) \end{aligned} \tag{12}$$

Solving for the deterministically unforced system, then

$$\begin{aligned} &E_o\{\boldsymbol{q}^T(i+1)\boldsymbol{P}\boldsymbol{q}(i+1)\} = \boldsymbol{v}^T(i)\boldsymbol{V}^T\boldsymbol{P}\boldsymbol{V}\boldsymbol{v}(i) + \\ &+\boldsymbol{q}^T(i)\big(\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F} + E_o\{o^2(i)\}\boldsymbol{U}^T\boldsymbol{P}\boldsymbol{U}\big)\boldsymbol{q}(i) + \\ &+\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{V}\boldsymbol{v}(i) + \boldsymbol{v}^T(i)\boldsymbol{V}^T\boldsymbol{P}\boldsymbol{F}\boldsymbol{q}(i) \end{aligned} \tag{13}$$

and, substituting (2) and (13) into (12), then also

$$\begin{aligned} &E_o\{p(\boldsymbol{q}(i+1))\} - p(\boldsymbol{q}(i)) = \\ &= \boldsymbol{q}^T(i)(-\boldsymbol{P} + \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F} + \boldsymbol{U}^T\boldsymbol{P}\boldsymbol{U} + \boldsymbol{C}^T\boldsymbol{C})\boldsymbol{q}(i) + \\ &+\boldsymbol{q}^T(i)\boldsymbol{F}^T\boldsymbol{P}\boldsymbol{V}\boldsymbol{v}(i) + \boldsymbol{v}^T(i)\boldsymbol{V}^T\boldsymbol{P}\boldsymbol{F}\boldsymbol{q}(i) + \\ &+\boldsymbol{v}^T(i)\boldsymbol{V}^T\boldsymbol{P}\boldsymbol{V}\boldsymbol{v}(i) - \gamma\boldsymbol{v}^T(i)\boldsymbol{v}(i) \end{aligned} \tag{14}$$

Defining the composed vector

$$\boldsymbol{q}^{\circ T}(i) = \begin{bmatrix} \boldsymbol{q}^T(i) & \boldsymbol{v}^T(i) \end{bmatrix} \tag{15}$$

the inequality

$$\boldsymbol{q}^{\circ T}(i)\boldsymbol{P}^{\circ}\boldsymbol{q}^{\circ}(i) < 0 \tag{16}$$

it is satisfied only if

$$\boldsymbol{P}^{\circ} = \begin{bmatrix} \boldsymbol{P}^{\circ}_{11} & \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{V} \\ \boldsymbol{V}^T\boldsymbol{P}\boldsymbol{F} & -\gamma\boldsymbol{I}_{r_v} + \boldsymbol{V}^T\boldsymbol{P}\boldsymbol{V} \end{bmatrix} < 0 \tag{17}$$

$$\boldsymbol{P}^{\circ}_{11} = -\boldsymbol{P} + \boldsymbol{F}^T\boldsymbol{P}\boldsymbol{F} + \boldsymbol{U}^T\boldsymbol{P}\boldsymbol{U} + \boldsymbol{C}^T\boldsymbol{C} \tag{18}$$

Writing (17) as follows

$$\begin{aligned} &\begin{bmatrix} -\boldsymbol{P} + \boldsymbol{U}^T\boldsymbol{P}\boldsymbol{U} + \boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} \end{bmatrix} + \\ &+ \begin{bmatrix} \boldsymbol{F}^T\boldsymbol{P} \\ \boldsymbol{V}^T\boldsymbol{P} \end{bmatrix} \boldsymbol{P}^{-1} \begin{bmatrix} \boldsymbol{P}\boldsymbol{F} & \boldsymbol{P}\boldsymbol{V} \end{bmatrix} < 0 \end{aligned} \tag{19}$$

and using the Schur complement property, (19) can be rewritten in the following form

$$\begin{bmatrix} -\boldsymbol{P}+\boldsymbol{U}^T\boldsymbol{P}\boldsymbol{U}+\boldsymbol{C}^T\boldsymbol{C} & \boldsymbol{0} & \boldsymbol{F}^T\boldsymbol{P} \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} & \boldsymbol{V}^T\boldsymbol{P} \\ \boldsymbol{P}\boldsymbol{F} & \boldsymbol{P}\boldsymbol{V} & -\boldsymbol{P} \end{bmatrix} < 0 \tag{20}$$

