Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Robust stabilization of linear uncertain plants with polynomially time varying parameters

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Abstract: The robust stabilization of uncertain linear time-varying continuous-time systems with a mode-switch dynamics is considered. Each mode is characterized by a dynamical matrix containing elements whose time behavior over bounded time intervals is sufficiently smooth to be well described by interval polynomials of arbitrary degree. The stability conditions of the switching closed-loop system are derived defining a switched Lyapunov function and involving the dwell time of the system over each single mode. An important feature of the paper is that, unlike all the other existing methods, each plant mode can be stabilized over arbitrarily large uncertain domains of parameters and their derivatives.

1. INTRODUCTION

In recent years analysis and synthesis of control systems for linear time-varying (LTV) plants with polytopic uncertainties have been widely investigating in different settings and from different points of view. Much attention has been recently devoting to the synthesis of control systems for linear parameter-varying (LPV) plants, (see e.g. Apkarian et al. [1995],Daafouz et al. [2008], Jetto et al. [2010a], Heemels et al. [2010], Oliveira et al. [2009] and references therein). A wide literature also exists on the robust analysis and synthesis of LTV uncertain polytopic system with no on-line available information on physical parameters (see e.g. Daafouz et al. [2001], Dong et al. [2008], Geromel et al. [2006],Jetto et al. [2009],Jetto et al. [2010b], Mao [2003], Rugh et al. [2000],Trofino et al. [2001] and references therein). A common assumption of all the above papers is that the unknown parameters belong to a bounded (and generally small) uncertainty domain. The stability analysis reported in Jetto et al. [2009] showed that it is possible to state stability conditions under arbitrarily large time varying parametric uncertainties with possibly arbitrarily large variations rates. This is possible under the assumption of plants with a time-varying dynamical matrix whose elements are described by interval polynomial functions.

The purpose of this paper is to extend the results of Jetto et al. [2009] to the controller synthesis problem for a plant with a dynamics switching among a finite number of modes. The physically meaningful modeling assumption allows us to transfer the uncertainty from the domain of the process parameter space to that of the relative polynomial coefficient space so that arbitrarily large uncertainty region can be obtained by increasing time. The stabilization of each "single mode" is obtained through an observer based controller with gain matrices polynomially depending on the time. A parameter dependent Lyapunov function whose matrix is itself polynomially depending on time is adopted and the stabilization problem is solved here defining a set of BMI's which reduce to a set of LMI's fixing two positive scalars.

The main evident theoretical interest of the present paper is that, unlike all the other approaches, it allows the synthesis of a stabilizing controller for uncertain non uniformly bounded dynamical matrices.

The overall controller is given by the switching among the family of observer-based controllers designed for each single mode. Closed-loop stability conditions are stated in terms of permanence time intervals of the plant dynamics over the same mode. This is accomplished by defining a suitable switched Lyapunov function.

2. "SINGLE MODE" PLANT

Consider the following uncertain polynomially time varying plant Σ

$$\Sigma = \begin{cases} \dot{x}(t) &= A(t,\alpha)x(t) + Bu(t) \\ y(t) &= Cx(t) \end{cases} \tag{1}$$

$$\text{with} \quad A(t,\alpha) \overset{\triangle}{=} A_0(\alpha_0) + \sum_{i=1}^{\ell} A_i(\alpha_i)t^i,\ t \geq 0, \tag{2}$$

where: $A_i(\alpha_i) \overset{\triangle}{=} \sum_{j=1}^{n_i} \alpha_{i,j} A_{i,j}$, $\sum_{j=1}^{n_i} \alpha_{i,j} = 1$, $\alpha_{i,j} \geq 0$, and $A_{i,j}, i = 0, \cdots, \ell; j = 1, \cdots, n_i$, are known square constant matrices.

Remark 1. The assumption of constant matrices B and C is not a loss of generality because, as shown in Apkarian et al. [1995], it can always be satisfied if proper LTI filters are applied to the original signals $u(t)$ and $y(t)$. This implies a controller of increased order. $\triangle$

The robust stabilization problem defined in this section consists in finding (if it exists) a dynamic output feedback controller Σ_c which guarantees the exponential γ-stability (as defined in Jetto et al. [2009]) of the closed-loop uncertain polynomially time varying system Σ_f given by the

feedback connection of Σ_c with Σ.
The two following preliminary results will be exploited to solve the aforementioned problem.

*PR*1: If G is a positive definite matrix, X and Y are matrices of appropriate dimensions and ε is a positive scalar, then the following direct consequence of the Schur complement holds

$$\begin{bmatrix} 0 & XY^T \\ YX^T & 0 \end{bmatrix} \leq \begin{bmatrix} \varepsilon XGX^T & 0 \\ 0 & \frac{1}{\varepsilon} YG^{-1}Y^T \end{bmatrix}.$$

*PR*2, Song et al. [2011]: Let Φ be a symmetric matrix and N, M be matrices of appropriate dimensions. The following statements are equivalent:

(1) $\Phi < 0$ and $\Phi + NM^T + MN^T < 0$;
(2) The LMI problem $\begin{bmatrix} \Phi & M + NF \\ M^T + F^T N^T & -F - F^T \end{bmatrix} < 0$ is feasible with respect to F.

