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On the Control Rate versus Quantizer-Resolution Trade Off in Networked Control

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Abstract: This paper addresses the control-rate versus quantizer-resolution trade-off in networked control. The case presented considers the situation where the bit rate between controller and plant is constrained to a fixed number of bits per unit of time and an underlying fixed fast sampling rate is deployed to take measurements. However, a variable control update rate can be used between the controller and the plant. Inspired by the practical problem of inner loop power control in WCDMA, we assume the plant is an integrator. This assumption covers more general plants when fast sampling is used. Also, a restricted architecture in which linear filters are used for the encoder/decoder is considered and a quantizer with linear feedback is deployed. These choices give maximal insights into the underlying problem. It is shown that, in this case, it is best to use one bit per sample, in which case, the control update frequency is equal to the bit rate.

Keywords: Communication Control Application, Linear Control Systems, Network Control, Minimum Variance Control, Quantization, Sampling Rates

1. INTRODUCTION

Control theory has traditionally ignored communication constraints but the recent developments in networked control systems, and the problems arising there from, have inspired considerable interest in the interplay between communication and control (Wong and Brockett, 1997; Antsaklis and Baillieul, 2004). A major focus in this literature has been on the effect of network constraints on performance and stability; typical constraints are limits on (average) data rate, random delays and lost packets. There has been important progress in several areas. Data rate constraints have been investigated in Nair and Evans (2004); Nair et al. (2007); Savkin (2006); Tatikonda and Mitter (2004) while the effect of packet loss and random delays has been studied in Ling and Lemmon (2004); Schenato et al. (2007); Seiler and Sengupta (2005) and Lian et al. (2003) respectively. A signal to noise ratio formulation of the problem has also been explored (see e.g. Goodwin et al. (2010)). A key recent result establishes necessary and sufficient conditions on the average channel data rate that ensures closed-loop stability; see Nair et al. (2007) and the many references therein.

A discrete time formulation, in which the sampling frequency (in both uplink and downlink) is a-priori fixed, has typically been adopted in the networked control literature. This formulation is well matched to problems where physical constraints dictate the rate at which samples can be taken and control updates sent, rather than the number of bits that can be transmitted. In this paper, an alternative point of view is taken. The assumption is that the constraint is on the bit rate in the communication channel between controller and plant rather than the control update rate. A related problem is discussed in Fulton et al. (1997) in connection with Kalman Filtering for speech coders. However, the conclusions are different from those presented here since the problem is posed differently. Also note that 1 bit digital to analogue converters are commonly used in consumer electronics Smith (1999) but, again, this is for different reasons. To the best of authors knowledge, the question posed here has not been considered previously in the networked control literature. We believe that the results obtained are both surprising and of practical relevance.

A general treatment of this question has been carried out in a recent paper (Goodwin et al., 2014). However, because of the complexity of the general problem, the final design is necessarily simulation based. The current paper considers a simplified case where the plant is assumed to be an integrator. In practice this also covers more general systems when fast sampling is used since all continuous systems of relative degree one act (locally) as a pure integrator under fast sampling- see Åström and Wittenmark (1997); Åström et al. (1984)

for further discussion. In this case, the design is relatively simple and further insights into the problem are obtained.

The control channel (between controller and plant) bit rate is the product of the number of bits per unit of time and the update frequency. Since p bits per sample permits 2^p quantization levels, a higher value for p reduces the quantization error but also reduces the update frequency and, therefore, the ability of the controller to reduce the effect of disturbances. This decomposition of the bit rate into the product of bits per sample and control update frequency results in an obvious trade-off and leads to the question: "What is the best allocation of a given bit rate into the number of bits per sample and the number of control updates per second?" This paper addresses this question. Because the control update frequency has to be chosen, the underlying system is modelled using the smallest possible update period i.e. the inverse of the control channel bit rate.

