

# Discrete-Time Immersion and Invariance Adaptive Control of a Slider-crank Mechanism

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**Abstract:** A discrete-time adaptive nonlinear control procedure is developed based on immersion and invariance control, and using back-stepping for the regulation of slider-crank system. A convenient parametric formulation of the system dynamics is established. A novel parameter estimator design is presented, and the back-stepping controller is constructed considering the certainty equivalence principle.

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## 1. INTRODUCTION

The slider-crank mechanism has a very wide usage in machine design. Some of the applications are found in internal combustion engines, in electrical switch gears, packaging and textile engineering (Lin et al. 1997, Ha et al. 2006). It is mainly used to convert rotary motion to a reciprocating motion or vice versa. Several control techniques have been presented for the system in the literature by sliding mode control (Fung and Chang, 2009), energy control (Komaïta and Furuta, 2008), generalized minimum variance control (Saitoh and Furuta, 2007) and computed torque technique (Lin et al. 1998).

Most of the mechanisms in the engineering applications consist of motors, gears and nonlinear components. The control performance of these systems is usually dominated by the effects of nonlinearities and uncertainties. Therefore, the nonlinear adaptive design procedure for the case study considered in this paper may be applicable for a wider range of nonlinear electro-mechanical systems.

On the other hand, to develop modelling and control techniques for discrete-time nonlinear systems have gained more importance as in the engineering practice computer-controlled systems are preferred because of the simplicity and flexibility of their implementation.

Immersion and Invariance (I&I) control is a recently presented new methodology in the literature proposed by (Astolfi *et al.*, 2007, Astolfi *et al.*, 2008, Karagiannis and Astolfi 2008) for designing nonlinear and adaptive controllers for nonlinear systems which relies upon the notion of system immersion and manifold invariance. In I&I control approach the behaviour that satisfies the control objective is formulated as a target system and the controller is constructed such that the plant dynamics immersed into the target dynamics.

In (Yalcin and Astolfi 2011, 2012) a discrete-time adaptive controller design via state-feedback for the adaptive regulation of linearly parameterised discrete-time systems in strict-feedback form is presented. In this paper, the adaptive regulation of the slider-crank mechanism is considered and a discrete time adaptive nonlinear control procedure is

proposed based on immersion and invariance control approach and using back-stepping. For the purpose, firstly, a parametric model for the system dynamics is obtained. The system dynamics are not directly linear in parameters but it is expressed in a form such that the intended control procedure can be developed utilizing it. Then, a parameter estimator is designed. Afterwards the back-stepping controller is constructed considering certainty equivalence principle.

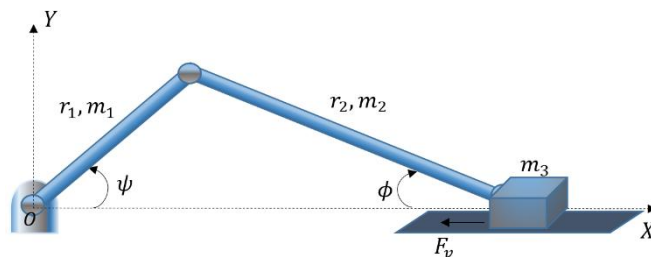


Fig. 1. Slider-crank systems

## 2. DERIVATION OF SYSTEM DYNAMIC EQUATIONS

In this section the dynamic equations for the considered crank-slide mechanism shown in Figure 1 are derived by the Euler-Lagrange technique and these equations are expressed in a form in the next section such that an immersion and invariance adaptive control algorithm can be developed and a back-stepping controller can be constructed.

In Figure 1,  $r_1$  and  $r_2$  are length of the arms,  $m_1$ ,  $m_2$  and  $m_3$  masses of the arms and the slider, respectively. The slider-crank system is a 2-dof under-actuated mechanical system. The actuation is performed through the base joint. The angular motion of base joint is transformed to a linear motion in the slider so that the slider moves accordingly.

Let's write the kinetic energy function of the system,

$$K = \frac{1}{2} I_1 \dot{\psi}^2 + \frac{1}{2} I_2 \dot{\phi}^2 + \frac{1}{2} m_2 \dot{\phi}^2 + \frac{1}{2} m_3 \dot{\phi}^2 \quad (1)$$

where  $I_1$  and  $I_2$  are the inertias of arms,  $\mathcal{G}_2$  is the linear velocity of the second arm and  $\mathcal{G}_3$  is linear velocity of the mass  $m_3$ . Besides, consider the potential energy of the system as

$$P = m_1 g \frac{r_1}{2} \sin \psi + m_2 g \frac{r_2}{2} \sin \phi. \quad (2)$$

Note that the following relation is valid for the angles  $\psi$ ,  $\phi$ ,

$$r_1 \sin \psi = r_2 \sin \phi. \quad (3)$$

Therefore,

$$r_1 \dot{\psi} \cos \psi = r_2 \dot{\phi} \cos \phi, \text{ namely } \dot{\phi} = \frac{r_1 \dot{\psi} \cos \psi}{r_2 \cos \phi}. \quad (4)$$