Consider the following observer based controller Σ_c

$$\Sigma_c = \begin{cases} \dot{\xi}(t) & = \overline{A}(t)\xi(t) + Bu(t) - L(t)(y(t) - C\xi(t)) \\ u(t) & = K(t)\xi(t) \end{cases} \quad (3)$$

where $\overline{A}(t)$ defines a sort of " nominal central plant" and is given by $\overline{A}(t) = \overline{A}_0 + \sum_{i=1}^{\ell} \overline{A}_i t^i,\ t \geq 0,\quad \overline{A}_i = \frac{\sum_{j=1}^{n_i} A_{i,j}}{n_i}$. The gains $K(t)$ and $L(t)$ are polynomially time varying matrices defined as

$$K(t) \triangleq K_\ell t^\ell \qquad L(t) \triangleq L_\ell t^\ell, \quad (4)$$

where K_ℓ and L_ℓ, are constant matrices to be computed. The assumed form of $K(t)$ and $L(t)$ will be justified in the light of Remark 3 reported later.
Applying the usual transformation matrix, the state space representation of the uncertain time varying closed loop system $\Sigma_f \triangleq (\hat{C}_f, \hat{A}_f(t,\alpha))$ is

$$\dot{\hat{x}}_f(t) = \begin{bmatrix} A(t,\alpha) + BK(t) & -BK(t) \\ \Delta A(t,\alpha) & \overline{A}(t) + L(t)C \end{bmatrix} \hat{x}_f(t) \quad (5)$$

$$y(t) = [\,C \;\; 0\,]\, \hat{x}_f(t) \quad (6)$$

where: $\dot{\hat{x}}_f(t) \triangleq \left[\dot{x}^T(t),\ \dot{x}(^Tt) - \dot{\xi}^T(t)\right]^T,\ \Delta A(t,\alpha) \triangleq A(t,\alpha) - \bar{A}(t) \triangleq \sum_{i=0}^{\ell} \Delta A_i(\alpha_i) t^i$, with $\Delta A_i(\alpha_i) \triangleq (A_i(\alpha_i) - \overline{A}_i) = \sum_{j=1}^{n_j} \alpha_{i,j} \Delta A_{i,j},\ \Delta A_{i,j} \triangleq A_{i,j} - \bar{A}_i,\ \sum_{j=1}^{n_j} \alpha_{i,j} = 1,\ \alpha_{i,j} \geq 0$.
To investigate the stability of Σ_f the following parameter dependent Lyapunov function is considered

$$V(\hat{x}_f(t),\alpha) = \hat{x}_f(t)^T R(t,\alpha) \hat{x}_f(t), \quad \alpha \in S \quad (7)$$

where

$$R(t,\alpha) = \begin{bmatrix} P(t,\alpha) & W(t,\alpha) \\ W^T(t,\alpha) & Q(t,\alpha) \end{bmatrix} = \sum_{i=0}^{\ell} R_i(\alpha_i) t^i \quad (8)$$

is a symmetric positive definite matrix $\forall t \geq 0,\ \forall \alpha \in S$, with

$$R_i(\alpha_i) = \begin{pmatrix} P_i(\alpha_i) & W_i(\alpha_i) \\ W_i^T(\alpha_i) & Q_i(\alpha_i) \end{pmatrix}, i = 0, \cdots, \ell - 1,$$

$$\text{and } R_\ell(\alpha_\ell) = \begin{pmatrix} P_\ell(\alpha_\ell) & 0 \\ 0 & Q_\ell(\alpha_\ell) \end{pmatrix}. \quad (9)$$

The time derivative of $V(\hat{x}_f(t),\alpha)$ is

$$\dot{V}(\hat{x}_f(t),\alpha) = \hat{x}_f^T(t) H(t,\alpha) \hat{x}_f(t) \quad (10)$$

with $H(t,\alpha) = \hat{A}_f^T(t,\alpha)R(t,\alpha) + R(t,\alpha)\hat{A}_f(t,\alpha) + \dot{R}(t,\alpha)$. By (5) and (8), $H(t,\alpha)$ is the symmetric polynomially time varying matrix given by

$$H(t,\alpha) = \begin{pmatrix} H^{(1,1)}(t,\alpha) & H^{(1,2)}(t,\alpha) \\ H^{(2,1)}(t,\alpha) & H^{(2,2)}(t,\alpha) \end{pmatrix}. \quad (11)$$

The form of each single block $H^{(i,j)}(t,\alpha)$ is reported in (12) at the top of page 3. Exploiting the polynomial time-dependence of matrices on the r.h.s. of (12) (see page 3), an equivalent representation of (10) is

$$\dot{V}(\hat{x}_f(t),\alpha) = \hat{x}_f^T(t) \left(H_0(\alpha) + \sum_{k=1}^{2\ell} H_k(\alpha) t^k \right) \hat{x}_f(t) \quad (13)$$

with $H_k(\alpha) = \begin{pmatrix} H_k^{(1,1)}(\alpha) & H_k^{(1,2)}(\alpha) \\ H_k^{(2,1)}(\alpha) & H_k^{(2,2)}(\alpha) \end{pmatrix},\ k = 0, \cdots, 2\ell$.
Exploiting (10)-(13), it can be shown that each single block $H^{(i,j)}(t,\alpha),\ i,j = 1,2$ of $H(t,\alpha)$ in (11) has the form: $H^{(i,j)}(t,\alpha) = H_0^{(i,j)}(\alpha) + \sum_{k=1}^{2\ell} H_k^{(i,j)}(\alpha) t^k$. Each term $H_k^{(i,j)}(\alpha)$ has the form reported in Appendix.

