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Local Stabilization of Markov Jump Nonlinear Quadratic Systems ⋆

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Abstract: This paper investigates the problem of local stabilization of Markov jump nonlinear quadratic systems. A method is presented for the synthesis of a static nonlinear quadratic state feedback control law that ensures the local exponential mean square stability of the zero equilibrium point of the closed-loop system in some polytopic region of the state-space with a guaranteed region of stability inside this polytope. The proposed control design is tailored in terms of linear matrix inequalities together with convex optimization to achieve an enlarged stability region. A numerical example is presented to illustrate the application of the stabilization method.

Keywords: Markov jump systems, nonlinear quadratic systems, local stabilization, stability region.

1. INTRODUCTION

Markov jump systems, namely dynamic systems that are subject to random abrupt parameters changes in their structure which are modeled via a Markov process, constitute an important class of dynamic systems and find applications in a number of scenarios, as for instance, to model plants subject to random components failures, sudden environment disturbances and changes of the operating point. Over the last ten or more years, intensive research effort has been direct towards Markov jump systems and important advances have been made to the topics of stability analysis, control design and filtering in the context of linear systems; see, the seminal works of Sworder [1969] and Wonham [1970], and the books [Mariton, 1990, Boukas, 2006, Costa *et al.*, 2013, 2014] and their references to cite a few. However, to date, few results have focused on the nonlinear counterpart. For instance, Boukas *et al.* [2003], Wei *et al.* [2008] and Wang *et al.* [2010] have dealt with control of Markov jump linear systems subject to unknown nonlinearities under either a global Lipschitz condition or linear growth condition, Liu *et al.* [2006] have proposed stabilization conditions for jump linear systems subject to sector bound nonlinearities, and in Wu *et al.* [2009] the classical backstepping technique of control design for nonlinear systems has been extended to Markov jump systems. Furthermore, de Souza and Coutinho [2006] have developed a linear matrix inequality (LMI) method of local robust stability analysis and estimation of domain of attraction for the class of Markov jump nonlinear rational systems under parametric uncertainty, and very recently Zhao *et al.* [2012] have studied the input-to-state stability of Markov jump nonlinear systems.

On the other hand, the class of bilinear systems and its extension, the so-called *nonlinear quadratic systems* (i.e. systems with quadratic nonlinearities in the state variables and bilinear terms in the state and control signal), have attracted the attention of control systems researchers in the last years due to its capability of adequately modeling a number of process dynamics while keeping the conditions for stability analysis and control synthesis numerically tractable. To cite a few, Amato *et al.* [2007] have derived conditions for designing locally stabilizing linear controllers with a guaranteed region of stability for quadratic systems, Valmoribida *et al.* [2010] have addressed the problem of actuator saturation, and Coutinho and de Souza [2012] have proposed designs of nonlinear quadratic controllers for local stabilization, quadratic cost control, and H_∞ control of quadratic system while providing a guaranteed region of stability. In spite of these developments, to the authors' knowledge, the design of locally stabilizing feedback controllers for Markov jump nonlinear quadratic systems has not yet been addressed in the specialized literature.

This paper deals with the problem of local mean square stabilization of Markov jump nonlinear quadratic systems. The motivations for considering this problem are as follows. First, Markov jump systems with quadratic nonlinearities can represent a large number of processes, and includes the so-called Markov jump bilinear systems as a special case. Secondly, stabilization conditions generally do not hold globally for nonlinear systems. We will develop an LMI based synthesis method of static nonlinear quadratic state feedback controllers to ensure the local exponential mean square stability of the zero equilibrium point of the closed-loop system with a guaranteed region of stability (in the mean-square sense) inside a given bounded polyhedral region of the state-space.