Applying twice the Schur complement property, (20) implies (10). This concludes the proof. ■

Lemma 2. (enhanced BRL for discrete-time stochastic systems with multiplicative noise) The deterministically unforced system (1), (2) is stable in the mean square sense and with the quadratic performance γ if there exist positive definite matrices $\boldsymbol{P},\boldsymbol{Q} \in I\!R^{n\times n}$ and a positive scalar $\gamma \in I\!R$ such that

$$\boldsymbol{P}=\boldsymbol{P}^T>0, \quad \boldsymbol{Q}=\boldsymbol{Q}^T>0, \quad \gamma>0 \tag{21}$$

$$\begin{bmatrix} -\boldsymbol{P} & * & * & * & * \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} & * & * & * \\ \boldsymbol{Q}\boldsymbol{F} & \boldsymbol{Q}\boldsymbol{V} & \boldsymbol{P}-2\boldsymbol{Q} & * & * \\ \boldsymbol{Q}\boldsymbol{U} & \boldsymbol{0} & \boldsymbol{0} & -0.5\boldsymbol{Q} & * \\ \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{22}$$

Proof. Since the unforced system (1) implies

$$\boldsymbol{F}\boldsymbol{q}(i)+\boldsymbol{U}o(i)\boldsymbol{q}(i)+\boldsymbol{V}\boldsymbol{v}(i)-\boldsymbol{q}(i{+}1)=\boldsymbol{0} \tag{23}$$

then, with an arbitrary symmetric positive definite matrix $\boldsymbol{Q} \in I\!R^{n\times n}$, it yields

$$\big(\boldsymbol{q}^T(i{+}1)+\boldsymbol{q}^T(i)\boldsymbol{U}^To(i)\big)\boldsymbol{Q}\begin{pmatrix}\boldsymbol{F}\boldsymbol{q}(i)+\boldsymbol{U}o(i)\boldsymbol{q}(i)+\\ +\boldsymbol{V}\boldsymbol{v}(i)-\boldsymbol{q}(i{+}1)\end{pmatrix}=\boldsymbol{0} \tag{24}$$

$$\begin{aligned} E_o\{\boldsymbol{q}^T(i{+}1)\}\boldsymbol{Q}\big(\boldsymbol{F}\boldsymbol{q}(i)+\boldsymbol{V}\boldsymbol{v}(i)-E_o\{\boldsymbol{q}(i{+}1)\}\big)+\\ +\boldsymbol{q}^T(i)\}\boldsymbol{U}^T\boldsymbol{Q}\boldsymbol{U}\boldsymbol{q}(i)=\boldsymbol{0}\end{aligned} \tag{25}$$

respectively. Adding (25) and its transposition to (12) gives

$$\begin{aligned} &E_o\{p(\boldsymbol{q}(i{+}1))\}-p(\boldsymbol{q}(i))=\\ &=E_o\{\boldsymbol{q}^T(i{+}1)\}\boldsymbol{P}E_o\{\boldsymbol{q}(i{+}1)\}+\\ &+E_o\{\boldsymbol{q}^T(i{+}1)\}\boldsymbol{Q}\big(\boldsymbol{F}\boldsymbol{q}(i)+\boldsymbol{V}\boldsymbol{v}(i)-E_o\{\boldsymbol{q}(i{+}1)\}\big)+\\ &+\big(\boldsymbol{F}\boldsymbol{q}(i)+\boldsymbol{V}\boldsymbol{v}(i)-E_o\{\boldsymbol{q}(i{+}1)\}\big)^T\boldsymbol{Q}E_o\{\boldsymbol{q}(i{+}1)\}+\\ &+2\boldsymbol{q}^T(i)\}\boldsymbol{U}^T\boldsymbol{Q}\boldsymbol{U}\boldsymbol{q}(i)-\gamma\boldsymbol{v}^T(i)\boldsymbol{v}(i)+\\ &+\boldsymbol{q}^T(i)\boldsymbol{C}^T\boldsymbol{C}\boldsymbol{q}(i)-\boldsymbol{q}^T(i)\boldsymbol{P}\boldsymbol{q}(i)<0\end{aligned} \tag{26}$$