The work described in the current paper was originally motivated by the problem of inner loop power control in WCDMA mobile communications (Cea and Goodwin, 2011, 2013). In this system the output is sampled without quantization at period $\Delta_1 = 0.667$[msec]. A control command is also sent from the controller (in the base station) to the user equipment every $\Delta_1 = 0.667$[msec] but is quantized to 1 bit. The plant is a pure integrator (since the user equipment simply treats the incoming control commands as increments in power). It would be possible to keep the same control channel bit rate but to send control commands every p samples and to deploy a p bit quantizer. Preliminary simulations carried out by the current authors suggests that the use of 1 bit is actually optimal. The current paper substantiates this choice.

A pragmatic view of network control in which linear filters are utilized for the encoder/decoder pair is taken. The single input single output case is considered and a quantizer with linear feedback is deployed to assign the control signal to the available bits. The analysis is restricted to open loop LTI stable systems. Subject to the above constraints, a design procedure in which, for each choice for the number of bits in the quantizer, the optimal controller, encoder/decoder, quantizer feedback and quantizer step size are chosen. Then, the optimal number of bits/sample is chosen. It is shown, surprisingly in our view, that one bit per sample is best. Consequently a sampling rate equal to the available bit rate is best. This choice corresponds to implementing the control law using a scaled sign function.

The layout of the remainder of the paper is as follows: Section 2 describes the class of models of interest. Section 3 presents the quantizer model, Section 4 describes the system structure and gives details of the filter design. Section 5 specializes the analysis to the case where the plant is an integrator. Section 6 describes the final closed loop system. Section 7 presents performance comparisons under a further simplifying assumption. Section 8 present performance comparisons without simplifying assumptions. Section 9 concludes the paper.

2. A CLASS OF MODELS

The work presented in this paper is based on the following assumptions:

A.1 The bit rate of the control channel (between controller and plant) is restricted to K bits/second so that $\Delta_1 = 1/K$ seconds is the smallest possible control update period.
A.2 The output of the system is sampled at fixed period Δ_1.
A.3 The controller is located near the plant output and no quantization of the output measurements occurs.
A.4 An appropriate anti-aliasing filter is used at the output.

For the minimal sample period Δ_1, the system can be described in innovations form (Anderson and Moore, 1979)

$$x_{k+1} = Ax_k + B\bar{u}_k + K\varepsilon_k \tag{1}$$
$$y_k = Cx_k + \varepsilon_k \tag{2}$$

where $x_k \in \mathbb{R}^n$, $\bar{u}_k \in \mathbb{R}^1$, $y_k \in \mathbb{R}^1$, $\varepsilon_k \in \mathbb{R}^1$ are the state, plant input, plant output and innovations sequence respectively.

Furthermore, assume that the discrete transfer function $C(zI - A)^{-1}B$ has relative degree $d + 1 < n$. Hence,

$$CA^iB = 0 \quad \forall i = 0, 1, \ldots, d-1 \tag{3}$$
$$CA^dB \neq 0 \tag{4}$$

The model (1), (2) can equivalently be expressed in the form of a stochastic ARMA model (Goodwin and Sin, 1984) as follows:

$$\mathcal{A}(z)y_k = \mathcal{B}(z)\bar{u}_k + \mathcal{C}(z)\varepsilon_k \tag{5}$$

where z is the forward shift operator and

$$\mathcal{B}(z)/\mathcal{A}(z) = C(zI - A)^{-1}B = P(z) \tag{6}$$
$$\mathcal{C}(z)/\mathcal{A}(z) = 1 + C(zI - A)^{-1}K = N(z) \tag{7}$$

Here,

$$\mathcal{A}(z) = 1 + a_1z^{-1} + \cdots + a_nz^{-n} \tag{8}$$
$$\mathcal{B}(z) = z^{-d-1}\bar{\mathcal{B}}(z) \tag{9}$$
$$\bar{\mathcal{B}}(z) = b_0 + b_1z^{-1} + \cdots + b_{n-1}z^{-n+1}; \quad b_0 \neq 0 \tag{10}$$
$$\mathcal{C}(z) = 1 + c_1z^{-1} + \cdots + c_nz^{-n} \tag{11}$$

The following additional assumptions are introduced:

A.5 The discrete time transfer function from u to y is stable and minimum phase.
A.6 A uniform-interval-nearest-neighbour quantizer with 2^p levels is deployed [1].