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Notation. $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n\times m$ real matrices, $\|\cdot\|$ is the Euclidean vector norm, 0_n and $0_{m\times n}$ are respectively the $n\times n$ and $m\times n$ matrices of zeros, I_n is the $n\times n$ identity matrix, and diag$\{\cdots\}$ denotes a block-diagonal matrix. For a real matrix S, S' is the transpose of S, He(S) denotes $S+S'$, and $S>0$ ($S\geq 0$) means that S is symmetric and positive definite (semi-definite). The symbol $\star$ in symmetric block matrices stands for the transpose of the blocks outside the main diagonal block and mathematical expectation is denoted by $\mathbf{E}[\cdot]$.

2. PROBLEM FORMULATION

Fix an underlying probability space $(\Omega,\mathscr{F},I\!P)$ and consider the stochastic system:

$$\dot{x}(t)=A_{\theta_t}(x(t))x(t)+B_{\theta_t}(x(t))u(t),\quad x(0)=x_0\in\mathscr{X} \tag{1}$$

where $x(t)\in\mathbb{R}^n$ is the state vector $u(t)\in\mathbb{R}^{n_u}$ is the control vector, $\{\theta_t\}$ is a homogeneous Markov process with right continuous trajectories and assuming values on a finite set $\Theta=\{1,2,\ldots,\sigma\}$ that represents the system operating modes, $\mathscr{X}\subset\mathbb{R}^n$ is a given polytopic region of the state-space containing the origin that defines the domain to be considered for local stability analysis (to be specified latter), and $A_{\theta_t}(x)$ and $B_{\theta_t}(x)$ for each possible values of $\theta_t=i,\ \forall\, i\in\Theta$, are affine matrix functions of x, namely

$$A_{\theta_t}(x)=A^{[0]}_{\theta_t}+\sum_{k=1}^{n}x_kA^{[k]}_{\theta_t},\quad B_{\theta_t}(x)=B^{[0]}_{\theta_t}+\sum_{k=1}^{n}x_kB^{[k]}_{\theta_t}, \tag{2}$$

where x_k denotes the k-th component of x, and $A^{[k]}_{\theta_t}$ and $B^{[k]}_{\theta_t}$, $k=1,\ldots,n$, for all $\theta(t)\in\Theta$, are given constant matrices.

The Markov process $\{\theta_t\}$ is assumed to satisfy the following assumptions:

Assumption 1.

(a) $\{\theta_t\}$ has a stationary transition rate matrix $\Lambda=[\lambda_{ij}]$, $i,\ j=1,\ldots,\sigma$, such that

$$I\!P\{\,\theta_{t+h}=j\,|\,\theta_t=i\,\}=\begin{cases}\lambda_{ij}h+o(h), & i\neq j\\ 1+\lambda_{ii}h+o(h), & i=j\end{cases}$$

where $h>0$, $\lim_{h\downarrow 0}\frac{o(h)}{h}=0$, $\lambda_{ij}\geq 0$ is the transition rate from the state i to the state j, $i\neq j$, and

$$\lambda_{ii}=-\sum_{\substack{j=1\\ j\neq i}}^{\sigma}\lambda_{ij}; \tag{3}$$

(b) $\{\theta_t\}$ is accessible.

Note that the joint process $\{\,(x(t),\,\theta_t),\ t\geq 0\}$ is a Markov process; see, for instance, Wonham [1970].

This paper focuses on designing a mode-dependent static nonlinear quadratic state feedback control law as follows:

$$\begin{cases}u=K_{\theta_t}(x)x;\quad K_{\theta_t}(x)=K_i(x),\ \text{when}\ \ \theta_t=i,\\ K_i(x)=K^{[0]}_i+\displaystyle\sum_{k=1}^{n}x_kK^{[k]}_i,\end{cases} \tag{4}$$

to locally stabilize the equilibrium solution $x=0$ of the closed-loop system, where $K^{[k]}_i$, $i=1,\ldots,\sigma$, $k=1,\ldots,n$, are constant matrices to be found. To this end, the notion of stability for the closed-loop system used in this paper is in the mean square sense. In the sequel we introduce the following concept of *local exponential mean square stability*.

