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Distributed tracking of first order systems using second-order sliding modes

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Abstract: In this paper, the second-order sliding mode super-twisting controller is applied to distributed tracking problem. The distributed tracking is ensured by the proposed controller after a finite-time transient. The controller ensures the exact tracking of leader's position in spite of external disturbances acting on each follower. The corresponding conditions for the distributed tracking and for the simultaneous consensus are studied. Simulations show the workability of the proposed method.

1. INTRODUCTION

1.1 *State of the art*

The multivehicle cooperative control has received a significant attention from the control society during the last twenty years (see, Ren and Beard [2008] and Bullo et al. [2009]). The main motivation behind cooperative control is that a group of agents behave in a collaborative way to reach a common objective, the last finds applications in several disciplines and not only on robotics.

The relation among the agents is modeled as a network and the analysis of their interaction can be made using an algebraic graph. This approach provides the possibility to use mathematical tools such as the graph Laplacian matrix to study the flow of information. Moreover, the graph Laplacian matrix provides the opportunity to use its properties to design the control law that ensures the desired distributed behavior.

A particular challenging problem is known as consensus tracking (consensus with a leader). In this problem a group of followers tracks the position of a leader using only local interactions. The solution to this problem has been studied using different approaches, to name some of them: In Hong et al. [2008] was presented a distributed tracking algorithm that combines the use of distributed observers, to estimate the velocity of the followers. A Proportional-Derivative-like algorithm is proposed in Ren [2010] for the directed network topology. Nonetheless, these algorithms provides asymptotic tracking and they are very sensitive to disturbances.

Robust control techniques such as sliding mode controllers (see Utkin [1992], Pisano [2012], Shtessel et al. [2013]) have been successfully applied to consensus and formation control. Their main properties such as insensitivity to matched disturbances and finite time convergence have been used to solve the consensus tracking problem. Conventional sliding mode controllers have been successfully applied to solve the distributed tracking problem. In Rao and Goshe [2010], the consensus amongst first order dynamics is obtained by means of first order sliding mode controllers. This technique is also applied in Rao and Ghose [2011] to ensure the finite time consensus for formation, swarm and pursuing. In Cao and Ren [2012] the conventional sliding-mode controllers are applied to guarantee finite time convergence of a consensus protocol for distributed tracking and distributed swarm. Even when the aforementioned techniques are successful in providing the finite time tracking, the control signal contains high-frequency components that are harmful for the most mechanical systems. Besides, the trajectory of the agents contain discontinuities due the high-frequency control. In Mirkin et al. [2012], the distributed tracking problem is approached from an adaptive point of view. In this case, the chattering effect is diminished by the adaptive gains of a first order sliding mode, and the asymptotic tracking is achieved for nonlinear delayed systems. Nonetheless that the chattering is almost eliminated, with the exception of some isolated periods of time, the distributed tracking is reached asymptotically.

To overcome the drawback associated to the chattering effect, the high-order sliding mode controllers have been successfully applied to reduce its harmful effects (see, for example, Levant [2003]).

High-order sliding mode techniques have been applied to consensus tracking problem. In Rao and Ghose [2011] a sliding surface of a conventional sliding mode is designed to ensure tracking for high-order dynamics, this application correspond to a high-relative degree sliding surface in a conventional sliding mode controller. In Galzi and Shtessel [2006] is addressed the flight formation problem using the super-twisting controller. In this article, the super-twisting controller was applied to ensure tracking of an exogenous command signal that enforces the formation of multiple UAV's. Despite its robustness, the last mentioned application does not corresponds to a distributed tracking algorithm. Finally, in the recent work by Pilloni et al. [2013], the sliding-mode twisting controller is applied for the consensus of a network of perturbed double integrators.

Even when the higher-order sliding mode controllers have been applied to solve the leader tracking problem, none of these techniques have studied the distributed tracking problem from a consensus perspective using the second order sliding mode techniques, this is the aim of this work.

1.2 Main contribution

In this paper a distributed tracking algorithm to first order dynamics is proposed. The main features with respect to the already existent methods are the following:

- The algorithm provides finite-time distributed consensus tracking by means of a differentiable control law.
- The distributed tracking is reached in spite of disturbances.
- The conditions for the simultaneous convergence of the follower's position to the leader's position are studied.

1.3 Paper structure

Preliminary concepts related to the consensus problem are presented in Section 2. The problem is stated in Section 3. Section 4 presents the application of the super-twisting algorithm to the distributed tracking problem. In this Section, the conditions for the simultaneous tracking convergence are studied. Finally, in Section 5 simulations show the effectiveness of the proposed controller.