[1] The use of non uniform quantizers is discussed in Section 3.

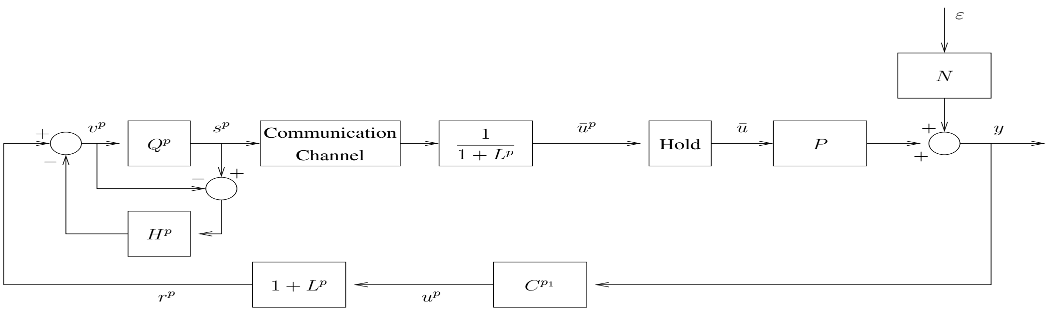

Fig. 1. Proposed three degree of freedom architecture.

A.7 All bits used in the quantizer are communicated over a serial link supporting K bits/second.
A.8 The communication channel is error free.

Note that a quantizer which allocates p bits/sample introduces a transmission delay of period $\Delta_p = p\Delta_1$ between controller and plant since the channel only allows 1 bit to be communicated in period Δ_1.

Remark 1. Note that the model (5) incorporates the usual model deployed in process control of a single lag plus pure delay as a special case. □

Remark 2. Assumption A.5 allows one to use Minimum Variance Control (Goodwin and Sin, 1984).

The proposed architecture for the feedback system is shown in Figure 1. In this figure, a superscript p denotes either a downsampled signal with period $p\Delta_1$ or a system that operates at period $p\Delta_1$. To implement effective control over the bit rate limited network, a multi-degree of freedom architecture is employed, see Figure 1. This architecture has the following degrees of freedom: C^{p1} (a controller which is driven by an unquantized signal with sample period Δ_1 and which outputs a quantized control every p^{th} sample of the input), $(1 + L_p)$ and $(1 + L_p)^{-1}$ (the channel coder and decoder), and H^p (providing feedback around the quantizer). For realizability, L^p and H^p are constrained to be strictly proper. To motivate this three degree of freedom controller, note that C^{p1} has the role of mitigating the effect of the process noise, ε, on the output y, L^p has the role of reducing the variance of the input to the quantizer which, in turn, allows one to minimize the variance of the quantization 'noise' and H^p has the role of reducing the effect of the resultant quantization errors on the output y.

3. THE QUANTIZER

Two types of quantizer are studied, namely a uniform quantizer and an optimal non-uniform quantizer both with 2^p levels.

$Q^p[\cdot]$ denotes the quantization operation and the quantizer errors are given by:

$$q^p = v^p - Q[v^p] = F^p_\lambda[v^p] \tag{12}$$

For a uniform quantizer with 2^p levels and steps λ, $F^p_\lambda[\cdot]$ takes the form

$$F^p_\lambda[v] = \begin{cases} v + (2p+1)\lambda \ \forall v \in (-\infty, -2\lambda(2^{p-1}-1)] \\ v + (2i-1)\lambda \ \ \forall v \in (-2\lambda j, -2\lambda(j-1)] \\ v - (2i-1)\lambda \ \ \forall v \in (2\lambda(j-1), 2\lambda j] \\ v - (2p+1)\lambda \ \forall v \in [2\lambda(2^{p-1}-1), \infty) \end{cases} \tag{13}$$

where $j = 1, 2, \ldots, 2^{p-1} - 1$.

Lemma 1. Say that the input to the quantizer is scaled by a factor α, then the error function is also scaled by the same factor [2] For all $(p, \lambda) \in \mathbb{I}_{\geq 1} \times \mathbb{R}_{\geq 0}$

$$F^p_{\alpha\lambda}(\alpha v) = \alpha F^p_\lambda(v) \tag{14}$$

for all $\alpha \in \mathbb{R}_{\geq 0}$.