Remark 2. A conceptually simple but algebraically tedious generalization of the results reported in the Appendix shows that even assuming $K(t) = \sum_{i=0}^{\ell} K_i t^i,\ L(t) = \sum_{i=0}^{\ell} L_i t^i,\ K_i, L_i \neq 0$, in any case $H_{2\ell}(\alpha)$ only depends on K_ℓ and L_ℓ $\triangle$.

As neither $R(t,\alpha)$, nor $A(t,\alpha)$ are uniformly bounded, the stability analysis of Σ_f requires the two following lemmas.

Lemma 1. Jetto et al. [2009] If there exists a finite $\bar{t} \geq 0$ such that

$$\dot{V}(\hat{x}_f(t),\alpha) < 0, \quad \forall \alpha \in S, \quad \forall t \geq \bar{t}, \quad (14)$$

then Σ_f is exponentially γ-stable.

Lemma 2. Jetto et al. [2009] If $\exists\ \bar{k} \in [0, 1, \cdots, 2\ell]$ such that $\forall \alpha \in S$, one has

$$H_{\bar{k}}(\alpha) < 0, \quad H_{\bar{k}+j}(\alpha) \leq 0,\ 1 \leq j \leq 2\ell - \bar{k}, \quad (15)$$

then condition (14) holds $\forall t \geq \bar{t} \triangleq \bar{t}(\bar{k}) \geq 0$, where $\bar{t}(\bar{k})$ is the minimum t such that

$$\sum_{k=0}^{\bar{k}-1} H_k(\alpha) t^{(k-\bar{k})} < -\sum_{k=\bar{k}}^{2\ell} H_k(\alpha) t^{(k-\bar{k})}. \quad (16)$$

Remark 3. The importance of Lemma 2 is that the stability of Σ_f can be guaranteed with no constraint on $H_k(\alpha)$, for $k < \bar{k}$. Hence a robust stabilizing output dynamic controller can be found by simply imposing the fulfillment of (15) for $\bar{k} = 2\ell$. This drastically reduce the numerical

$$
\begin{aligned}
H^{(1,1)}(t,\alpha) &= P(t,\alpha)(A(t,\alpha)+BK(t)) + (A(t,\alpha)+BK(t))^T P(\alpha) + \dot{P}(t,\alpha) + W(t,\alpha)\Delta A(t,\alpha) + \Delta A^T(t,\alpha)W^T(t,\alpha) \\
H^{(2,1)}(t,\alpha) &= W^T(t,\alpha)(A(t,\alpha)+BK(t)) + Q(t,\alpha)\Delta A(t,\alpha) - K^T(t)B^T P(t,\alpha) + (\overline{A}(t)+L(t)C)^T W^T(t,\alpha) + \dot{W}^T(t,\alpha) \\
H^{(2,2)}(t,\alpha) &= Q(t,\alpha)(\overline{A}(t)+L(t)C) + (\overline{A}(t)+L(t)C)^T Q(t,\alpha) + \dot{Q}(t,\alpha) - W^T(t,\alpha)BK(t) - K^T(t)B^T W(t,\alpha) \qquad (12)
\end{aligned}
$$

complexity of the synthesis procedure because $H_{2\ell}(\alpha) \triangleq H_{2\ell}(\alpha_\ell) < 0$ only involves the following matrices: $A_\ell(\alpha_\ell)$, $\Delta A_\ell(\alpha_\ell)$, $R_\ell(\alpha_\ell) = \begin{pmatrix} P_\ell(\alpha_\ell) & 0 \\ 0 & Q_\ell(\alpha_\ell) \end{pmatrix}$, K_ℓ and L_ℓ. The diagonal structure of $R_\ell(\alpha_\ell)$ introduces some conservatism but on the other hand allows the simultaneous design of K_ℓ and L_ℓ under the assumption of parametric uncertainties arbitrarily increasing with time. The counterpart is that $\dot{V}(\hat{x}_f(t),\alpha) < 0$, $\forall \alpha \in S$, is not satisfied for $t \geq 0$ and hence both a slower convergence to the null equilibrium point and a more conservative estimate of the dwell time are obtained. Nevertheless, as shown later, the conservatism can be greatly reduced through an iterative constrained minimization procedure (ICMP) based on semidefinite programming. The ICMP exploits the degrees of freedom introduced by the full block matrices $R_k(\alpha_k)$, $k = 0,\cdots,\ell-1$, of $R(t,\alpha)$ to minimize the time instant $\bar{t} = \bar{t}(\bar{k}) = \bar{t}(2\ell)$ such that (16) for $\bar{k} = 2\ell$ is satisfied (or equivalently (14) hold). $\triangle$

Remark 4. In the light of Remark 3, Remark 2 justifies the assumption on the form of $K(t)$ and $L(t)$ given by (4). $\triangle$

Theorem 1. The condition (15) of Lemma 2 holds for $\bar{k} = 2\ell$ if there exist positive definite symmetric matrices $\tilde{P}_{\ell,j}$, $Q_{\ell,j}$, $j = 1,\cdots,n_\ell$, Z, $\tilde{J}_\ell$ and matrices V_ℓ, X_ℓ, Y_ℓ, such that for some fixed positive scalars β_ℓ, γ_ℓ, conditions (17)-(20)(reported at the top of page 4) are satisfied. $\triangle$
Proof of Theorem 1. Not reported for brevity.