Now consider  $x_{c2} = r_1 \cos \psi + \frac{r_2}{2} \cos \phi$  and  $y_{c2} = \frac{r_2}{2} \sin \phi$  together with (3), then

$$\bar{\mathcal{G}}_2 = \mathcal{G}_{2x} i + \mathcal{G}_{2y} j \quad (5)$$

where

$$\mathcal{G}_{2x} = \dot{x}_{c2} = -r_1 \dot{\psi} \sin \psi - \frac{r_2}{2} \dot{\phi} \sin \phi = -r_1 \dot{\psi} \sin \psi - \frac{r_2}{2} \dot{\phi} \sin \psi, \quad (6)$$

$$\mathcal{G}_{2y} = \dot{y}_{c2} = \frac{r_2}{2} \dot{\phi} \cos \phi = \frac{r_2}{2} \dot{\psi} \cos \psi \quad (7)$$

therefore,

$$\mathcal{G}_2^2 = r_1^2 \sin^2 \psi (\dot{\psi} + \frac{1}{2} \dot{\phi})^2 + \frac{r_2^2}{4} \dot{\phi}^2 \cos^2 \phi. \quad (8)$$

Here note that, using  $\cos^2 \phi = 1 - \sin^2 \phi$  and relations (3-4), the following expression for  $\dot{\phi}$  which is utilized in the sequel is obtained.

$$\dot{\phi} = \frac{r_1 \dot{\psi} \cos \psi}{\sqrt{r_2^2 - r_1^2 \sin^2 \psi}} \text{ equivalently } \dot{\phi}^2 = \frac{r_1^2 \dot{\psi}^2 \cos^2 \psi}{r_2^2 - r_1^2 \sin^2 \psi}. \quad (9)$$

Thus  $\mathcal{G}_2^2$  can be written as,

$$\begin{aligned} \mathcal{G}_2^2 &= r_1^2 \sin^2 \psi (\dot{\psi} + \frac{1}{2} \dot{\phi})^2 + \frac{r_2^2}{4} \dot{\phi}^2 \cos^2 \phi \\ &= r_1^2 \sin^2 \psi \dot{\psi}^2 + \frac{r_1^3 \dot{\psi}^2}{c} \sin^2 \psi \cos \psi + \frac{r_1^2 r_2^2 \dot{\psi}^2}{4c^2} \cos^2 \psi \\ &= r_1^2 \dot{\psi}^2 [\sin^2 \psi (1 + \frac{r_1}{c} \cos \psi) + \frac{r_2^2}{4c^2} \cos^2 \psi] \end{aligned} \quad (10)$$

where

$$c = \sqrt{r_2^2 - r_1^2 \sin^2 \psi}$$

as in (Ha et al. 2006).

Finally, since  $x_{c3} = x = r_1 \cos \psi + r_2 \cos \phi$  and  $y_{c3} = 0$ , then

$$\begin{aligned} \mathcal{G}_3 &= \dot{x} = -r_1 \dot{\psi} \sin \psi - r_2 \dot{\phi} \sin \phi \\ &= -r_1 \dot{\psi} \sin \psi - r_1 \dot{\phi} \sin \psi \\ &= -r_1 \dot{\psi} \sin \psi (1 + \frac{r_1}{c} \cos \psi) \end{aligned} \quad (11)$$

and when (9) is considered,

$$\begin{aligned} \mathcal{G}_3^2 &= \dot{x}^2 = r_1^2 \sin^2 \psi (\dot{\psi} + \dot{\phi})^2 \\ &= r_1^2 \dot{\psi}^2 \sin^2 \psi + \frac{2r_1^3 \dot{\psi}^2}{c} \sin^2 \psi \cos \psi \\ &\quad + \frac{r_1^4 \dot{\psi}^2}{c^2} \sin^2 \psi \cos^2 \psi. \end{aligned} \quad (12)$$

Moreover, the potential energy of the system can be rewritten as,

$$P = m_1 g \frac{r_1}{2} \sin \psi + m_2 g \frac{r_2}{2} \sin \phi = (m_1 + m_2) g \frac{r_1}{2} \sin \psi. \quad (13)$$