Definition 1. The closed-loop system of (1) with the control law in (4) is said to be *locally exponentially mean square stable* if for any x_0 in a neighborhood of $x=0$ and $\theta_0\in\Theta$, there exist positive scalars α and β such that $x(t)$ satisfies

$$\mathbf{E}[\,\|x(t)\|^2\,]\leq\beta\|x_0\|^2e^{-\alpha t},\ \forall\, t>0.$$

The equilibrium point $x=0$ of system (1) with $u(t)\equiv 0$ is allowed to be mean square unstable and the following assumption is adopted:

Assumption 2. The Markov jump linear system

$$\dot{x}(t)=A_{\theta_t}(0)x(t)+B_{\theta_t}(0)u(t)$$

is mean square stabilizable via a mode-dependent static linear state feedback $u(t)=K_{\theta_t}x(t)$.

Note that Assumption 2 is required in order for determining a mean square stabilizing control law as in (4) for the system (1).

For notation simplicity, in the sequel the argument t of $x(t)$ and $u(t)$ will be often omitted. In addition, the value of a function R_{θ_t} for each possible values of $\theta_t=i,\ \forall\, i\in\Theta$, will be denoted by $R_i,\ i=1,\ldots\ \sigma$.

In the paper, we will address the following stabilization problem for system (1): Determine a control law as in (4) that ensures the local exponential mean square stability of the closed-loop system and a guaranteed *stability region* inside a given polytopic region $\mathscr{X}$ of the state-space, namely a set of initial states x_0 for the closed-loop system inside $\mathscr{X}$ such that $\mathbf{E}[\,\|x(t)\|^2\,]\to 0$ as $t\to\infty$ for all $\theta_0\in\Theta$. The latter problem of local stabilization while providing a stability region will be referred to as *regional mean square stabilization*.

The polytopic state-space region $\mathscr{X}$ plays an important role on deriving an LMI based solution to the regional mean square stabilization problem. For simplicity of presentation, it is assumed that $\mathscr{X}$ is a given symmetric polytope (with respect to the origin). Depending on the context, $\mathscr{X}$ will be represented either in terms of the convex hull of its n_v vertices as below:

$$\mathscr{X}\ =\ \text{Co}\{\,v_1,v_2,\ldots,v_{n_v}\,\}, \tag{5}$$

where $v_k\in\mathbb{R}^n$, $k=1,\ldots,n_v$ are the vertices of $\mathscr{X}$, or in terms of its faces, that is

$$\mathscr{X}=\left\{x\in\mathbb{R}^n:\,|\,c_k'x\,|\leq 1,\ k=1,\ldots,n_f\right\}, \tag{6}$$

with $c_k\in\mathbb{R}^n$, $k=1,\ldots,n_f$ defining the faces of $\mathscr{X}$. Note that the results of this paper can be easily extended to handle a non-symmetric polytope $\mathscr{X}$.

To conclude this section, we present a version of Finsler's lemma to handle constrained inequalities (see, e.g., de Oliveira and Skelton [2001]).

Lemma 1. Given matrix functions $H(v)\in\mathbb{R}^{s\times n_\sigma}$, $S(v)=S(v)'\in\mathbb{R}^{n_\eta\times n_\eta}$ and $\eta(v)\in\mathbb{R}^{n_\eta}$, with $v\in\mathbb{V}\subseteq\mathbb{R}^{\nu}$, then

$$\eta(v)'S(v)\eta(v)<0,\ \forall\, v\in\mathbb{V}:H(v)\eta(v)=0,\ \ \eta(v)\neq 0$$

if there exists a matrix L such that

$$S(v)+\text{He}\big(LH(v)\big)<0,\ \forall\, v\in\mathbb{V}.$$

3. REGIONAL STABILIZATION

First, we write the closed-loop system of (1) with the control law in (4) in the following form:

$$\dot{x} = \left[A_{\theta_t}(x) + \left(B_{\theta_t}^{[0]} + \Pi(x)'\mathbb{B}_{\theta_t}\right) K_{\theta_t}(x)\right] x, \quad (7)$$

where

$$\mathbb{B}_{\theta_t} = \begin{bmatrix} B_{\theta_t}^{[1]} \\ \vdots \\ B_{\theta_t}^{[n]} \end{bmatrix}, \quad \Pi(x) = \begin{bmatrix} x_1 I_n \\ \vdots \\ x_n I_n \end{bmatrix}. \quad (8)$$