Defining the composed vector

$$\boldsymbol{q}^{\bullet T}(i)=\big[\,\boldsymbol{q}^T(i)\;\boldsymbol{v}^T(i)\;E_o\{\boldsymbol{q}^T(i{+}1)\}\,\big] \tag{27}$$

then (26) can be written as

$$\boldsymbol{q}^{\bullet T}(i)\boldsymbol{P}^{\bullet}\boldsymbol{q}^{\bullet}(i)<0 \tag{28}$$

where

$$\boldsymbol{P}^{\bullet}=\begin{bmatrix} -\boldsymbol{P}+\boldsymbol{C}^T\boldsymbol{C}+2\boldsymbol{U}^T\boldsymbol{Q}\boldsymbol{U} & \boldsymbol{0} & \boldsymbol{F}^T\boldsymbol{Q} \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} & \boldsymbol{V}^T\boldsymbol{Q} \\ \boldsymbol{Q}\boldsymbol{F} & \boldsymbol{Q}\boldsymbol{V} & \boldsymbol{P}-2\boldsymbol{Q} \end{bmatrix} < 0 \tag{29}$$

To find the LMI form, (29) can be rewritten using Schur complement property as (22). This concludes the proof. ■

The inequality (22) is an enhanced representation of BRL for the given class of stochastic systems. It is linear with respect to the system variables but does not involve any product of the Lyapunov matrix $\boldsymbol{P}$ and the system matrices $\boldsymbol{F}$, $\boldsymbol{V}$. This offers its possibility to be applied in a singular task analysis.

4. STATE FEEDBACK CONTROL

This section addresses the problem of finding the state-feedback control law (4) that stabilizes the system (1), (2) and achieves the optimal level of unknown input disturbance attenuation.

Theorem 1. The control (4) to the system (1), (2) exists if there exists a symmetric positive definite matrix $\boldsymbol{X} \in I\!R^{n\times n}$, a matrix $\boldsymbol{Y} \in I\!R^{r\times n}$ and a positive scalar $\gamma \in I\!R$ such that

$$\boldsymbol{X}=\boldsymbol{X}^T>0, \quad \gamma>0 \tag{30}$$

$$\begin{bmatrix} -\boldsymbol{X} & * & * & * & * \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} & * & * & * \\ \boldsymbol{F}\boldsymbol{X}-\boldsymbol{G}\boldsymbol{Y} & \boldsymbol{V} & -\boldsymbol{X} & * & * \\ \boldsymbol{U}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{X} & * \\ \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{31}$$

When the above conditions hold,

$$\boldsymbol{K}=\boldsymbol{Y}\boldsymbol{X}^{-1} \tag{32}$$

Proof. Replacing $\boldsymbol{F}$ in (10) by the closed-loop system matrix

$$\boldsymbol{F}_c=\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K} \tag{33}$$

and defining the transform matrix

$$\boldsymbol{T}_A=\operatorname{diag}\big[\,\boldsymbol{P}^{-1}\;\boldsymbol{I}_{r_v}\;\boldsymbol{P}^{-1}\;\boldsymbol{P}^{-1}\;\boldsymbol{I}_m\,\big] \tag{34}$$

then, pre-multiplying and post-multiplying the left-hand and the right-hand side of (10) by (34), it yields

$$\begin{bmatrix} -\boldsymbol{P}^{-1} & * & * & * & * \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} & * & * & * \\ (\boldsymbol{F}-\boldsymbol{G}\boldsymbol{K})\boldsymbol{P}^{-1} & \boldsymbol{V} & -\boldsymbol{P}^{-1} & * & * \\ \boldsymbol{U}\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{P}^{-1} & * \\ \boldsymbol{C}\boldsymbol{P}^{-1} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{35}$$

Thus, with the notations

$$\boldsymbol{X}=\boldsymbol{P}^{-1}, \quad \boldsymbol{Y}=\boldsymbol{K}\boldsymbol{P}^{-1} \tag{36}$$

(35) implies (31). This concludes the proof. ■

Considering the enhanced BRL form (21), (22), the next design condition can be obtained.