2. PRELIMINARIES

In this section the main concepts of graph theory are introduced (more detail on graph theory can be found in, for example, Olfati-Saber et al. [2007], Ren and Cao [2011] and Antonelli [2013]).

Graph theory is useful when it is necessary to model the interaction among a set of agents. For example, if a set of agents is equipped with a limited set of sensors, the graph may be used to represent the connections established by means of the sensors among each one of the agents.

A graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, is a pair that consists of a set of nodes $\mathcal{V} = \{v_1, ..., v_n\}$, and a set of edges $\mathcal{E} \subseteq \{(i, j) : i, j \in \mathcal{V}, j \neq i\}$. The edge (i, j) in the edge set of a directed graph denotes that the agent j can obtain information from agent i, but not necessarily the opposite. The graph $\mathcal{G}$ is said to be undirected if $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. The quantities $|\mathcal{V}|$ and $|\mathcal{E}|$ are called order and size of the graph and represent the number of nodes and number of edges, respectively. The graph $\mathcal{G}$ is also defined as a Network Topology.

The adjacency matrix $\mathcal{A} = [a_{ij}]$ of a directed graph is defined such that a_{ij} is positive weight if $(j, i) \in \mathcal{E}$, and $a_{ij} = 0$ if $(j, i) \notin \mathcal{E}$. When the elements of the adjacency matrix are only 0 or 1, then graph is called un-weighted. The adjacency matrix for an undirected graph is defined analogously except that $a_{ij} = a_{ji}$ for all $i \neq j$ because $(j, i) \in \mathcal{E}$ implies $(i, j) \in \mathcal{E}$.

The degree matrix of $\mathcal{G}$ is a diagonal matrix $\Delta = \Delta(\mathcal{A})$ with diagonal elements $\sum_{j=1}^{n} a_{ij}$ that are row-sums of $\mathcal{A}$.

The graph Laplacian $\mathcal{L} = [l_{ij}]$ is an $n \times n$ matrix associated with the $\mathcal{G}$ that is defined as $\mathcal{L} = \Delta(\mathcal{A}) - \mathcal{A}$. An important property of the Laplacian matrix is the following:

Property 1. (see Mesbahi and Egerstedt [2010]). The scalar zero is always an eigenvalue of $\mathcal{L}$. The vector $\mathbf{1}$ (a column vector composed by the constant 1 in all the rows) is always the corresponding eigenvector, i.e.,

$$\mathcal{L}\mathbf{1} = 0$$

A path is a sequence of edges in a directed or undirected graph, i.e., (i_1, i_2), (i_2, i_3), ... where $i_j \in \mathcal{V}$. When there exist a direct path between every arbitrary pair of nodes in a graph, it is said that the graph is strongly connected. A graph is connected, or weakly connected, when there exist an undirected path connecting every arbitrary pair of distinct nodes.

The set of neighbors of node i is defined by the set $\mathcal{N}_i = \{j \in \mathcal{V} : a_{i,j} \neq 0\} = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$.

3. PROBLEM STATEMENT

Let us consider that there exist a set of "n"vehicles, labeled as vehicles (or agents) 1 to n, called followers, and a vehicle labeled 0 called leader. The position and velocity of the leader is time variant and it possesses a bounded acceleration (notice that in any application, this proposition implies a physical restriction, but this always exist by nature), it is also assumed that the leader does not have any information about the followers.

The follower's first-order kinematics is given by:

$$\dot{x}_i = w_i + u_i, \qquad i = 0, ..., n \tag{1}$$

where x_i is the position of the i-th vehicle and u_i is the control associated to the i-th agent and w_i is a differentiable disturbance affecting the dynamics of the vehicle. Let us consider the set of vertex $\mathcal{V} = \{v_0, v_1, ..., v_n\}$ and a set of edges $\mathcal{E}$ which define the network topology. Then the Graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is used to modeling the interaction among the followers and the leader.

The following assumptions are made on the system dynamics, the disturbances and the network topology.

Assumption 1. The leader's acceleration is bounded, i.e.,

$$||\ddot{x}_0|| \leq \gamma_{2_0}^{+}$$

Assumption 2. The disturbance is a differentiable signal with bounded first-order derivative, i.e.,

$$||\dot{w}_i|| \leq \gamma_1^{+}$$

Assumption 3. The graph Laplacian matrix $\mathcal{L}$, corresponding to $\mathcal{G}$, has a single zero eigenvalue.

It is important to remark that Assumption 3 is not restrictive. This condition holds, for example, for a connected network (a network in which exists a directed path between each node and the leader). A Graph that contains a spanning tree also possess this property.