For the sake of easier readability, the results related to local stabilization and stability region will be separately presented.

3.1 Local Stabilization

Consider the following mode-dependent Lyapunov function candidate for the closed-loop system in (7):

$$V_{\theta_t}(x(t)) = x(t)' P_{\theta_t} x(t), \quad (9)$$

where P_{θ_t} for each possible value of $\theta_t = i, \ \forall\ i \in \Theta$, is a symmetric positive-definite matrix to be determined.

Let $\mathscr{A}$ be the infinitesimal generator of the Markov process $\{(x(t), \theta_t), \ t \geq 0\}$, where $x(t)$ satisfies (7). Then, it can be readily obtained that (see, e.g. Kushner [1967]):

$$\mathscr{A} \cdot V_{\theta_t}(x) = 2x' P_{\theta_t}\left[A_{\theta_t}(x) + B_{\theta_t}^{[0]} K_{\theta_t}(x) + \Pi(x)'\mathbb{B}_{\theta_t} K_{\theta_t}(x)\right] x + x' \sum_{j=1}^{\sigma} \lambda_{\theta_t, j} P_j x. \quad (10)$$

Define the following variable transformations:

$$\xi_{\theta_t} = P_{\theta_t} x, \quad Q_{\theta_t} = P_{\theta_t}^{-1}. \quad (11)$$

Then, $\mathscr{A} \cdot V_{\theta_t}(x)$ in (10) can be written as

$$\mathscr{A} \cdot V_{\theta_t}(x) = 2\xi_{\theta_t}'\left[A_{\theta_t}(x) + B_{\theta_t}^{[0]} K_{\theta_t}(x) + \Pi(x)'\mathbb{B}_{\theta_t} K_{\theta_t}(x)\right] Q_{\theta_t}\xi_{\theta_t} + \xi_{\theta_t}' Q_{\theta_t} \sum_{j=1}^{\sigma} \lambda_{\theta_t, j} Q_j^{-1} Q_{\theta_t}\xi_{\theta_t}. \quad (12)$$

Next, introducing the parametrization

$$Y_{\theta_t}(x) := Y_{\theta_t}^{[0]} + \sum_{i=1}^{n} x_i Y_{\theta_t}^{[i]} = K_{\theta_t}(x) Q_{\theta_t}, \quad (13)$$

we can recast (12) as

$$\mathscr{A} \cdot V_{\theta_t}(x) = \eta_{\theta_t}'\left[\Phi_{\theta_t}(x) + N' Q_{\theta_t} \Lambda_{\theta_t} \widetilde{Q}_{\theta_t}^{-1} \Lambda_{\theta_t}' Q_{\theta_t} N\right] \eta_{\theta_t}, \quad (14)$$

where

$$\eta_{\theta_t} = \begin{bmatrix} I_n \\ \Pi(x) \end{bmatrix} \xi_{\theta_t}, \quad N = [\, I_n \quad 0_{n \times n^2}\,], \quad (15)$$

$$\Phi_i(x) = \begin{bmatrix} \mathrm{He}\left(A_i(x) + B_i^{[0]} Y_i(x)\right) + \lambda_{ii} Q_i & \star \\ \mathbb{B}_i Y_i(x) & 0 \end{bmatrix}, \quad (16)$$

$$\Lambda_i = [\, \lambda_{i1} I_n \ \cdots \ \lambda_{i(i-1)} I_n \quad \lambda_{i(i+1)} I_n \ \cdots \ \lambda_{i\sigma} I_n \,] \quad (17)$$

$$\widetilde{Q}_i = \mathrm{diag}\{\lambda_{i1} Q_1, \ldots, \lambda_{i(i-1)} Q_{i-1}, \lambda_{i(i+1)} Q_{i+1}, \ldots, \lambda_{i\sigma} Q_\sigma\}. \quad (18)$$