Proof. From equation (13),

$$\begin{aligned} F^p_{\alpha\lambda}(\alpha v) &\triangleq \alpha[v - (2j+1)\lambda], \\ &\quad \alpha v \in [2(j-1)\alpha\lambda,\ 2j\alpha\lambda], \\ &\quad j = 1, 2, \ldots, 2^{p-1} - 1 \\ &\triangleq \alpha[v - (2p+1)\lambda], \\ &\quad \alpha v \in [2\alpha\lambda(2^{p-1}-1), \infty) \end{aligned} \tag{15}$$

A similar procedure can be used for negative values of v. Hence,

$$F^p_{\alpha\lambda}(\alpha v) = \alpha F^p_\lambda(v) \tag{16}$$

for all $\alpha \in \mathbb{R}_{\geq 0}$. □

The next question is how to design the quantizer step $\lambda = \lambda^p$. Say that the quantizer step is chosen, for each p, so as to minimize the mean square quantization error (V_p). The best choice for λ^p depends upon the probability distribution of the signal being quantized. Table 1 gives the optimal uniform quantizer (UQ) spacing λ^0_p and associated mean square error (V^0_p: UQ MSE) for the special case when input to the quantizer is zero mean gaussian [3] with unit variance. Table I also gives the mean square errors (NUQ MSE) when an optimal non uniform quantizer is used based on a minimal distortion design (Gersho and M.Gray, 1992).

The results in Table 1 show that the use of an optimal non-uniform quantizer leads to only marginal changes in the mean square quantization error for the case of a gaussian input. Hence, the use of a uniform quantizer is emphasized in the sequel.

[2] This result is used later to simplify the presentation of results.
[3] This restriction is removed later. However, this assumption is used in Theorem 1.

p	1	2	3	4
λ_p^0	0.80	0.50	0.295	0.17
V_p^0: UQ M.S.E	0.363	0.119	0.0374	0.0116
NUQ M.S.E.	0.363	0.117	0.0345	0.0095

Table 1. λ_p^0 and V_p^0 uniform quantizer, and M.S.E with non-uniform quantizer vs p when Var$\{\sigma_2^p\} = 1$

4. PERFORMANCE CRITERIA AND DESIGN GUIDELINES

At sample time $k = \ell p$, $\ell \in \mathbb{Z}^+$, the controller has knowledge of past outputs (sampled at period Δ_1), i.e. $y_{\ell p}$, $y_{\ell p-1}$, $y_{\ell p-2}\ldots$. The controller then generates an input signal u_ℓ^p. This input signal is held for p samples. The fast sampled input is denoted u_k and $u_{\ell p+i} = u_\ell^p$, $i = 0, 1, \ldots, p-1$.

After filtering by $(1+L^p)$ the control is quantized to 2^p levels which leads to a p bit representation. It takes $p\Delta_1$ seconds to transmit these p bits over the communication channel to the plant input. On arrival, the signal is passed through $(1 + L^p)^{-1}$, then a series to parallel conversion is applied followed by D/A conversion. This process produces an analogue signal constrained to the same 2^p levels. This is then passed through a p sample hold so that the plant input, $\bar{u}_k$, is constant for p successive samples. During this period, the next p bits are received, allowing the next plant input to be reconstructed, and so on.

In the above description, the total delay between sample time, $k = \ell p$, and the first time that the resultant control, $u_{\ell p}^p$, effects the output of the plant is $(d+p+1)$ samples. Thus, $u_{\ell p}$ effects the output at sample $\ell p+p+d+1, \ldots, \ell p+p+d+p$, and so on. The focus is on the output over the period $\ell p+p+d+i$; $i = 1, \ldots, p$. The variance at the end point of the above period is used as the performance criteria:

$$J_E(p) = E\left\{y^2_{(\ell p+p+d+p)}\right\} \tag{17}$$

5. PLANT AS AN INTEGRATOR

The case when the plant is an integrator contains all of the key features of the general case [4] but has the advantage of clarity of exposition. The core conclusion regarding the efficiency of $p = 1$ is clear in this case.