For the sake of simplicity, in the results presented here it is considered only the scalar (one-dimensional) case. However, the extension of the proposed methodology for the m-dimensional case is straightforward by using the Kronecker product.

The aim of this paper is to design a distributed consensus controller under a fixed network topology, that provides an exact tracking of the leader in the presence of disturbances in the followers. It is desired that the control signal be differentiable.

The super-twisting algorithm Levant [1993] plays an important role in the proposed distributed consensus algorithm. Differential equations are understood in the Filippov's sense (see Filippov [1988]) in order to provide for the possibility to use discontinuous signals in controls. Filippov solutions coincide with the usual solutions, when the right-hand sides are Lipschitzian. It is assumed also that all considered inputs allow the existence of solutions and their extension to the whole semi-axis $t \geq 0$.

4. DISTRIBUTED CONSENSUS TRACKING

Let model the interaction among the followers, excluding all the information of the leader x_0, as a reduced Graph. To do that, let the adjacency matrix among the followers be defined as $\mathcal{A}_f$ and its corresponding graph Laplacian as $\mathcal{L}_f$. Let the matrix M be defined as $M = \mathcal{L}_f + diag\{a_{10}, a_{20}, ..., a_{n0}\}$.

In order to obtain distributed consensus, the control signal is proposed as:

$$u_i = -\beta_1 \left| \sum_{j=0}^{n} a_{ij}(x_i - x_j) \right|^{1/2} sign\left(\sum_{j=0}^{n} a_{ij}(x_i - x_j) \right) + v_i \tag{2}$$

$$\dot{v}_i = -\beta_2 sign\left(\sum_{j=0}^{n} a_{ij}(x_i - x_j) \right) \tag{3}$$

where β_1, $\beta_2 > 0$ are scalar constants designed such that:

(1) The matrix $\bar{\Gamma}$ is defined as:
$$\bar{\Gamma} = \begin{bmatrix} -\frac{1}{2}\beta_1 M & \frac{1}{2}M \\ M(\gamma_{2_0}^{+} + \gamma_1^{+}) - \beta_2 I_n & 0 \end{bmatrix},$$
is positive semidefinite.

(2) The Lyapunov equation:
$$\bar{\Gamma}^T P + P\bar{\Gamma} = -Q \tag{4}$$
has a solution for positive definite matrices $P = P^T > 0$ and $Q = Q^T > 0$.

The following Theorem summarizes the main result:

Theorem 1. Suppose that the graph $\mathcal{G}$ satisfies Assumption 3. The application of the distributed control (2)-(3) in the followers kinematics (1) ensures that after a finite-time transient $x_i \to x_0$. In particular, $x_i = x_0$ in a finite-time.

Under conditions of Theorem 1, the proposed controller ensures the finite time convergence of the followers's trajectories to the position of the leader. However, the satisfaction of Assumption 3 does not guarantees the simultaneous convergence of all the agents to the leader's position.

With this aim, the following corollary is introduced using the reduced order Graph $\mathcal{G}_f$ corresponding to all the followers without the information of the leader.

Corollary 2. If in addition to the conditions of Theorem 1 the corresponding induced graph to the subset of vertices $\mathcal{V}_f = \{v_1, ..., v_n\}$ is strongly connected, then the followers converge simultaneously to the leader's position.

5. EXAMPLE

5.1 Distributed tracking

Consider a set of five followers and a leader. The followers and the leader are connected according to the following graph:

$$\mathcal{G} = (\mathcal{V}, \mathcal{E})$$

where

$$\mathcal{V} = \{v_0, v_1, v_2, v_3, v_4, v_5\}$$
$$\mathcal{E} = \{(0,5), (5,4), (4,2), (2,3), (3,2), (2,1)\}$$

The corresponding topology is presented in Fig. 1.

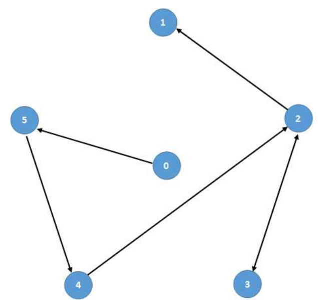

Fig. 1. Network topology for distributed tracking.