Thus, the Lagrangian for the system is obtained as follows,

$$\begin{aligned} L = K - P &= \frac{1}{2} I_1 \dot{\psi}^2 + \frac{1}{2} I_2 \frac{r_1^2 \dot{\psi}^2 \cos^2 \psi}{c^2} \\ &\quad + \frac{1}{2} m_2 r_1^2 \dot{\psi}^2 [\sin^2 \psi (1 + \frac{r_1}{c} \cos \psi) + \frac{r_2^2}{4c^2} \cos^2 \psi] \\ &\quad - \frac{1}{2} m_3 r_1 \dot{\psi} \sin \psi (1 + \frac{r_1}{c} \cos \psi) - (m_1 + m_2) g \frac{r_1}{2} \sin \psi. \end{aligned} \quad (14)$$

Hence, considering the inertias as  $I_1 = \frac{1}{\lambda_1} m_1 r_1^2$  and

$I_2 = \frac{1}{\lambda_2} m_2 r_2^2$  to capture the various arm-joint structures and

uncertainties, and  $F_v = -\text{sgn}(\dot{x}) f_v \dot{x} = -\text{sgn}(\dot{x}) \mu_v m_3 g \dot{x}$  the Lagrange Equations of the system is obtained as given below,

$$\begin{aligned} \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\psi}} \right) - \frac{\partial L}{\partial \psi} &= (k_1 + P_1(\psi)) \cos^2 \psi + P_2(\psi) \sin^2 \psi \dot{\psi} \\ &\quad + [(2k_3 \sin \psi \cos^2 \psi \dot{\psi} - 2P_1(\psi) \dot{\psi} + 2P_2(\psi)) \sin \psi \cos \psi \\ &\quad - k_5 \dot{\psi} \sin^3 \psi - \dot{\psi} \sin \psi P_3(\psi)] \dot{\psi} + k_{10} \cos \psi = \tau - F_v \end{aligned} \quad (15)$$

Here,  $\tau$  is the torque applied to base joint,  $F_v$  is the viscous friction,  $\mu_v$  is the viscous friction coefficient,  $\lambda_1$ ,  $\lambda_2$  are positive scalars and

$$\begin{aligned} P_1(\psi) &= k_2 + k_3 \sin^2 \psi, \\ P_2(\psi) &= k_4 + k_5 \cos \psi, \end{aligned} \quad (16)$$

$$\begin{aligned} P_3(\psi) &= k_6 \cos \psi (c + r_1^2 \cos^2 \psi) + k_4 \cos \psi + k_7 (2 \cos^2 \psi \\ &\quad - \sin^2 \psi) - k_8 \cos \psi + k_9 \cos \psi (\cos^2 \psi - \sin^2 \psi), \end{aligned}$$

$$F_v = -\text{sgn}(\dot{x}) f_v \dot{x} = -k_{11} \text{sgn}(\dot{x}) \sin \psi (1 + \frac{r_1}{c} \cos \psi) \dot{\psi},$$

with

$$k_1 = \frac{m_1 r_1^2}{\lambda_1}, \quad k_2 = \frac{5m_2 r_2^2 r_1^2}{4c^2}, \quad k_3 = \frac{r_1^4 m_3}{c^2}, \quad k_4 = r_1^2 (m_2 + m_3),$$

$$k_5 = \frac{r_1^3 (m_2 + 2m_3)}{c}, \quad k_6 = -\frac{r_1^2 r_2^2 m_2}{c^4 \lambda_2}, \quad k_7 = \frac{r_1^3}{2c} (m_2 + m_3), \quad (17)$$

$$k_8 = \frac{m_2 r_1^2 r_2^2}{4c^2}, \quad k_9 = \frac{m_3 r_1^4}{2c}, \quad k_{10} = (m_1 + m_2) g \frac{r_1}{2},$$

$$k_{11} = f_v r_1 = \mu_v m_3 g r_1.$$

Note that, the equations (15)-(17) can be written in the form of

$$A(\psi)\ddot{\psi} + B(\psi, \dot{\psi})\dot{\psi} + C(\psi) = \tau \quad (18)$$

where

$$A(\psi) = k_1 + P_1(\psi) \cos^2 \psi + P_2(\psi) \sin^2 \psi,$$

$$B(\psi, \dot{\psi}) = (2k_3 \sin \psi \cos^2 \psi \dot{\psi} - 2P_1(\psi)\dot{\psi} + 2P_2(\psi)) \sin \psi \cos \psi$$

$$- k_5 \dot{\psi} \sin^3 \psi - \dot{\psi} \sin \psi P_3(\psi) + k_{11} \operatorname{sgn}(\dot{x}) \sin \psi \times (1 + \frac{r_1}{c} \cos \psi),$$

$$C(\psi) = k_{10} \cos \psi.$$

Finally, considering (18) for  $x_1 = \psi$ ,  $x_2 = \dot{\psi}$ , the dynamic equations of the system can be written as,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{B(x_1, x_2)}{A(x_1)} x_2 - \frac{C(x_1)}{x_1 A(x_1)} x_1 + v. \quad (19)$$

where  $\tau = A(x_1)v$ .