For an integrator the model linking controller output, u_k, and the plant output y_k is as follows (for a p bit quantizer and when the plant has delay d):

$$\begin{aligned} y_{k+p+d+1} = &\, b_0[u_k + u_{k-1} + \ldots] \\ &+ [\varepsilon_{k+p+d+1} + \varepsilon_{k+p+d} + \ldots] \end{aligned} \tag{18}$$

Lemma 2. (Design of C_E^{p1}). Consider a p-sample hold input implemented as described in Section 3. Then,

(i) The variance of $y_{pl+d+2p}$ is minimized by the control law:

$$\begin{aligned} u_{pl} = \frac{-1}{pb_0}&\left(y_{pl} + b_0\left(pu_{p(l-1)} + \ldots\right.\right. \\ &\left.\left.+ r_p u_{p(l-2-\tilde{d})}\right)\right) \end{aligned} \tag{19}$$

where $\tilde{d} = \left\lfloor \frac{d}{p} \right\rfloor$ and $r_p = d - p\tilde{d}$.

(ii) The resultant form of $y_{pl+d+2p}$ is

$$y_{p(\ell+2)+d} = \varepsilon_{p(\ell+2)+d} + \cdots + \varepsilon_{pl+1} \tag{20}$$

which has variance

$$J_E(p) = E\left\{y^2_{p(\ell+2)+d}\right\} = (2p+d)\sigma_\varepsilon^2 \tag{21}$$

(iii) The control signal (19) is equivalent to

$$u_{pl} = \frac{-1}{b_0 p}(\varepsilon_{pl} + \cdots + \varepsilon_{p(l-1)+1}) \tag{22}$$

which is a white noise sequence having variance:

$$E\{u_{pl}\} = \frac{1}{b_0^2 p}\sigma_\varepsilon^2 \tag{23}$$

Proof. The design is obtained by using a minimum variance controller. The details have been omitted. □

Next consider the design of L^p. Recall that the goal when designing L^p is to minimize the variance (due to $\{\varepsilon_k\}$) of the signal appearing at the input of the quantizer. Also recall from (iii) of Lemma 2 that, using the controller, C_E^{p1}, u_{pl} is a white noise sequence (which has minimal variance). Hence one can immediately conclude:

Lemma 3. The optimal choice for L^p is

$$L^p = 0 \tag{24}$$

Proof. Note that (23) is a direct consequence of the use of the minimum variance control law. □

Next consider the design of H^p. This is particularly simple when $d = 0$.

Lemma 4. (Design of H^p). Based on the working hypothesis that q^p is a white noise sequence and setting $d = 0$, then the variance of $y_{\ell p+2p}$, due to the effect of q^p, is minimized by the choice

$$H^p = \frac{z^{-p}}{1+z^{-p}} \tag{25}$$

Proof. Iterating the model, and noting that $u_{\ell p}$ is held constant over p samples yields

$$y_{(\ell+2)p} = y_{\ell p} + pb_0\bar{u}_{(\ell+1)p} + pb_0\bar{u}_{\ell p} + \bar{\varepsilon}_{(\ell+1)p} + \bar{\varepsilon}_{(\ell+2)p} \tag{26}$$

where $\bar{u}_{\ell p}$ is the past input and

$$\bar{\varepsilon}_{\ell p} = \sum_{i=0}^{p-1} \varepsilon_{\ell p+i} \tag{27}$$

Now, from Figure 1 and equation (12) we see that

$$\bar{u}_{(\ell+1)p} = u_{\ell p} + (1-H^p)q_{\ell p} \tag{28}$$

Hence

$$z^{2p}y_{\ell p} = y_{\ell p} + (1+z^{-p})pb_0\bar{u}_{(\ell+1)p} + z^{2p}(1+z^{-p})\bar{\varepsilon}_{\ell p} \tag{29}$$

[4] The general case is the subject of a separate paper Goodwin et al. (2014).

$$z^{2p}y_{\ell p} = y_{\ell p} + (1+z^{-p})pb_0(u_{\ell p} + (1-H^p)q_{\ell p}) + z^{2p}(1+z^{-p})\bar{\varepsilon}_{\ell p} \tag{30}$$