If the set of LMIs (17)-(20) admits a solution for a fixed pair (γ_ℓ,β_ℓ) then (15) holds for $\bar{k} = 2\ell$. By Lemmas 1 and 2, the closed loop system Σ_f results to be exponentially γ-stable for some $\gamma > 0$. The gain matrices characterizing the dynamic output controller (3)-(4) are:

$$
K_\ell = Y_\ell Z^{-1} \qquad L_\ell = V_\ell^{-T} X_\ell. \qquad (21)
$$

The respective Lyapunov function (7)-(9) is characterized by $R_\ell(\alpha_\ell) \triangleq \begin{pmatrix} P_\ell(\alpha_\ell) & 0 \\ 0 & Q_\ell(\alpha_\ell) \end{pmatrix}$ with vertices $R_{\ell,j} = \begin{pmatrix} Z^{-T}\tilde{P}_{\ell,j}Z^{-1} & 0 \\ 0 & Q_{\ell,j} \end{pmatrix}$, $j = 1,\cdots,n_\ell$ and by the full block matrices $R_i(\alpha_i)$'s, $i = 0,\cdots,2\ell-1$ with vertices $R_{i,j} = \begin{pmatrix} P_{i,j} & W_{i,j} \\ W_{i,j}^T & Q_{i,j} \end{pmatrix}$ still to be determined. According to Remark 3, they are degrees of freedom exploited by the ICMP for minimizing $\bar{t} = \bar{t}(2\ell)$, as explained beneath, and hence to reduce the introduced conservatism imposing $\dot{V}(\hat{x}_f(t),\alpha) < 0$ for $t \geq \bar{t} > 0$.

2.1 Minimization of $\bar{t}$

As $\bar{k} = 2\ell$, by Lemma 2, $\bar{t}$ is the minimum t such that

$$
\sum_{i=0}^{2\ell-1} \frac{H_i(\alpha)}{t^{2\ell-i}} < -H_{2\ell}(\alpha). \qquad (22)
$$

Taking into account that: 1) $H_{2\ell}(\alpha) = H_{2\ell}(\alpha_\ell) < -v_{2\ell}I$ for some known positive scalar $v_{2\ell}$, 2) for each $H_i(\alpha)$, $i = 0,\cdots,2\ell-1$, there exists a scalar v_i, to be determinated, such that $H_i(\alpha) < v_i I$, inequality (22) makes it evident that to reduce $\bar{t}$ it is necessary to minimize the maximum eigenvalue v_i of each $H_i(\alpha)$, $i = 0,\cdots,2\ell-1$, $\alpha \in S$. Unfortunately, the ℓ degrees of freedom $R_i(\alpha_i)$, $i = 0,\cdots,\ell-1$, allow us to only solve ℓ problems of minimization with respect to the maximum eigenvalues of ℓ matrices $H_i(\alpha)$ with $i \in [0,\cdots,2\ell-1]$. Inequality (22) suggests that a more significant reduction of $\bar{t}$ is obtained minimizing the maximum eigenvalues of the ℓ matrices $H_{\ell+k}(\alpha)$, for $k = 0,\cdots,\ell-1$.
As the ℓ matrices $H_{\ell+k}(\alpha)$, for $k = 0,\cdots,\ell-1$, are functions of the unknown $R_i(\alpha_i)$, $i = 0,\cdots,\ell-1$, according to the following relation for $k = 0,\cdots,\ell-1$:

$$
H_{\ell+k}(\alpha) = f_{\ell+k}(R_k(\alpha_k), R_{k+1}(\alpha_{k+1},\cdots,R_{\ell-1}(\alpha_{\ell-1})),
$$

the ICMP has the following structure:

(1) put $k = \ell-1$
(2) min $v_{\ell+k}$ subject to

$$
H_{\ell+k}(\alpha) < v_{\ell+k} \cdot I \qquad (23)
$$

Constraint (23) corresponds to the finite set of LMIs whose unknowns to be determinated are $v_{\ell+k}$ and the vertices of the polytopic $R_k(\alpha_k)$
(3) put $k = k-1$, if $k \geq 0$ go to step (2) otherwise end.

At the end of the ICMP, all the vertices of $R_i(\alpha_i)$'s, $i = 1,\cdots,\ell-1$, have been computed. This means that all available degrees of freedom have been used and, as a consequence, the matrices $H_i(\alpha)$, $i = 0,\cdots,\ell-1$, (namely those ones which have not been taken into account in the minimization procedure) are automatically determinated at the vertices as well as their respective maximum eigenvalues v_i.
In conclusion, by (22) one has that the finite $\bar{t} = \bar{t}(2\ell)$ such that $\dot{V}(\hat{x}_f(t),\alpha) < 0$, $\forall t \geq \bar{t}(2\ell)$ is obtained in the following way:

$$
\bar{t} = \bar{t}(2\ell) = \min_{t} : \sum_{i=0}^{2\ell-1} \frac{v_i}{t^{2\ell-i}} < v_{2\ell}. \qquad (24)
$$