### 3. CONTROLLER DESIGN

In this section, a discrete-time adaptive nonlinear control procedure is developed. The nonlinear system (23) can be considered as a time varying linear system. If all the parameters are known, an adaptive linear controller can be designed. In this case  $A(\theta)$ ,  $B(\theta, \dot{\theta})$  and  $C(\theta)$  can be calculated at each sample and the linear controller to be designed can be accordingly adapted. If there is uncertainties in the parameters of the system, a nonlinear adaptive controller is needed or preferred to design, since it may not be possible to directly calculate the values of A, B and C polynomials.

Here, it is assumed that  $r_1$  and  $r_2$  are known and  $m_1, m_2, m_3$ ,  $I_1$  and  $I_2$  (namely  $\lambda_1$  and  $\lambda_2$ ) are unknowns and a discrete-time adaptive nonlinear controller based on immersion and invariance control approach is designed.

Therefore, let's rewrite system dynamics (19) in the following form,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{\hat{\Phi}_{2BC}(x_1, x_2)\theta}{\hat{\Phi}_{2A}(x_1, x_2)\theta} + v, \quad (20)$$

with

$$\theta = [\hat{k}_1 \quad \hat{k}_2 \quad \hat{k}_3 \quad \hat{k}_4 \quad \hat{k}_5]^T \quad (21)$$

where

$$\hat{\Phi}_{2A}(x_1) = [r_1^2, \frac{5r_2^2 r_1^2}{4c^2} \cos^2 x_1 + r_1^2 \sin^2 x_1 (1 + \frac{r_1}{c} \cos x_1),$$

$$r_1^2 \sin^2 x_1 (\frac{r_1^2}{c^2} \cos^2 x_1 + 1) + \frac{2r_1^3}{c} \cos x_1 \sin^2 x_1, 0, 0] \quad (22)$$

and,

$$\hat{\Phi}_{2BC}(x_1, x_2) = [0, \frac{r_1^2 x_2}{2c^2} \sin x_1 \cos x_1 (-5r_2^2 x_2 + 2c^2(2 - x_2))$$

$$+ 2cr_1(1 - x_2) \cos x_1 + \frac{c^2 r_2^2 x_2^2}{2} + \frac{r_1^3 x_2^2}{2c} \sin^3 x_1,$$

$$r_1 \cos x_1 \sin x_1 (\frac{2r_1^3 x_2^2}{c^2} \sin x_1 \cos^2 x_1 + r_1 x_2 (2 - x_2))$$

$$- 2x_2^2 \sin^2 x_1 + \frac{2r_1^2}{c} x_2 \cos x_1 - \frac{r_1^3 x_2^2}{c} (\sin^2 x_1$$

$$- \cos^2 x_1) + (2 - x_2)x_2 \cos x_1 + \frac{3r_1^3 x_2^2}{2c} \sin^3 x_1$$

$$+ \mu_v g r_1 \operatorname{sgn}(\dot{x}) \sin \psi (1 + \frac{r_1}{c} \cos \psi), \frac{r_1^2 r_2^2 x_2^2}{c^4} \times$$

$$\sin x_1 \cos x_1 (c + r_1^2 \cos^2 x_1), \frac{r_1}{2} g \cos x_1].$$

Note that,

$$\hat{k}_1 = \frac{m_1}{\lambda_1}, \quad \hat{k}_2 = m_2, \quad \hat{k}_3 = m_3, \quad \hat{k}_4 = \frac{m_2}{\lambda_2}, \quad \hat{k}_5 = m_1 + m_2,$$

and,

$$k_1 = \hat{k}_1 r_1^2, \quad k_2 = \hat{k}_2 \frac{5r_2^2 r_1^2}{4c^2}, \quad k_3 = \hat{k}_3 \frac{r_1^4}{c^2}, \quad k_4 = (\hat{k}_2 + \hat{k}_3) r_1^2,$$

$$k_5 = (\hat{k}_2 + 2\hat{k}_3) \frac{r_1^3}{c}, \quad k_6 = -\hat{k}_4 \frac{r_2^2}{c^4}, \quad k_7 = (\hat{k}_2 + \hat{k}_3) \frac{r_1}{2c}, \quad k_8 = \hat{k}_2 \frac{r_2^2}{4c^2},$$

$$k_9 = \hat{k}_3 \frac{r_1^2}{2c}, \quad k_{10} = g \frac{r_1}{2} \hat{k}_5, \quad k_{11} = \hat{k}_3 \mu_v g r_1.$$