Note that the vector η_{θ_t} satisfies the following equality constraint:

$$\Omega(x)\eta_{\theta_t} = 0, \quad \Omega(x) = \begin{bmatrix} \Pi(x) & -I_{n^2} \\ 0_{n(n-1)\times n} & \mathscr{N}(x) \end{bmatrix}, \quad (19)$$

where $\mathscr{N}(x)$ is a linear matrix annihilator [1] of the matrix function $\Pi(x)$, which is given by:

$$\mathscr{N}(x) = \begin{bmatrix} x_2 I_n & -x_1 I_n & 0_n & \cdots & 0_n \\ 0_n & x_3 I_n & -x_2 I_n & \cdots & 0_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_n & \cdots & 0_n & x_n I_n & -x_{n-1} I_n \end{bmatrix}. \quad (20)$$

In light of the representation of $\mathscr{A} \cdot V_{\theta_t}(x)$ in (14) and the equality constraint in (19), by Lemma 1 we are able to derive the following local mean square stabilization result for the system (1):

Theorem 1. Consider the system (1) and let $\mathscr{X}$ be a given polytopic region defined by (5). Suppose there exist real matrices $L_i, Q_i, Y_i^{[0]}, \ldots, Y_i^{[n]}$, $i = 1, \ldots, \sigma$, satisfying the following LMIs:

$$\Xi_i(v_j) < 0, \quad j = 1, \ldots, n_v, \quad i = 1, \ldots, \sigma, \quad (21)$$

where

$$\Xi_i(x) = \begin{bmatrix} \Phi_i(x) + \mathrm{He}\,(L_i \Omega(x)) & \star \\ \Lambda_i' Q_i N & -\widetilde{Q}_i \end{bmatrix}. \quad (22)$$

Then, the control law $u = K_{\theta_t}(x)x$, with $K_{\theta_t}(x) = Y_i(x) Q_i^{-1}$ when $\theta_t = i$, where $Y_i(x) = Y_i^{[0]} + x_1 Y_i^{[1]} + \ldots + x_n Y_i^{[n]}$, ensures that the controlled system is locally exponentially mean square stable.

Proof. Firstly, it follows that $V_{\theta_t}(x)$ in (9) satisfies

$$\underline{\lambda}_i \|x\|^2 \leq V_i(x) \leq \overline{\lambda}_i \|x\|^2, \quad \forall\, x \in \mathbb{R}^n, \ i = 1, \ldots, \sigma \quad (23)$$

where $\underline{\lambda}_i$ and $\overline{\lambda}_i$ are respectively the minimum and maximum eigenvalues of P_i.

Secondly, (21) implies that $Q_i > 0$, $i = 1, \ldots, \sigma$. Hence, the state feedback gain $K_{\theta_t}(x) = Y_i(x) Q_i^{-1}$ when $\theta_t = i$, is well defined and it follows from (13).

Next, note that if the LMIs in (21) are feasible, then by convexity they are also satisfied for all $x \in \mathscr{X}$. Thus, consider (21) with $v_i = x$, $\forall\, x \in \mathscr{X}$. Applying Schur's complement, these LMIs are equivalent to

$$\Phi_i(x) + N' Q_i \Lambda_i \widetilde{Q}_i^{-1} \Lambda_i' Q_i N + \mathrm{He}\,(L_i \Omega(x)) < 0, \quad \forall\, x \in \mathscr{X}, \ i = 1, \ldots, \sigma.$$