The adjacency matrix for the followers is given by the following matrix:

$$\mathcal{A} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&1&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 1&0&0&0&0&0 \end{bmatrix}$$

The Graph Laplacian associated to the network shown in Figure 1 has associated the following eigenvalues:

$\lambda_1 = 1,\quad \lambda_2 = 0.3820,\quad \lambda_3 = 2.6180,\quad \lambda_4 = 1,\quad \lambda_5 = 1,$
$\lambda_6 = 0.$

The five followers have the following dynamics:

$$\dot{x}_i = w_i + u_i$$

For simulation purposes, the disturbances are given by:

$$w_i = b_1 \sin\left((\omega_1 + b_2 \sin(\omega_2 t))t\right) + b_3$$

where the constants b_1, $b_2 \in \mathbb{R}$ and ω_1, ω_2 are positive scalar given in the following table:

follower	b_1	b_2	b_3	ω_1	ω_2
1	1.6	1	0	1	3
2	1	2	2	1.8	1
3	2	1	0	1.6	0.1
4	2.1	1	0	0.5	1
5	3	1	1	2	0.5

Table 1. Disturbance parameters for simulation purposes.

It is clear that Assumptions 1, 2 and 3 are satisfied.

The control signal is designed according to (3). All controllers, corresponding to each one of the five followers, have the same set of parameters given by $\beta_2 = 1.1(25)$ and $\beta_1 = 1.5(25)^{1/2}$.

In this simulation the following initial conditions have been considered: $x_1(0) = 9$, $x_2(0) = -10$, $x_3(0) = 0$, $x_4(0) = -8$, $x_5(0) = 2$. The leader initial condition is assumed $x_0(0) = 0$.

Figure 2 shows the convergence of all the followers to the leader's value. Notice that in Figure 3 the value of the leader is reached after a finite time transient, and notice also that once the leader's position is reached, the followers follows the value exactly in spite of the disturbances. In order to verify the condition for simultaneous convergence, this topology does not satisfy the conditions of Corollary 2 (notice that the agent 5 and the agent 1 are not strongly connected), hence the convergence to the leader's position is not simultaneous. In Figure 3 the convergence of x_5 to x_0 takes place before the rest of the followers.

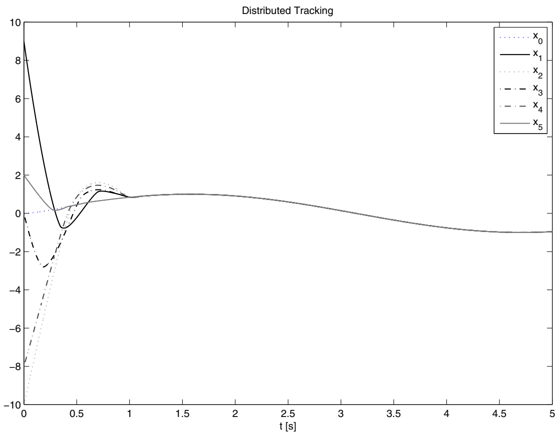

Fig. 2. Distributed tracking.

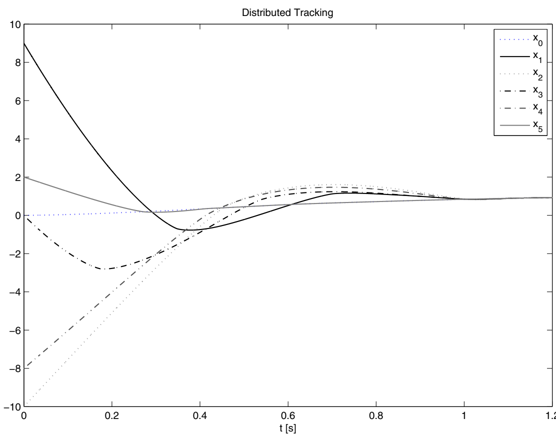

Fig. 3. Distributed tracking (Zoom).

The disturbances for the followers 2, 3 and 5 are presented in Figure 4. It is important to remark that the control signals are differentiable and that does not contain high-frequency. In Figure 5 the control signals for the dynamics x_2, , x_3 and x_5 is presented.

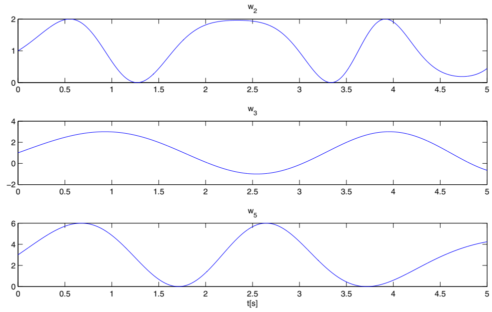

Fig. 4. Disturbances w_2, w_3, and w_5.