Theorem 2. The control (4) to the system (1), (2) exists if there exist symmetric positive definite matrices $\boldsymbol{X},\boldsymbol{T} \in I\!R^{n\times n}$, a matrix $\boldsymbol{Y} \in I\!R^{r\times n}$ and a positive scalar $\gamma \in I\!R$ such that

$$\boldsymbol{X}=\boldsymbol{X}^T>0, \quad \boldsymbol{T}=\boldsymbol{T}^T>0 \quad \gamma>0 \tag{37}$$

$$\begin{bmatrix} -\boldsymbol{T} & * & * & * & * \\ \boldsymbol{0} & -\gamma\boldsymbol{I}_{r_v} & * & * & * \\ \boldsymbol{F}\boldsymbol{X}-\boldsymbol{G}\boldsymbol{Y} & \boldsymbol{V} & \boldsymbol{T}-2\boldsymbol{X} & * & * \\ \boldsymbol{U}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -0.5\boldsymbol{X} & * \\ \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{38}$$

When the above conditions hold, the control law gain matrix is given as

$$\boldsymbol{K}=\boldsymbol{Y}\boldsymbol{X}^{-1} \tag{39}$$

Proof. Since it was supposed that $\boldsymbol{Q}$ is a positive definite matrix, defining so the transform matrix

$$\boldsymbol{T}_B = \text{diag}\left[\, \boldsymbol{Q}^{-1} \;\boldsymbol{I}_{r_v} \;\boldsymbol{Q}^{-1} \;\boldsymbol{Q}^{-1} \;\boldsymbol{I}_m \,\right] \tag{40}$$

then, substituting (33) in (22) and subsequently pre-multiplying the left-hand side and the right-hand side of (22) by (40) results in

$$\begin{bmatrix} -\boldsymbol{Q}^{-1}\boldsymbol{P}\boldsymbol{Q}^{-1} & * & * & * & * \\ \boldsymbol{0} & -\gamma \boldsymbol{I}_{r_v} & * & * & * \\ \boldsymbol{F}\boldsymbol{Q}^{-1}-\boldsymbol{G}\boldsymbol{K}\boldsymbol{Q}^{-1} & \boldsymbol{V} & \boldsymbol{P}^{\bullet}_{33} & * & * \\ \boldsymbol{U}\boldsymbol{Q}^{-1} & \boldsymbol{0} & \boldsymbol{0} & -0.5\boldsymbol{Q}^{-1} & * \\ \boldsymbol{C}\boldsymbol{Q}^{-1} & \boldsymbol{0} & \boldsymbol{0} & 0 & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{41}$$

$$\boldsymbol{P}^{\bullet}_{33} = \boldsymbol{Q}^{-1}\boldsymbol{P}\boldsymbol{Q}^{-1} - 2\boldsymbol{Q}^{-1} \tag{42}$$

Thus, with the notations

$$\boldsymbol{T} = \boldsymbol{Q}^{-1}\boldsymbol{P}\boldsymbol{Q}^{-1}, \quad \boldsymbol{X} = \boldsymbol{Q}^{-1}, \quad \boldsymbol{Y} = \boldsymbol{K}\boldsymbol{Q}^{-1} \tag{43}$$

(41) implies (38). This concludes the proof. ∎

5. CONSTRAINED STATE CONTROL

In practice (Cakmakci and Ulsoy (2009), Debiane et al. (2004)), the deterministic ratio control can be used to maintain the relationship between two state variables q_h, q_k, defined compactly as

$$\boldsymbol{l}_h^T \boldsymbol{q}(i{+}1) = 0 \tag{44}$$

where $a_{hk} \in I\!R$ is a positive ratio value and

$$\boldsymbol{l}_h^T = [\, 0_1 \;\cdots\; 1_h \;\cdots\; -a_{hk} \;\cdots\; 0_n \,] \tag{45}$$

The task formulation above means that the problem of the interest is generally defined as an equality constrain conditioned closed-loop system design of the state feedback control law (4), where $\boldsymbol{L}$ in (5) reflects a prescribed fixed ratios of two or more state variables.