As the latter inequalities are strict, it follows that there exist sufficient small scalars $\varepsilon_i > 0$, $i = 1, \ldots, \sigma$, such that

$$\Phi_i(x) + N' Q_i \Lambda_i \widetilde{Q}_i^{-1} \Lambda_i' Q_i N + \mathrm{He}\,(L_i \Omega(x)) + \varepsilon_i N' Q_i Q_i N < 0, \quad \forall\, x \in \mathscr{X}, \ i = 1, \ldots, \sigma. \quad (24)$$

Since for η_{θ_t} in (15) we have $\Omega(x)\eta_{\theta_t} = 0$, then in view of Lemma 1 the inequalities in (24) imply that

$$\eta_i'\left[\Phi_i(x) + N' Q_i \Lambda_i \widetilde{Q}_i^{-1} \Lambda_i' Q_i N + \varepsilon_i N' Q_i Q_i N\right] \eta_i < 0, \quad \forall\, x \in \mathscr{X}, \, x \neq 0, \ \eta_i \neq 0, \ i = 1, \ldots, \sigma. \quad (25)$$

[1] A matrix $\mathscr{N}(x)$ is a linear matrix annihilator of a matrix function $\Upsilon(x)$ if it is linear in x and such that $\mathscr{N}(x)\Upsilon(x) \equiv 0$; this is an extension of the notion of linear annihilator as proposed in Trofino [2000].

Considering (14) and that $Q_i N\eta_i = x$, (25) leads to

$$\mathscr{A} \cdot V_i(x) < -\varepsilon_i \|x\|^2, \quad \forall\, x \in \mathscr{X}, \ x \neq 0, \ i = 1, \ldots, \sigma. \quad (26)$$

Now, taking into account (23) and (26), we get

$$\frac{\mathscr{A} \cdot V_i(x)}{V_i(x)} < -\alpha, \quad \forall\, x \in \mathscr{X}, \ x \neq 0, \ i = 1, \ldots, \sigma,$$

where $\alpha := \min_{i \in \Theta} \{ \varepsilon_i / \overline{\lambda}_i \}$.

Applying Dynkin's formula Kushner [1967] and the Gronwall-Bellman's lemma (Desoer and Vidyasagar [1975]) to the latter inequality and using similar arguments as in the proof of Theorem 1 in Ji and Chizeck [1990], it follows that:

$$\mathbf{E}\big[V_{\theta_t}(x(t)) \,|\, x_0, \theta_0 \big] \leq V_{\theta_0}(x_0) e^{-\alpha t}, \ \forall\, t \geq 0, \ \forall\, \theta_0 \in \Theta, \ (27)$$

and for all x_0 in a neighborhood of the origin. Thus, the closed-loop system is locally exponentially mean square stable. $\nabla\nabla\nabla$

3.2 Stability Region

Assuming that the conditions of Theorem 1 hold, consider the Lyapunov function $V_{\theta_t}(x)$ as in (9) that proves local exponential mean square stability of the closed-loop system and let the sets:

$$\mathscr{R}_i = \big\{ x \in \mathbb{R}^n : V_i(x) \leq 1 \big\}, \quad i = 1, \ldots, \sigma, \qquad (28)$$

subject to $\mathscr{R}_i \subset \mathscr{X}$, $i = 1, \ldots, \sigma$. Note that in view of (27), the intersection $\mathscr{R}_0$ of all $\mathscr{R}_i$, i.e.

$$\mathscr{R}_0 := \bigcap_{i \in \Theta} \mathscr{R}_i \qquad (29)$$

is a contractive and positively invariant set in the mean square sense, that is:

$$\mathbf{E}[V_i(x(t)) \,|\, x_0, \theta_0] \leq 1, \ \forall\, t \geq 0, \quad i = 1, \ldots, \sigma, \quad \text{and}$$
$$\lim_{t \to \infty} \mathbf{E}[\, \|x\|^2 \,] = 0,$$

for all $x_0 \in \mathscr{X}$ and $\theta_0 \in \Theta$. In addition, it follows that $\mathbf{E}[x(t)\,] \in \mathscr{R}_0$, $\forall\, t \geq 0$ and $\lim_{t \to \infty} \mathbf{E}[\, \|x\| \,] = 0$.