The discrete-time dynamics of the system is required to be able to perform a discrete-time design procedure. Forward Euler approximation to derivatives is used to obtain discrete-time dynamics of the system as given below,

$$x_1^+ = x_1 + T x_2$$

$$x_2^+ = x_2 + T \frac{\hat{\Phi}_{2BC}(x_1, x_2)\theta}{\hat{\Phi}_{2A}(x_1, x_2)\theta} + T v. \quad (24)$$

The following notation,

$$x_1 = \tilde{x}_1, \quad x_2 = \frac{\tilde{x}_2 - \tilde{x}_1}{T},$$

yields

$$\begin{aligned}\tilde{x}_1^+ &= \tilde{x}_2 \\ \tilde{x}_2^+ &= T^2 \frac{\hat{\Phi}_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta}{\hat{\Phi}_{2A}(\tilde{x}_1, \tilde{x}_2)\theta} + 2\tilde{x}_2 - \tilde{x}_1 + T^2 v,\end{aligned}\quad (25)$$

and when  $u = 2\tilde{x}_2 - \tilde{x}_1 + T^2 v$ ,  $\Phi_{2BC} = T^2 \hat{\Phi}_{2BC}$  and  $\Phi_{2A} = \hat{\Phi}_{2A}$  are considered, the expression for the discrete-time dynamics are obtained in the following form,

$$\begin{aligned}\tilde{x}_1^+ &= \tilde{x}_2 \\ \tilde{x}_2^+ &= \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta} + u.\end{aligned}\quad (26)$$

In the sequel, an indirect adaptive controller construction is derived using immersion and invariance control approach. For the estimation of the parameters, following estimation error is considered.

$$z = \hat{\theta} - \theta + \gamma \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \theta \beta(\tilde{x}_1^-) \tilde{x}, \quad (27)$$

where  $0 < \gamma < 1$  and is such that  $\bar{\sigma}(\gamma \beta(\tilde{x}_1^-) x \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-)) < 1$ , namely  $(I - \gamma \beta(\tilde{x}_1^-) x \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-))^{-1}$  exists. Note that,

$$\theta = (I - \gamma \beta(\tilde{x}_1^-) \tilde{x} \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-))^{-1} (\hat{\theta} - z), \quad (28)$$

and for  $\Gamma = (I - \gamma \beta(\tilde{x}_1^-) x \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-))^{-1}$ , it is rewritten as

$$\theta = \Gamma (\hat{\theta} - z).$$

hence parameter estimations are obtained as

$$\theta_{est} = \Gamma \hat{\theta}. \quad (29)$$

Also note that,

$$z^+ = \hat{\theta}^+ - \theta + \gamma \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \theta \beta(\tilde{x}_1) \tilde{x}^+. \quad (30)$$

Therefore,

$$\begin{aligned}z^+ - \alpha z &= \hat{\theta}^+ + (1 - \alpha)\theta + \gamma \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \theta \beta(\tilde{x}_1) \tilde{x}^+ - \alpha \hat{\theta} \\ &\quad - \alpha \gamma \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \theta \beta(\tilde{x}_1^-) \tilde{x} \\ &= \hat{\theta}^+ + (1 - \alpha)\theta + \gamma \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \theta \beta(\tilde{x}_1) \begin{bmatrix} \tilde{x}_1^+ \\ \tilde{x}_2^+ \end{bmatrix} - \alpha \hat{\theta} \\ &\quad - \alpha \gamma \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \theta \beta(\tilde{x}_1^-) \tilde{x}, \\ &= \hat{\theta}^+ - \alpha \hat{\theta} + \gamma \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \theta \beta(\tilde{x}_1) \begin{bmatrix} \tilde{x}_2 \\ \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta} + u \end{bmatrix} \\ &\quad - \alpha \gamma \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \theta \beta(\tilde{x}_1^-) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + (1 - \alpha)\theta, \\ &= \hat{\theta}^+ - \alpha \hat{\theta} + \gamma \beta(\tilde{x}_1) \begin{bmatrix} \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \theta \tilde{x}_2 \\ \Phi_{2BC}(\tilde{x}_1, \tilde{x}_2) \theta + \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \theta u \end{bmatrix} \\ &\quad - \alpha \gamma \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \theta \beta(\tilde{x}_1^-) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + (1 - \alpha)\theta,\end{aligned}$$

$$\begin{aligned}&= \hat{\theta}^+ - \alpha \hat{\theta} + \gamma \beta(\tilde{x}_1) \begin{bmatrix} \tilde{x}_2 \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \Gamma (\hat{\theta} - z), \\ \Phi_{2BC}(\tilde{x}_1, \tilde{x}_2) \Gamma (\hat{\theta} - z) \\ + u \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \Gamma (\hat{\theta} - z) \end{bmatrix} \\ &\quad - \alpha \gamma \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \Gamma (\hat{\theta} - z) \beta(\tilde{x}_1^-) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + (1 - \alpha) \Gamma (\hat{\theta} - z),\end{aligned}$$

is obtained. Let's select the update law as

$$\begin{aligned}\hat{\theta}^+ &= \alpha \hat{\theta} - \gamma \beta(\tilde{x}_1) \begin{bmatrix} \tilde{x}_2 \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \Gamma \hat{\theta} \\ \Phi_{2BC}(\tilde{x}_1, \tilde{x}_2) \Gamma \hat{\theta} + u \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \Gamma \hat{\theta} \end{bmatrix} \\ &\quad + \alpha \gamma \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \Gamma \hat{\theta} \beta(\tilde{x}_1^-) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} + (\alpha - 1) \Gamma \hat{\theta}.\end{aligned}\quad (31)$$