Using the control law (4), the equilibrium control equation takes the form

$$\boldsymbol{q}(i+1) = (\boldsymbol{F} + \boldsymbol{U}o(i) - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{V}\boldsymbol{v}(i) \tag{46}$$

and, considering the design constraint (5), it can be set

$$\begin{aligned} &E_o\{\boldsymbol{L}\boldsymbol{q}(i{+}1)\} = \\ &= \boldsymbol{L}\boldsymbol{V}\boldsymbol{v}(i) + \boldsymbol{L}(\boldsymbol{F}{-}\boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) = \boldsymbol{L}\boldsymbol{V}\boldsymbol{v}(i) \end{aligned} \tag{47}$$

where it is supposed that the matrix $\boldsymbol{L}$ is introduced in such a way that

$$\boldsymbol{L}(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \boldsymbol{0} \quad \Rightarrow \quad \boldsymbol{L}\boldsymbol{F} = \boldsymbol{L}\boldsymbol{G}\boldsymbol{K} \tag{48}$$

as well as that the closed-loop system matrix (33) is stable.

Therefore, $\mathcal{N}_{\boldsymbol{L}}$ is the constrain subspace, the states will be constrained in this subspace, and the system state stays within the constrain subspace (see, e.g., Ko and Bitmead (2007), Krokavec and Filasová (2009)).

Since (7) implies all solutions of $\boldsymbol{K}$ as

$$\boldsymbol{K} = (\boldsymbol{L}\boldsymbol{G})^{\ominus 1}\boldsymbol{L}\boldsymbol{F} + (\boldsymbol{I}_m - (\boldsymbol{L}\boldsymbol{G})^{\ominus 1}\boldsymbol{L}\boldsymbol{G})\boldsymbol{K}^{\circ} \tag{49}$$

where $\boldsymbol{K}^{\circ}$ is an arbitrary matrix of the appropriate dimension and $(\boldsymbol{L}\boldsymbol{G})^{\ominus 1}$ is the pseudoinverse of $\boldsymbol{L}\boldsymbol{G}$, then (4) results in the control law

$$\boldsymbol{u}(i) = -\boldsymbol{M}\boldsymbol{q}(i) + \boldsymbol{N}\boldsymbol{u}^{\circ}(i) \tag{50}$$

where

$$\boldsymbol{u}^{\circ}(i) = -\boldsymbol{K}^{\circ}\boldsymbol{q}(i) \tag{51}$$

$$\boldsymbol{M} = (\boldsymbol{L}\boldsymbol{G})^{\ominus 1}\boldsymbol{L}\boldsymbol{F}, \quad \boldsymbol{N} = \boldsymbol{I}_m - (\boldsymbol{L}\boldsymbol{G})^{\ominus 1}\boldsymbol{L}\boldsymbol{G} \tag{52}$$

The task is to design the feedback control gain matrix $\boldsymbol{K}^{\circ}$ in such a way that the closed-loop is stable and (29) is satisfied for (33).

Since (8) implies that such a design problem is a singular task, the design conditions are derived using Theorem 2.

Theorem 3. (design condition for an in state constrained control) Given the system (1), (2), the equality constraint (5) and the matrices (52), then the constrained control of the form (50), (51) exists if there exist symmetric positive definite matrices $\boldsymbol{X}, \boldsymbol{T} \in I\!R^{n\times n}$, a matrix $\boldsymbol{Y} \in I\!R^{r\times n}$ and a positive scalar $\gamma \in I\!R$ such that

$$\boldsymbol{X} = \boldsymbol{X}^T > 0, \quad \boldsymbol{T} = \boldsymbol{T}^T > 0 \quad \gamma > 0 \tag{53}$$

$$\begin{bmatrix} -\boldsymbol{T} & * & * & * & * \\ \boldsymbol{0} & -\gamma \boldsymbol{I}_{r_v} & * & * & * \\ \boldsymbol{F}^{\circ}\boldsymbol{X}-\boldsymbol{G}^{\circ}\boldsymbol{Y} & \boldsymbol{V} & \boldsymbol{T}-2\boldsymbol{X} & * & * \\ \boldsymbol{U}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & -0.5\boldsymbol{X} & * \\ \boldsymbol{C}\boldsymbol{X} & \boldsymbol{0} & \boldsymbol{0} & 0 & -\boldsymbol{I}_m \end{bmatrix} < 0 \tag{54}$$

where

$$\boldsymbol{F}^{\circ} = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{M}, \quad \boldsymbol{G}^{\circ} = \boldsymbol{G}\boldsymbol{N} \tag{55}$$