3. SWITCHING MODE PLANTS

The results of Section 2 are here exploited to investigate stabilizability conditions for the class of switching systems $\Sigma_{\sigma(t)}$ that, according to the introductory considerations, are described by

$$
\Sigma_{\sigma(t)} = \begin{cases} \dot{x}(t) = A_{\sigma(t)}(t,\alpha^{(\sigma(t))})x(t) + Bu(t), \\ y(t) = Cx(t) \end{cases} \qquad (25)
$$

where $\sigma(t) : [0,\infty) \to \mathcal{P} \triangleq [1,\cdots,\bar{p}]$ is a piecewise constant function which represents the switching signal,

$$\begin{bmatrix} -2\gamma_\ell \tilde{P}_{\ell,j} + \tilde{J}_\ell & \tilde{P}_{\ell,j} + Z^T A_{\ell,j}^T + Y_\ell^T B^T + \gamma_\ell Z^T & \tilde{P}_{\ell,j} \\ \tilde{P}_{\ell,j} + A_{\ell,j} Z + BY_\ell + \gamma_\ell Z & -Z - Z^T & 0 \\ \tilde{P}_{\ell,j} & 0 & -Z^T \end{bmatrix} < 0, \quad j = 1, \cdots, n_\ell, \tag{17}$$

$$\begin{bmatrix} -\tilde{J}_\ell & Z^T \Delta A_{\ell,j}^T \\ \Delta A_{\ell,j} Z & -I \end{bmatrix} < 0, \quad j = 1, \cdots, n_\ell, \tag{18}$$

$$\begin{bmatrix} -Z & Y_\ell^T B^T \\ BY_\ell & -Z \end{bmatrix} < 0, \tag{19}$$

$$\left[\begin{array}{ccc|c} -2\beta_\ell Q_{\ell,j} & Q_{\ell,j} + \overline{A}_\ell^T V_\ell + C^T X_\ell^T + \beta_\ell V_\ell & Q_{\ell,j} & I \\ Q_{\ell,j} + V_\ell^T \overline{A}_\ell + X_\ell C + \beta_\ell V_\ell^T & -V_\ell - V_\ell^T & 0 & 0 \\ Q_{\ell,j} & 0 & -I & 0 \\ \hline I & 0 & 0 & -Z^T \end{array}\right] < 0, \quad j = 1, \cdots, n_\ell. \tag{20}$$

its value $\sigma(t)$ identifies the particular mode acting at time t. Any interval over which a particular mode p is active is denoted by $T_k^{(p)} \triangleq [t_k^{(p)}; t_m^{(q)})$, $k, m \in \mathbb{Z}^+$. This notation means that over $T_k^{(p)}$ the p-th mode occurs for the k-th time (since $t = 0$) and it is followed by the m-th occurrence (since $t = 0$) of the q-th mode. Hence, $\forall t \in T_k^{(p)}$, each $A_p(t, \alpha^{(p)})$ is of the same kind of $A(t, \alpha)$ given by (2) and can be written as

$$A_p(t, \alpha^{(p)}) = A_{p,0}(\alpha_0^{(p)}) + \sum_{i=1}^{\ell_p} A_{p,i}(\alpha_i^{(p)})(t - t_k^{(p)})^i \tag{26}$$

where $A_{p,i}(\alpha_i^{(p)}) = \sum_{j=1}^{n_{p_i}} \alpha_{i,j}^{(p)} A_{i,j}^{(p)}$.
The following assumption is made:

A1). Both the switching instant and the new configuration assumed by the switched plant are assumed to be known.

The Lyapunov function associated to each mode is defined as : $V_p(\hat{x}_f(t), \alpha^{(p)}) = \hat{x}_f^T(t) R_p(t, \alpha^{(p)}) \hat{x}_f(t)$, $p \in \mathcal{P}$, where, accordingly to (26), the time is reset at every switching instant defining $R_p(t, \alpha^{(p)})$, $\forall t \in T_k^{(p)}$ as

$$R_p(t, \alpha^{(p)}) = R_{p,0}(\alpha_0^{(p)}) + \sum_{i=1}^{\ell_p} R_{p,i}(\alpha_i^{(p)})(t - t_k^{(p)})^i. \tag{27}$$

Each $R_{p,i}(\alpha_i^{(p)}), p \in \mathcal{P}$, is of the same kind of $R_i(\alpha_i)$ given by (9). Moreover it is assumed :

A2). $R_{p,0}(\alpha_0^{(p)}) = \mathcal{R}_{\mathbf{0}}(\alpha_{\mathbf{0}}), \forall p \in \mathcal{P}$, for a suitably defined constant vector $\alpha_{\mathbf{0}} \in R^{\mathbf{n_0}}$ where $\mathbf{n_0} = \max_p \{n_{p_0}\}$.

Assumption *A2)* implies that the degree of freedom $\mathcal{R}_{\mathbf{0}}(\alpha_{\mathbf{0}})$ is common to all the $\bar{p}$ Lyapunov matrices $R_p(t, \alpha^{(p)})$. As it will be explained later, this hypothesis will allow us to derive stabilizability conditions for the switching plant $\Sigma_{\sigma(t)}$.
The derivative of each $V_p(\hat{x}_f(t), \alpha^{(p)})$ is $\dot{V}_p(\hat{x}_f(t), \alpha^{(p)}) = \hat{x}_f^T(t) H_p(t, \alpha^{(p)}) \hat{x}_f(t)$, where analogously to (13),$\forall t \in T_k^{(p)}$, $H_p(t, \alpha^{(p)})$ is given by

$$H_p(t, \alpha^{(p)}) = H_{p,0}(\alpha^{(p)}) + \sum_{i=1}^{2\ell_p} H_{p,i}(\alpha^{(p)})(t - t_k^{(p)})^i. \tag{28}$$