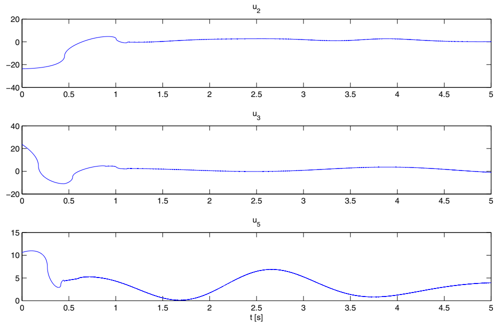

Fig. 5. Control signals u_2, u_3, and u_5.

5.2 Simultaneous convergence

Let us consider the same set of vertex than in the previous example, but let us consider the new set of edges given by

$$\mathcal{E}_2 = \{(0,5),(5,4),(4,2),(2,3),(3,2),(2,1),(3,1),(1,5)\}$$

Then, a new graph can be defined as:

$$\mathcal{G}_2 = (\mathcal{V}, \mathcal{E}_2)$$

This new set of edges induces a different topology shown in Figure 6.

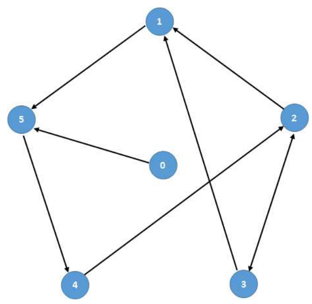

Fig. 6. Network topology for simultaneous convergence.

The corresponding adjacency matrix for the graph $\mathcal{G}_2$ is now given by:

$$\mathcal{A} = \begin{bmatrix} 0&0&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&1&1&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&1 \\ 1&1&0&0&0&0 \end{bmatrix}$$

This new adjacency matrix satisfies the conditions of Theorem 1 with eigenvalues in:

$$\lambda_1 = 2.8668, \quad \lambda_2 = 1.5 + 0.6067i, \quad \lambda_3 = 1.5 - 0.6067i,$$
$$\lambda_4 = 0.1332, \quad \lambda_5 = 2, \quad \lambda_6 = 0.$$

Furthermore, it satisfies the conditions of Corollary 2 and the followers are strongly connected. Using the same control parameters as in the previous example, the distributed tracking is illustrated in Figure 7, here is shown the convergence of the followers to the leader's position. In Figure 8, the simultaneous convergence of all the followers to the leader's position is illustrated. Notice that all the trajectories converges simultaneously in a finite-time.

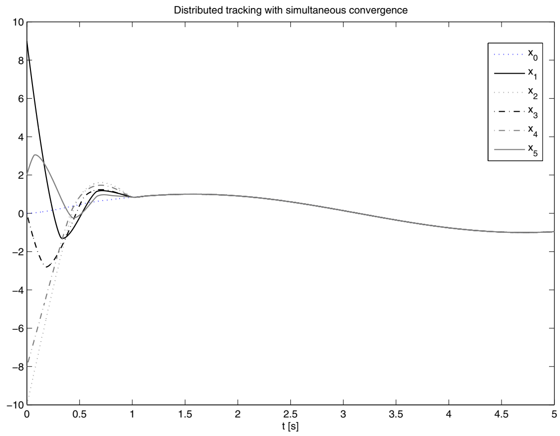

Fig. 7. Distributed tracking with simultaneous convergence.

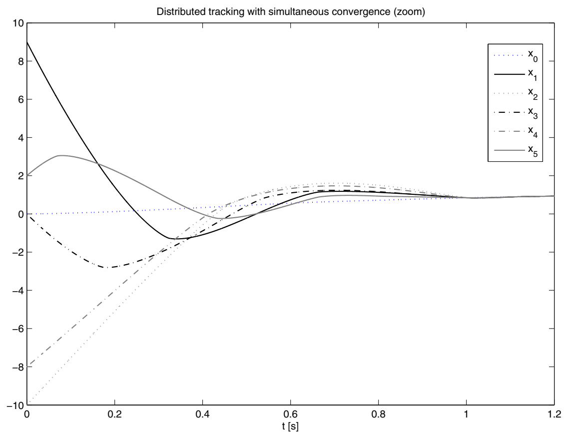

Fig. 8. Distributed tracking with simultaneous convergence (Zoom).

6. CONCLUSIONS

The distributed tracking problem has been solved by means of the second-order sliding mode super-twisting controller. The distributed tracking is achieved in a finite time in spite of disturbances in the follower's dynamics. The proposed controller ensures the exact tracking of leader's position using a differentiable control law, and the distributed tracking has been ensured under structural conditions on the network graph. The conditions for distributed tracking and for simultaneous consensus have been studied and illustrated by means of simulations. The workability of the proposed method has been shown in simulations.

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