When the above conditions hold, the control law gain matrices are given as

$$\boldsymbol{K}^{\circ} = \boldsymbol{Y}\boldsymbol{X}^{-1}, \quad \boldsymbol{K} = \boldsymbol{M} + \boldsymbol{N}\boldsymbol{K}^{\circ} \tag{56}$$

Proof. Substituting (50) into (1) gives

$$\begin{aligned} &\boldsymbol{q}(i+1) = \\ &= (\boldsymbol{F}{-}\boldsymbol{G}\boldsymbol{M}{+}\boldsymbol{U}o(i))\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{N}\boldsymbol{u}^{\circ}(i) + \boldsymbol{V}\boldsymbol{v}(i) \end{aligned} \tag{57}$$

and using (55) then (57) can be rewritten as

$$\boldsymbol{q}(i+1) = (\boldsymbol{F}^{\circ} + \boldsymbol{U}a(i))\boldsymbol{q}(i) + \boldsymbol{G}^{\circ}\boldsymbol{u}^{\circ}(i) + \boldsymbol{V}\boldsymbol{v}(i) \tag{58}$$

Since the structure of (58) is the same as of (1), following the similar approach as used in the proof of Theorem 2, the equivalent inequality for (58) can be obtained by inserting $\boldsymbol{F} \leftarrow \boldsymbol{F}^{\circ}$, $\boldsymbol{G} \leftarrow \boldsymbol{G}^{\circ}$ into (31). This concludes the proof. ∎

Remark 1. Since

$$\begin{aligned} \boldsymbol{F}_c^{\circ} &= \boldsymbol{F}^{\circ} - \boldsymbol{G}^{\circ}\boldsymbol{K}^{\circ} = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{M} - \boldsymbol{G}\boldsymbol{N}\boldsymbol{K}^{\circ} = \\ &= \boldsymbol{F} - \boldsymbol{G}\boldsymbol{M} - \boldsymbol{G}(\boldsymbol{K} + \boldsymbol{M}) = \boldsymbol{F} - \boldsymbol{G}\boldsymbol{K} = \boldsymbol{F}_c \end{aligned} \tag{59}$$

the eigenvalues spectra of the closed-loop system matrices $\boldsymbol{F}_c^{\circ}$, $\boldsymbol{F}_c$ are equal. As (8), (48) implies the closed-loop system matrix $\boldsymbol{F}_c$ is singular, a singular problem has to be solved.

Because plants generally do not work in autonomous mode, the constrained forced mode is also analyzed in the following.

Definition 1. The forced regime for the system (1), (2) is given by the control policy

$$\boldsymbol{u}(i) = -\boldsymbol{K}\boldsymbol{q}(i) + \boldsymbol{W}\boldsymbol{w}(i) \tag{60}$$

where $r = m$, $\boldsymbol{w}(i) \in I\!R^m$ is a desired output signal vector, and $\boldsymbol{W} \in I\!R^{m\times m}$ is the signal gain matrix.

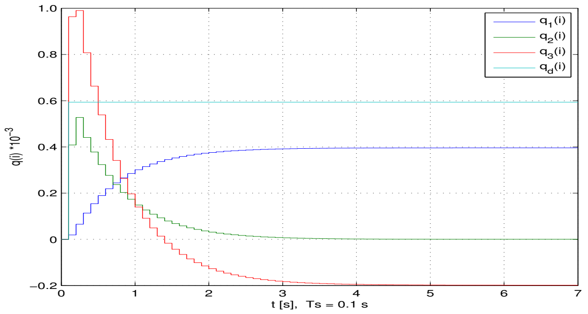

Fig. 1. Response of the noise- and disturbance-free system ($q_d = \boldsymbol{L}\boldsymbol{W}\boldsymbol{w}_s$)