Let $\Sigma_{f,p}$ be the feedback connection of the mode Σ_p with the corresponding stabilizing observer based controller $\Sigma_{c,p}$ computed as explained in section 2. By Lemma 2, each $V_p(\hat{x}_f(t), \alpha^{(p)})$ is negative definite $\forall t \geq \bar{t}_p(2\ell_p)$.
By *A2)*, the $\bar{p}$ ICMP's are independent of each other until $k = 1$. For $k = 0$, the $\bar{p}$ constraints of the kind of (23) must be simultaneously satisfied. Hence putting all the $v_{\ell_p}^{(p)}$'s, $p = 1, \cdots, \bar{p}$, equal to $\mathbf{v}$ one has

$$min \quad \mathbf{v} \qquad subject \quad to \quad \begin{cases} H_{1,\ell_1}(\alpha^{(1)}) < \mathbf{v} I \\ \vdots \\ H_{\bar{p},\ell_{\bar{p}}}(\alpha^{(\bar{p})}) < \mathbf{v} I \end{cases} \tag{29}$$

where the unknowns are the scalar $\mathbf{v}$ and the vertices of $\mathcal{R}_{\mathbf{0}}(\alpha_{\mathbf{0}})$.
Analogously to (24), each $\bar{t}_p = \bar{t}_p(2\ell_p) > 0$, $p \in \mathcal{P}$, can be obtained as:

$$\bar{t}_p(2\ell_p) = \min_t : \sum_{i=0}^{\ell_p - 1} \frac{v_i^{(p)}}{t^{2\ell_p - i}} + \frac{\mathbf{v}}{t^{\ell_p}} + \sum_{i=\ell_p+1}^{2\ell_p - 1} \frac{v_i^{(p)}}{t^{2\ell_p - i}} < v_{2\ell_p}^{(p)}. \tag{30}$$

3.1 Stabilizability conditions for the switching mode plant

Provided that the conditions of Theorem 1 are satisfied, each controller $\Sigma_{c,p}$ stabilizing the corresponding Σ_p guarantees that inside each $T_k^{(p)}$, $p \in \mathcal{P}$, there exists a $\bar{t}_k^{(p)}$, with $\bar{t}_k^{(p)} - t_k^{(p)} \triangleq \bar{t}_p$ independent of k, such that $\dot{V}_p(\hat{x}_f(t), \alpha^{(p)}) < 0$, $\forall t \geq \bar{t}_k^{(p)}$, $t \in T_k^{(p)}$. In practice $\bar{t}_k^{(p)}$ has the same meaning of $\bar{t}_p$ in (30) for a single mode of the plant. Stability conditions for the switching closed loop system $\Sigma_{f,\sigma(t)}$ are stated in terms of minimum dwell time, as stated in the following Theorem.

Theorem 2. Provided that each $\Sigma_{f,p}$ is exponentially γ_p stable for some $\gamma_p > 0$, the switching system $\Sigma_{f,\sigma(t)}$ is asymptotically stable if the length $L_k^{(p)}$ of each $T_k^{(p)} = [t_k^{(p)}; t_m^{(q)})$, $\forall p \in \mathcal{P}$, $\forall k, m \in \mathbb{Z}^+$ is such that:

$$L_k^{(p)} \geq \tau_p > \bar{t}_p. \quad \text{where } \tau_p \text{ is the minimum time} \tag{31}$$

$$\text{such that } \int_{t_k^{(p)}}^{t_k^{(p)} + \tau_p} \dot{V}_p(\hat{x}_f(t), \alpha^{(p)}) dt < 0. \tag{32}$$

Proof of Theorem 2. Not reported for brevity.

4. NUMERICAL RESULTS

Consider the following dynamical plant $\Sigma_{\sigma(t)}$ dependent on the switching signal $\sigma(t) : [0, \infty) \to \mathcal{P} = [1, 2]$, described by the following triplet $(C, A_p(t, \alpha^{(p)}), B)$: $C = [0\ 1]$, $A_p(t, \alpha^{(p)}) = \begin{bmatrix} 8 & -9 \\ 120 & -18 \end{bmatrix} + \theta^{(p)}(t) \begin{bmatrix} -108 & -9 \\ -120 & 17 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}$. The dynamical matrix is borrowed from Montagner et al. [2003], where the stability is proved in the uncertainty range $\theta(t) \in [0, 1]$. The parameter $\theta(t)$ is here assumed to be a switching interval polynomial function $\theta^{(p)}(t)$ of degree $\ell_p = 2$, more precisely: $\theta^{(1)}(t) = [0,\ \ 0.2] + [0.01,\ \ 0.02]\, t + [0.005,\ \ 0.01]\, t^2$ and $\theta^{(2)}(t) = [0,\ \ 0.2] + [-0.02,\ \ -0.01]\, t + [-0.002,\ \ -0.001]\, t^2$. Unlike all the existing literature both the parameter and its derivative are allowed to vary over theoretically unbounded uncertainty sets. For each mode $p \in \mathcal{P}$, by varying both $\gamma_{\ell_p} = \gamma_2$ and $\beta_{\ell_p} = \beta_2$ with a logarithmic scale inside $[10^{-1},\ 10^1]$, conditions of Theorem 1 result to be satisfied for $\gamma_2 = \beta_2 = 1$. This means that $H_{p,2\ell_p}(\alpha^{(p)}) < 0$, for each $p \in \mathcal{P}$. More precisely the solution is given by: $H_{1,4}(\alpha^{(1)}) < -v_4^{(1)} \cdot I = -0.1879 \cdot I$ $(p = 1)$ and $H_{2,4}(\alpha^{(2)}) < -v_4^{(2)} \cdot I = -0.0078 \cdot I$ $(p = 2)$. By Lemma 2, $\dot{V}_p(\hat{x}_f(t), \alpha^{(p)}) < 0$, $\forall t \geq \bar{t}_p(2\ell_p)$ for some $\bar{t}_p(2\ell_p) > 0$. By Lemma 1, each $\Sigma_{f,p}$, $p \in \mathcal{P}$ is exponentially stable and the gain matrices, obtained through (21), of the respective controller $\Sigma_{c,p}$ are:

$$K_2 = \begin{pmatrix} -0.5649 & -0.0269 \\ 0.8802 & -1.6784 \end{pmatrix},\ L_2 = \begin{pmatrix} 0.8996 \\ -1.9163 \end{pmatrix},\ (p = 1),$$

$$K_2 = \begin{pmatrix} -0.9430 & 0.0590 \\ -0.1912 & -1.6097 \end{pmatrix},\ L_2 = \begin{pmatrix} -22.7356 \\ -4.3827 \end{pmatrix}\ (p = 2).$$

By Remark 3, as $\ell_p = 2$, $\forall p \in \mathcal{P}$, two degrees of freedom $R_{p,1}(\alpha_1^{(p)})$ and $R_{p,0}(\alpha_0^{(p)}) = \mathcal{R}_{\mathbf{0}}(\alpha_{\mathbf{0}})$ are available for minimizing $\bar{t}_p(2\ell_p)$.
Each ICMP consists of two iterations described as reported at the top of page 6.
By *A2)*, the two ICMP's are simultaneously solved in the last iteration. The solution is given by: $v_3^{(1)} = v_3^{(2)} = -1$ and $\mathbf{v} = -17.3116$. As mentioned in Section 2, the maximum eigenvalues $v_1^{(p)}$ and $v_0^{(p)}$ of $H_{p,1}(\alpha^{(p)})$'s and $H_{p,0}(\alpha^{(p)})$'s respectively $p \in \mathcal{P}$, are automatically determinated and their values are $v_1^{(1)} = 479.5$, $v_1^{(2)} = 11,112$, $v_0^{(1)} = 29,415$ and $v_0^{(2)} = 29,433$. The Lyapunov functions $R_p(t, \alpha^{(p)})$'s, $p \in \mathcal{P}$ are not reported to save space. Applying (30) one has:
$\dot{V}_1(\hat{x}_f(t), \alpha^{(1)}) < 0$, $t \geq \bar{t}_1 = 20$ and $\dot{V}_2(\hat{x}_f(t), \alpha^{(2)}) < 0$, $t \geq \bar{t}_2 = 26$.
Assuming to know that $\|x(0)\| \leq 1$, some calculations (not reported for brevity) show that condition (32) is satisfied for $\tau_1 = 29$, $(p = 1)$ and $\tau_2 = 41$, $(p = 2)$. Recalling that L_k^p is the length of $T_k^{(p)}$, by (31), the switching closed loop system $\Sigma_{f,\sigma(t)}$ is asymptotically stable if $L_k^1 \geq 29$ and $L_k^2 \geq 41$.
A simulation has been performed starting from $x(0) = [0.1, 0.1, 0, 0]^T$. The trajectories of $\theta^{(1)}(t)$ and $\theta^{(2)}(t)$ have been generated according to $\theta^{(1)}(t) = 0.1 + 0.01 \cdot t + 0.01 \cdot t^2$ $(p = 1)$ and $\theta^{(2)}(t) = 0.2 - 0.02 \cdot t - 0.001 \cdot t^2$ $(p = 2)$ respectively. Over a simulation interval of amplitude 80, the switching plant $\Sigma_{\sigma(t)}$ is such that: $\Sigma_{\sigma(t)} = \Sigma_1(t)$, $\forall t \in [0, 30] \triangleq T_1^{(1)}$ and $\Sigma_{\sigma(t)} = \Sigma_2(t)$, $\forall t \in (30, 80] \triangleq T_1^{(2)}$. The output response of the switching closed loop system $\Sigma_{f,\sigma(t)}$ practically converges to zero for $t \leq 1$. The plot is not reported for brevity.

5. CONCLUSIONS

The main novelty of this paper is a controller synthesis method for uncertain plants whose parameters are allowed to vary inside arbitrarily large domains. Another interesting by-product of the paper is the way each single-mode stabilizing controller is derived. The dynamic output-feedback controller is obtained through a set of LMI's by fixing two positive scalars. If the set is feasible, it provides both the observer and the static gains in only one step for parametric uncertainty arbitrarily increasing with time.

Appendix

Explicit expression of $H_k^{(i,j)}(\alpha)$: Exploiting (10)-(13), it can be shown that each term $H_k^{(i,j)}(\alpha)$ has the form reported on page 6.