Then the estimation error dynamics becomes

$$\begin{aligned}z^+ &= \alpha z - \gamma \beta(\tilde{x}_1) \begin{bmatrix} \tilde{x}_2 \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \Gamma \\ \Phi_{2BC}(\tilde{x}_1, \tilde{x}_2) \Gamma + u \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \Gamma \end{bmatrix} z \\ &\quad + \alpha \gamma \beta(\tilde{x}_1^-) \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-) \Gamma z + (\alpha - 1) \Gamma z.\end{aligned}\quad (32)$$

Using the following notation

$$\Phi = \begin{bmatrix} \tilde{x}_2 \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \\ \Phi_{2BC}(\tilde{x}_1, \tilde{x}_2) + u \Phi_{2A}(\tilde{x}_1, \tilde{x}_2) \end{bmatrix}, \quad (33)$$

$$\Pi = \begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \end{bmatrix} \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-), \quad (34)$$

(34) can be rewritten as,

$$z^+ = (\alpha I - \gamma \beta(\tilde{x}_1) \Phi \Gamma + \alpha \gamma \beta(\tilde{x}_1^-) \Pi \Gamma + (\alpha - 1) \Gamma) z. \quad (35)$$

Let's select,

$$\beta = \beta(x_1) = (I + M^{+T} M^+ + K_1^+ + K_2^+)^{-1} M^{+T} \quad (36)$$

and

$$\begin{aligned}0 < \alpha < 1 - \frac{0.5\kappa_1}{1 + \bar{\sigma}(\Gamma)}, \quad 0 < \gamma < \frac{0.5\kappa_2}{1 + \alpha}, \\ 0 < \kappa_1, \kappa_2 < 1, \quad \kappa_1 > \kappa_2,\end{aligned}\quad (37)$$

where

$$M = \gamma x \Phi_{2A}(\tilde{x}_1^-, \tilde{x}_2^-), \quad (38)$$

$$K_1^+ = M^T M + \bar{\sigma}(K_1) I_p + K_2, \quad K_1(0) \geq 0, \quad (39)$$

$$K_2 = \bar{\sigma}(M^T \Pi) I_p + \bar{\sigma}(M^{+T} \Phi) I_p. \quad (40)$$

Note that,

$$\beta = \beta(\tilde{x}_1) = (R^+)^{-1} M^{+T} \quad \text{and} \quad \beta^- = \beta(\tilde{x}_1^-) = R^{-1} M^T$$

where

$$R = I + M^T M + K_1 + K_2, \quad (41)$$

$$L = I + K_1 + K_2, \quad (42)$$

hence,

$$z^+ = (\alpha I - \gamma(R^+)^{-1}M^{+T}\Phi L^{-1}R + \alpha\gamma R^{-1}M^T\Pi L^{-1}R + (\alpha - 1)\Gamma)z \quad (43)$$

*Proposition:* Consider the system (43) in which  $\alpha$  and  $\gamma$  satisfy (37). The zero equilibrium of the system is asymptotically stable, namely  $\lim_{k \rightarrow \infty} z(k) = 0$ .

*Proof:* Consider the candidate Lyapunov function  $V = z^T z$  and,

$$\begin{aligned} \Delta V &= z^{T+} z^+ - z^T z \\ &= z^T [\Lambda^T \Lambda - I] z \end{aligned} \quad (44)$$

where

$$\begin{aligned} \Lambda &= \alpha I - \gamma(R^+)^{-1}M^{+T}\Phi L^{-1}R + \alpha\gamma R^{-1}M^T\Pi L^{-1}R \\ &\quad + (\alpha - 1)\Gamma. \end{aligned} \quad (45)$$

In (44),  $\bar{\sigma}((R^+)^{-1}M^{+T}\Phi L^{-1}R) \leq 1$ ,  $\bar{\sigma}(R^{-1}M^T\Pi L^{-1}R) \leq 1$ . Let