If the pair $(\boldsymbol{F}, \boldsymbol{G})$ of (1), (2) is controllable, the pair $(\boldsymbol{F}, \boldsymbol{C})$ is observable and

$$\text{rank}\begin{bmatrix} \boldsymbol{F} & \boldsymbol{G} \\ \boldsymbol{C} & \boldsymbol{0} \end{bmatrix} = n + m \tag{61}$$

then the matrix $\boldsymbol{W}$ designed by using the static decoupling principle (see, e.g., Wang (2003)) takes the form

$$\boldsymbol{W} = \left(\boldsymbol{C}\left(\boldsymbol{I}_n - (\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K})\right)^{-1}\boldsymbol{G}\right)^{-1} \tag{62}$$

Theorem 4. Using the control policy (60) designed with respect to the equality constrain (48), the controlled system state variables satisfy at all time instants the condition

$$\boldsymbol{q}_d(i) = \boldsymbol{L}E_o\{\boldsymbol{q}(i+1)\} = \boldsymbol{L}\boldsymbol{G}\boldsymbol{W}\boldsymbol{w}(i) + \boldsymbol{L}\boldsymbol{V}\boldsymbol{v}(i) \tag{63}$$

Proof. Since a forced motion of (1), (2) can be written as

$$\begin{aligned} &\boldsymbol{q}(i+1) = \\ &(\boldsymbol{F} + \boldsymbol{U}o(i) - \boldsymbol{G}\boldsymbol{K})\boldsymbol{q}(i) + \boldsymbol{G}\boldsymbol{W}_w\boldsymbol{w}(i) + \boldsymbol{V}\boldsymbol{v}(i) \end{aligned} \tag{64}$$

applying (47), so (64) implies (63). This concludes the proof. ∎

It is important to note that this theorem is a direct generalization of the technique presented in Filasová and Krokavec (2012b).

6. ILLUSTRATIVE EXAMPLE

To demonstrate the control properties, the system described by state-space equations (1), (2) was considered where the sampling period was $t_s = 0.1s$ and

$$\boldsymbol{F} = \begin{bmatrix} 0.9993 & 0.0987 & 0.0042 \\ -0.0212 & 0.9612 & 0.0775 \\ -0.3875 & -0.7187 & 0.5737 \end{bmatrix}, \ \boldsymbol{C} = \begin{bmatrix} 0 & 0.1 & 0.1 \\ 0.1 & 0 & 0 \end{bmatrix}$$

$$\boldsymbol{U} = \begin{bmatrix} 0 & 0 & 0.0004 \\ 0 & 0 & 0 \\ 0.0388 & 0 & 0 \end{bmatrix}, \ \boldsymbol{G} = \begin{bmatrix} 0.0051 & 0.0050 \\ 0.1029 & 0.0987 \\ 0.0387 & -0.0388 \end{bmatrix}$$

$$\boldsymbol{V}^T = [\,1 \ 3 \ 7\,], \quad \sigma_o^2 = \sigma_v^2 = 10^{-3}$$

Because the design conditions are tied to the requirement (3), in the following the state, input and output variables as well as disturbance are normalized by the norm factor $n_o = 10^3$.

The space variables constrain was specified as

$$\frac{2q_1(t) + q_3(t)}{q_2(t)} = 1$$

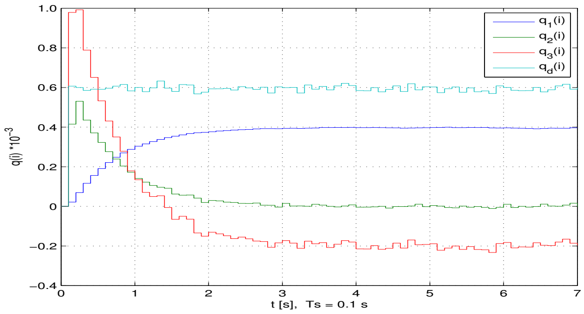

Fig. 2. Response of the system with multiplicative noise and disturbance ($q_d = \boldsymbol{L}E_o\{\boldsymbol{q}(i+1)\} = \boldsymbol{L}\boldsymbol{W}\boldsymbol{w}_s + \boldsymbol{L}\boldsymbol{V}\boldsymbol{v}(i)$)

which implies

$$\boldsymbol{L} = [\,2 \ -1 \ 1\,]$$

and subsequently it is

$$\boldsymbol{M} = \begin{bmatrix} -4.5975 & 4.1756 & -1.4212 \\ -10.8552 & 9.8590 & -3.3557 \end{bmatrix}$$

$$\boldsymbol{N} = \begin{bmatrix} 0.8479 & -0.3591 \\ -0.3591 & 0.1521 \end{bmatrix}$$