From Lemma (2), the minimum variance control law takes the form:

$$(1+z^{-p})b_0 p u_{\ell p} = -y_{\ell p} \tag{31}$$

Substituting (31) into (30) yields

$$z^{2p}y_{\ell p} = pb_0(1+z^{-p})(1-H^p)q_{\ell p} + z^{2p}(1+z^{-p})\bar{\varepsilon}_{\ell p} \tag{32}$$

Hence the variance of $y_{(\ell+2)p}$ due to $q_{\ell p}$ (under the working hypothesis that it is a white sequence) is minimized by the choice

$$(1+z^{-p})(1-H^p) = 1, \textit{ or } H^p = \frac{z^{-p}}{1+z^{-p}} \tag{33}$$

□

6. THE FINAL CLOSED LOOP

The various components are brought together to evaluate the performance. In the process, the optimal quantizer is designed for each p. It is important to note that here we no longer make the working hypothesis that q^p is a white noise sequence but, instead, model it correctly as being the result of the quantization process. For simplicity of presentation the delay is chosen to be zero ($d = 0$).

Note that the design described above ensures that

$$y_{(\ell+2)p} = \sum_{i=1}^{2p} \varepsilon_{\ell p+i} + pb_0 q_{\ell p}; \quad l = 0, 1, \ldots \tag{34}$$

This result holds true no matter how one models $q_{\ell p}$. Also note that

$$q_{\ell p} = F_{\lambda^p}(v^p) \tag{35}$$

where $F_{\lambda^p}(\cdot)$ is the error function of the quantizer, as discussed in Section 3.

Lemma 5. An important consequence of the choice of minimum variance control is that the output variance can be decomposed as:

$$E\left\{y^2_{(\ell+2)p}\right\} = E\left\{\left[y^{\varepsilon}_{(\ell+2)p}\right]^2\right\} + E\left\{\left[y^{q}_{(\ell+2)p}\right]^2\right\} \tag{36}$$

where $l = 0, 1, \ldots,$ and $y^{\varepsilon}_{(\ell+2)p}$ describes the output due to the direct effect of ε and $y^{q}_{(\ell+2)p}$ describes the output due to the indirect effect of ε via q^p.

Proof. The result follows immediately since the minimum variance controller ensures that $y^{\varepsilon}_{(\ell+2)p}$ is a function of $\varepsilon_{\ell p+i}$; $i = 1, \ldots, 2p$ whereas $y^{q}_{(\ell+2)p}$ is a function of past values of $\varepsilon_{\ell p+i}$; $i = 0, -1, -2, \ldots$ □

Also note that from Lemma 2 when $d = 0$,

$$E\left\{\left[y^{\varepsilon}_{(\ell+2)p}\right]^2\right\} = [2p]\sigma^2_{\varepsilon} \tag{37}$$

Hence, the remaining task is to evaluate $E\left\{\left[y^{q}_{(\ell+2)p}\right]^2\right\}$.

Lemma 6. Given the design choices described above, then the sequence of quantization errors, $\{q_{\ell p}\}$, are related to ε_k as follows: For $l = 0, 1, 2, \ldots$

$$v_{\ell p} = -\frac{1}{b_0 p}\bar{\varepsilon}_{\ell p} - q_{(\ell-1)p} \tag{38}$$

$$q_{\ell p} = F^p_{\lambda}(v_{p\ell}) \tag{39}$$

where $\bar{\varepsilon}_{\ell p} = \sum_{i=0}^{p-1} \varepsilon_{\ell p+i}$.