$\tilde{I} \triangleq (R^+)^{-1}M^{+T}\Phi L^{-1}R$ ,  $\hat{I} \triangleq R^{-1}M^T\Pi L^{-1}R$  and rewrite (43) as

$$\Lambda = \alpha(I + \Gamma) - \Gamma \pm \gamma(\tilde{I} \pm \alpha\hat{I})$$

consider  $\alpha$  and  $\gamma$  satisfy (37), then

$$\Lambda = I - (I + \Gamma) \frac{0.5\kappa_1 + \xi_1}{1 + \bar{\sigma}(\Gamma)} \pm \frac{0.5\kappa_2 - \xi_2}{1 + \alpha} (\tilde{I} \pm \alpha\hat{I})$$

where  $\xi_1, \xi_2 > 0$  are such that  $\alpha, \gamma > 0$ . Thus  $\bar{\sigma}(\Lambda) < 1$  and this makes  $\Delta V < 0$ . Therefore  $\lim_{k \rightarrow \infty} z(k) = 0$ .  $\square$

*Remark:* In the estimation of the parameters the ‘‘look-ahead’’ values of the state variable  $x_i$  is required. These values are calculated utilizing the system equations (26) with parameter values obtained in the previous estimations.  $\square$

The back-stepping construction of the controller is performed in the sequel.

Consider system equations (26) and denote  $\hat{x}_1 = \tilde{x}_1 - \tilde{x}_1^*$ , then,

$$\hat{x}_1^+ = \tilde{x}_2 - \tilde{x}_1^{*+}$$

and assume  $\tilde{x}_2$  as virtual input, note that  $\hat{x}_2 = \tilde{x}_2 - \tilde{x}_2^*$  and select  $\tilde{x}_2^* = \tilde{x}_1^{*+}$ , thus

$$\hat{x}_1^+ = \hat{x}_2$$

and note that

$$\hat{x}_2^+ = \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta} - \tilde{x}_1^{*++} + u$$

Taking the control signal  $u$  as

$$u = -\frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta_{est}}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}} + \tilde{x}_1^{*++}$$

yields

$$\hat{x}_2^+ = \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta} - \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta_{est}}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}}.$$

This can be rewritten as,

$$\hat{x}_2^+ = \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est} - \Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta_{est}\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}},$$

$$\begin{aligned} \hat{x}_2^+ &= \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)(\theta_{est} - \Gamma z)\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}} \\ &\quad - \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta_{est}\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)(\theta_{est} - \Gamma z)}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}}, \end{aligned}$$

$$\hat{x}_2^+ = \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta_{est}\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\Gamma z - \Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\Gamma z\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}},$$

hence, the closed loop system dynamics are obtained as,

$$\hat{x}_1^+ = \hat{x}_2$$

$$\hat{x}_2^+ = \frac{\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\theta_{est}\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\Gamma z - \Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}\Phi_{2BC}(\tilde{x}_1, \tilde{x}_2)\Gamma z}{\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta\Phi_{2A}(\tilde{x}_1, \tilde{x}_2)\theta_{est}}.$$

Note that, when  $z=0$ , these dynamics become,

$$\hat{x}_1^+ = \hat{x}_2$$

$$\hat{x}_2^+ = 0.$$

Thus, considering the Proposition, it can be claimed that if the update law (33) together with (36) is used, the control objective is achieved. Note that,

$$\tau = A(\theta_{est})\tilde{\tau} = A(\theta + z)\tilde{\tau}|_{z=0} = A(\theta)\tilde{\tau}.$$

#### 4. SIMULATION RESULTS

In this paper basically the regulation problem is considered therefore the simulation results for  $\tilde{x}_1^* = 0$  are presented in the sequel. In the simulations, system parameters are taken as  $g = 9.81 \text{ kgm}^2/s$ ,  $r_1 = 0.3 \text{ m}$ ,  $r_2 = 1 \text{ m}$ ,  $m_1 = 0.45 \text{ kg}$ ,  $m_2 = 1.5 \text{ kg}$ ,  $m_3 = 0.5 \text{ kg}$ ,  $\lambda_1 = 3$ ,  $\lambda_2 = 3$  which yields  $\theta = [0.225, 1.5, 0.5, 0.75, 1.95]$  and  $T = 0.01$  is used to obtain discrete-time dynamics. Estimator parameters are selected as  $\kappa_1 = 0.9$ ,  $\kappa_2 = 0.5$ ,  $\gamma = 0.1$  and  $\alpha = 0.3$ . The simulations are run with the initial conditions  $\tilde{x} = [3, 0.5]$  and  $\hat{\theta}(0) = [10, 10, 10, 10, 10]$ . Figure 2 illustrates the time histories of the parameter estimation errors  $z = [z_1, z_2, z_3, z_4, z_5]$  which converge to zero, while Figure 3, Figure 4 and Figure 5 illustrate the time history of the state angular position of first joint  $x_1$ , angular velocity of first joint  $x_2$  and the edge position  $x$  for both the adaptive control case and the known parameter control case (Dashed-dot lines corresponds to adaptive control, solid lines corresponds to known parameter control). Note that, the edge position is calculated using the relation  $x = r_1 \cos x_1 + c$ .