Solving (53), (54) with respect to the LMI variables $\boldsymbol{X}$, $\boldsymbol{T}$, $\boldsymbol{Y}$ and γ using Self-Dual-Minimization (SeDuMi) package for MATLAB (Peaucelle et al. (2002)), the feedback gain matrix design problem in the state-constrained stochastic system control was tackled with the results

$$\boldsymbol{X} = \begin{bmatrix} 0.7022 & -0.5322 & -1.2200 \\ -0.5322 & 9.9907 & 3.7945 \\ -1.2200 & 3.7945 & 14.3577 \end{bmatrix}, \quad \gamma = 17.4838$$

$$\boldsymbol{T} = \begin{bmatrix} 0.7319 & -0.4465 & -1.0576 \\ -0.4465 & 9.4713 & 1.0427 \\ -1.0576 & 1.0427 & 11.2210 \end{bmatrix}$$

$$\boldsymbol{Y} = \begin{bmatrix} 23.0884 & -3.4118 & 3.0856 \\ -9.7773 & 1.4452 & -1.3059 \end{bmatrix}$$

Using (56), the feedback gain matrices are

$$\boldsymbol{K}^\circ = \begin{bmatrix} 39.1693 & 0.4438 & 3.4258 \\ -16.5870 & -0.1879 & -1.4507 \end{bmatrix}$$

$$\boldsymbol{K} = \begin{bmatrix} 34.5710 & 4.6194 & 2.0045 \\ -27.4442 & 9.6710 & -4.8066 \end{bmatrix}$$

The control law for the above defined insures in the mean square sense the closed-loop dynamics determined by the system matrix eigenvalues spectrum

$$\rho(\boldsymbol{F} - \boldsymbol{G}\boldsymbol{K}) = \{0.0000 \ 0.8525 \ -0.0513\,\}$$

Note that one eigenvalue of $\boldsymbol{F}_c$ is zero (rank$(\boldsymbol{L})$ = 1)) as Preliminary 2 implies that the state-constrained design task is a singular problem.

The simulation presents the closed-loop system state variables properties in the forced mode. The desired steady-state values of signal, the signal gain matrix and the initial system state vector were set as

$$\boldsymbol{w}_s = \begin{bmatrix} -0.2 \\ 0.4 \end{bmatrix}, \boldsymbol{W} = \begin{bmatrix} 70.2546 & 395.8506 \\ -107.7310 & -324.2188 \end{bmatrix}, \boldsymbol{q}(0) = \begin{bmatrix} 0.0 \\ -0.1 \\ 0.0 \end{bmatrix}$$

It can be easily verified that (63) is satisfied in such way a that

$$\boldsymbol{q}_{ds} = L\boldsymbol{W}\boldsymbol{w}_s = 0.5936$$

(see the common variable $q_d(i)$ in the Fig. 1 and Fig. 2). To improve the visualization, the values of variables are displayed in SH mode.

Note that, as a rule, the power of a desired signal must be greater than the mean power of the disturbance and noise to satisfy the prescribed equality constraints.

7. CONCLUDING REMARKS

The paper presents the control design principle for discrete-time linear stochastic multi-variable dynamic systems with state-multiplicative noise where the state variables are tied up in the mean by equality constraints. The stability of the control scheme in the mean square sense and with the quadratic performance bounded by H_∞ norm of the disturbance transfer function matrix is established using the enhanced representation of the bounded real lemma for discrete-time stochastic systems with state-multiplicative noise to circumvent singular design task, where the resulting LMIs do not involve any product of the Lyapunov matrix and the system matrices. This provides a suitable way for determination of state control by solving this singular LMI problems. It is determined, however that the just found control law solves the problem in the constrained forced regime.

The proposed method poses the problem as a stabilization problem with a state feedback controller whose gain matrix takes no special structure. Comparing with the previous results, the number of assumptions is reduced while a singular solution is guaranteed and no iteration steps are needed. The presented applications can be considered as a task concerned the class of H_∞ stabilization control problems where the conditions were newly formulated. This formulation allows to find a solution to the control law without restrictive assumptions and additional specifications of the design parameters.

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