Proof. One has

$$z^{2p}y_{pl} = pb_0 q_{pl} + z^{2p}(1+z^{-p})\bar{\varepsilon}_{pl} \tag{40}$$

Next using the equation for the control law,

$$b_0 p[1+z^{-p}]u_{\ell p} = -y_{\ell p} \tag{41}$$

Hence

$$b_0 p[1+z^{-p}]u_{\ell p} = -pb_0 z^{-2p}q_{\ell p} - (1+z^{-p})\bar{\varepsilon}_{\ell p} \tag{42}$$

or

$$u_{\ell p} = \frac{-z^{-2p}}{1+z^{-p}}q_{\ell p} - \frac{1}{b_0 p}\bar{\varepsilon}_{\ell p} \tag{43}$$

Also, note that the input to the quantizer satisfies.

$$\begin{aligned} v_{pl} &= u_{pl} - H^p q_{pl} \\ &= \frac{-z^{-2p}}{1+z^{-p}}q_{\ell p} - \frac{z^{-p}}{1+z^{-p}}q_{\ell p} - \frac{1}{b_0 p}\bar{\varepsilon}_{\ell p} \\ &= -\left[\frac{z^{-2p}+z^{-p}}{1+z^{-p}}\right]q_{\ell p} - \frac{1}{b_0 p}\bar{\varepsilon}_{\ell p} \\ &= z^{-p}q_{\ell p} - \frac{1}{b_0 p}\bar{\varepsilon}_{\ell p} \end{aligned} \tag{44}$$

This completes the proof. □

Two alternatives are described to evaluate the variance of $y^q_{\ell p}$. An approximate approach is considered (see Theorem 1 below) and an exact analysis is provided by means of numerical simulation in section 8.

7. PERFORMANCE ANALYSIS: SIMPLIFIED ANALYSIS

Given that the quantizer is designed optimally one can anticipate that the quantization errors will be small relative to $\bar{\varepsilon}_{\ell p}$. This suggests the following simplifying assumption:

SA.1 The lower feedback path in equation (44) only effects the results in a minimal fashion and is thus removed.

Theorem 1. Based on Lemma 6, Lemma 5 and simplifying assumption SA.1 the output variance is:

$$E\left\{y^2_{(\ell+2)p}\right\} = [2p]\sigma^2_{\varepsilon} + V^0_p \tag{45}$$

Proof. Under simplifying assumption SA.1 the input to the quantizer becomes the white Gaussian noise sequence $\bar{\varepsilon}_{\ell p}$. □

Under these conditions, one can immediately use Table 1 to compute λ^0_p and to evaluate the MSE due to quantization errors (V^0_p) and the total MSE using (45). The results are presented in the top 2 lines of Table 2.

Table 2. Experimental output variances for different values of p using λ_p^0 and $\lambda_p^{C.L.}$.

p	1	2	3	4
λ_p^0	0.80	0.50	0.295	0.17
(a)	2.36	4.12	6.04	8.01
$\lambda_p^{C.L}$	0.97	0.553	0.388	0.31
(b)	2.61	4.10	6.03	8.13

(a) ignoring the feedback path and using λ_p^0.
(b) simulating the system using $\lambda_p^{C.L.}$.

Theorem 1 gives an approximate value for the output variance of a system using p bits. Then, from Table 1 it can be seen that 1 bit is the best choice. However, this has been obtained using simplifying assumption SA.1.

8. PERFORMANCE EVALUATION: MONTE CARLO SIMULATION

Here, simplifying assumption SA.1 is removed and, in this case, Monte Carlo techniques are used to simulate the complete feedback circuit described in Lemma 6. One can then search numerically for the optimal values of λ_p. The results are reported in the third line of Table 2 as $\lambda_p^{C.L.}$. Next, the total output mean square error is evaluated as a function of p. The results are summarized in the last line of Table 2. Table 2 shows that the approximate analysis of section 7 gives a very close approximation to the true results. Also, $p = 1$ is unequivocally the best choice in all cases. Moreover, the gap between the results for $p = 1$ and $p = 2$ is almost two to one. Hence, one can anticipate that the same conclusion (namely that $p = 1$ is best) is likely to hold under rather general circumstances.

9. CONCLUSIONS

This paper has studied the scenario where a control signal is implemented over a bit rate constrained communication channel but when the control update rate is, otherwise, unconstrained. The special case of fast sampling has been studied, and it has been shown that it is best to use 1 bit per sample and hence to choose the control update rate equal to the inverse of the bit rate. In separate work, not reported here due to space limitations, it has been shown that the conclusion that $p = 1$ is best holds under quite general conditions.

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