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ICMP's	(p=1)	(p=2)
i)	$\min \; v_3^{(1)}$ subject to $H_{1,3}(\alpha^{(1)}) < v_3^{(1)} I$	$\min \; v_3^{(2)}$ subject to $H_{2,3}(\alpha^{(2)}) < v_3^{(2)} I$
ii)	$\min \; \mathbf{v}$ subject to $\begin{cases} H_{1,2}(\alpha^{(1)}) < \mathbf{v} I \\ H_{2,2}(\alpha^{(2)}) < \mathbf{v} I \end{cases}$	

- $H_k^{i,j}(\alpha), \quad i = j = 1$

$$\begin{cases} \underset{0\le k\le \ell-1}{H^{(1,1)}_{k}}(\alpha) = \displaystyle\sum_{\substack{i+j=k\\ i,j\ge 0}} [P_i(\alpha_i)A_j(\alpha_j) + A_j^T(\alpha_j)P_i(\alpha_i) + W_i(\alpha_i)\Delta A_j(\alpha_j) + \Delta A_j^T(\alpha_j)W_i^T(\alpha_i)] + (k+1)P_{k+1}(\alpha_{k+1}), \\ \underset{0\le k\le \ell-1}{H^{(1,1)}_{\ell+k}}(\alpha) = P_k(\alpha_k)(A_\ell(\alpha_\ell) + BK_\ell) + (A_\ell(\alpha_\ell) + BK_\ell)^T P_k(\alpha_k) + P_\ell(\alpha_\ell)A_k(\alpha_k) + A_k^T(\alpha_k)P_\ell(\alpha_\ell) \\ \qquad + W_k(\alpha_k)\Delta A_\ell(\alpha_\ell) + \Delta A_\ell^T(\alpha_\ell)W_k^T(\alpha_k) \\ \qquad + \displaystyle\sum_{\substack{k+1\le i\le \ell-1\\ i+j=\ell+k}} [P_i(\alpha_i)A_j(\alpha_j) + A_j^T(\alpha_j)P_i(\alpha_i) + W_i(\alpha_i)\Delta A_j(\alpha_j) + \Delta A_j^T(\alpha_j)W_i^T(\alpha_i)], \\ H^{(1,1)}_{2\ell}(\alpha) \quad = P_\ell(\alpha_\ell)(A_\ell(\alpha_\ell) + BK_\ell) + (A_\ell(\alpha_\ell) + BK_\ell)^T P_\ell(\alpha_\ell) \end{cases}$$

- $H_k^{i,j}(\alpha), \quad i = 2, \; j = 1$

$$\begin{cases} \underset{0\le k\le \ell-1}{H^{(2,1)}_{k}}(\alpha) = \displaystyle\sum_{\substack{i+j=k\\ i,j\ge 0}} [W_i^T(\alpha_i)A_j(\alpha_j) + Q_i(\alpha_i)\Delta A_j(\alpha_j) + \overline{A}_j^T W_i^T(\alpha_i)] + \begin{cases} (k+1)W_{k+1}^T(\alpha_{k+1}), & k = 0, \cdots, \ell-2 \\ 0, & k = \ell-1 \end{cases} \\ \underset{0\le k\le \ell-1}{H^{(2,1)}_{\ell+k}}(\alpha) = W_k^T(\alpha_k)(A_\ell(\alpha_\ell) + BK_\ell) + Q_k(\alpha_k)\Delta A_\ell(\alpha_\ell) + Q_\ell(\alpha_\ell)\Delta A_k(\alpha_k) \\ \qquad + (\overline{A}_\ell + L_\ell C)^T W_k^T(\alpha_k) - K_\ell^T B^T P_k(\alpha_k) \\ \qquad + \displaystyle\sum_{\substack{k+1\le i\le \ell-1\\ i+j=\ell+k}} [W_i^T(\alpha_i)A_j^T(\alpha_j) + \overline{A}_j W_i^T(\alpha_i) + Q_i(\alpha_i)\Delta A_j(\alpha_j)] \\ H^{(2,1)}_{2\ell}(\alpha) \quad = Q_\ell(\alpha_\ell)\Delta A_\ell(\alpha_\ell) - K_\ell^T B^T P_\ell(\alpha_\ell) \end{cases}$$

- $H_k^{i,j}(\alpha), \quad i = j = 2$

$$\begin{cases} \underset{0\le k\le \ell-1}{H^{(2,2)}_{k}}(\alpha) = \displaystyle\sum_{\substack{i+j=k\\ i,j\ge 0}} [Q_i(\alpha_i)\overline{A}_j + \overline{A}_j^T Q_i(\alpha_i)] + (k+1)Q_{k+1}(\alpha_{k+1}), \\ \underset{0\le k\le \ell-1}{H^{(2,2)}_{\ell+k}}(\alpha) = Q_k(\alpha_k)(\overline{A}_\ell + L_\ell C) + (\overline{A}_\ell + L_\ell C)^T Q_k(\alpha_k) + Q_\ell(\alpha_\ell)\overline{A}_k + \overline{A}_k^T Q_\ell(\alpha_\ell) \\ \qquad - W_k^T(\alpha_k)BK_\ell - K_\ell^T B^T W_k(\alpha_k) + \displaystyle\sum_{\substack{k+1\le i\le \ell-1\\ i+j=\ell+k}} [Q_i(\alpha_i)\overline{A}_j + \overline{A}_j^T Q_i(\alpha_i)] \\ H^{(2,2)}_{2\ell}(\alpha) \quad = Q_\ell(\alpha_\ell)(\overline{A}_\ell + L_\ell C) + (\overline{A}_\ell + L_\ell C)^T Q_\ell(\alpha_\ell) \end{cases}$$

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