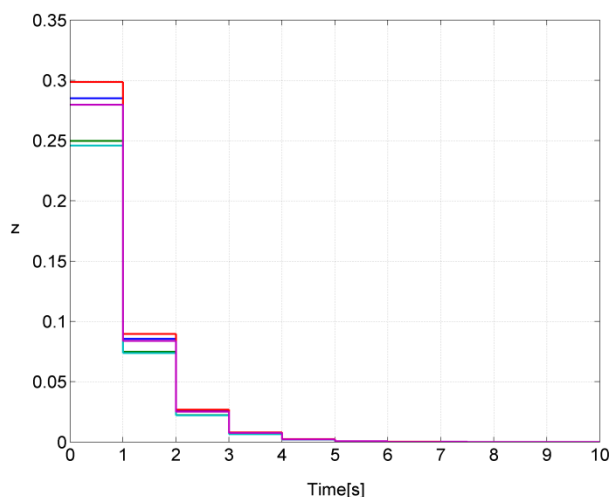


Fig. 2. Time history of the parameter estimation error  $z$ .

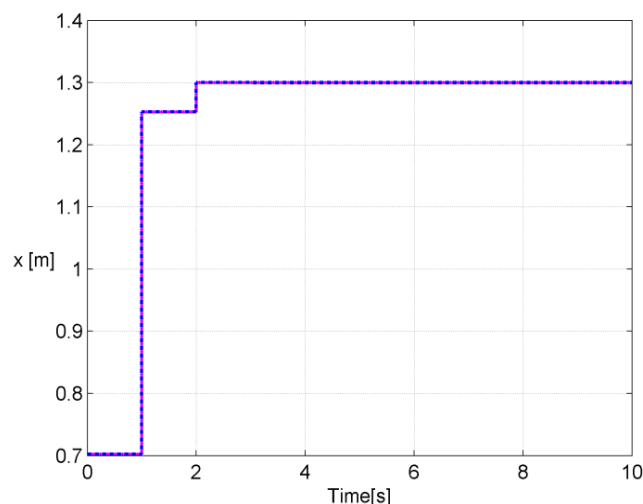


Fig. 5. Time history of the edge position  $x$ .

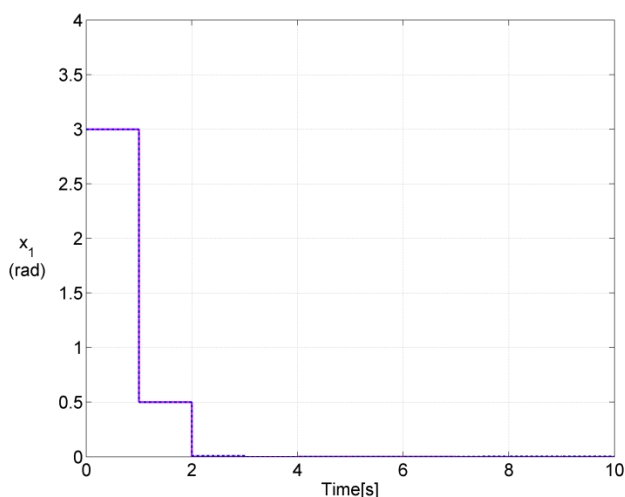


Fig. 3. Time history of the variable  $x_1$  (angular position)

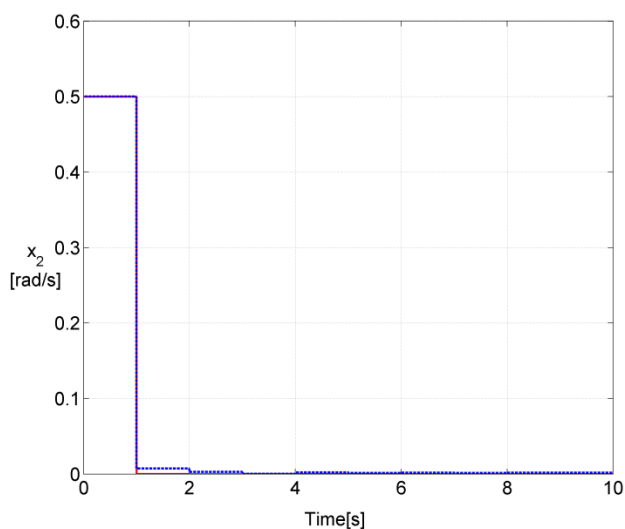


Fig. 4. Time history of the variable  $x_2$  (angular velocity)

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