Thus, the set $\mathscr{R}_0$ subject to the inclusion conditions $\mathscr{R}_i \subset \mathscr{X}$, $i = 1, \ldots, \sigma$, could be adopted as a stability region in the mean square sense. Note that it is desirable to obtain a stability region as large as possible, i.e., the largest possible volume $\mathscr{R}_0$ inside $\mathscr{X}$. However, it turns out that the problem of maximizing the volume of $\mathscr{R}_0$ is numerically hard. To overcome this difficulty, the stability region considered in this paper is the largest ellipsoid

$$\mathscr{R} = \big\{ x \in \mathbb{R}^n : x' P x \leq 1 \big\}, \quad P > 0 \qquad (30)$$

contained in $\mathscr{R}_0$ and subject to $\mathscr{R}_i \subset \mathscr{X}$, $i = 1, \ldots, \sigma$.

In light of (28) and (29), a sufficient condition to ensure $\mathscr{R} \subseteq \mathscr{R}_0$ is given by:

$$P - Q_i^{-1} \geq 0, \ i = 1, \ldots, \sigma, \qquad (31)$$

which clearly implies that $\mathscr{R} \subseteq \mathscr{R}_i$, $\forall\, i \in \Theta$.

Defining $P = Q^{-1}$, by Schur's complement (31) is equivalent to the following set of LMIs

$$\begin{bmatrix} Q_i & Q \\ Q & Q \end{bmatrix} \geq 0, \quad i = 1, \ldots, \sigma. \qquad (32)$$

On the other hand, the inclusions $\mathscr{R}_i \subset \mathscr{X}$, $i = 1, \ldots, \sigma$, are equivalent to (see Boyd *et al.* [1994]):

$$1 - c_j' Q_i c_j \geq 0, \quad j = 1, \ldots, n_f, \quad i = 1, \ldots, \sigma, \qquad (33)$$

where c_j, $j = 1, \ldots, n_f$ are the vectors defining the faces of $\mathscr{X}$ as given in (6).

As for maximizing the size of $\mathscr{R}$, since the volume of $\mathscr{R}$ is proportional to $\sqrt{\det(Q)}$, this maximization can be achieved by solving the convex problem of minimizing $-\log(\det(Q))$ (see, e.g., Boyd *et al.* [1994]).

The above arguments lead to the following theorem that presents a method based on an LMI optimization problem to determine a stability region $\mathscr{R}$ with maximized volume for a given polytopic state-space domain $\mathscr{X}$.

Theorem 2. Consider the system (1) and let $\mathscr{X}$ be a given polytopic region defined by either (5) or (6). Suppose there exist real matrices Q and L_i, Q_i, $Y_i^{[0]}, \ldots, Y_i^{[n]}$, $i = 1, \ldots, \sigma$, solving the following LMI optimization problem:

$$\begin{array}{cl} \displaystyle\min_{\substack{Q, Y_i^{[0]}, \ldots, Y_i^{[n]}, \\ L_i, Q_i, i = 1, \ldots, \sigma}} & -\log\det(Q) \\ \text{subject to } (21),\ (32),\ (33). & \end{array} \qquad (34)$$

Then, the control law $u = K_{\theta_t}(x)x$, with $K_{\theta_t}(x) = Y_i(x) Q_i^{-1}$ when $\theta_t = i$, where $Y_i(x) = Y_i^{[0]} + x_1 Y_i^{[1]} + \ldots + x_n Y_i^{[n]}$, ensures that the controlled system is locally exponentially mean square stable. Moreover, the set $\mathscr{R}$ defined in (30) with $P = Q^{-1}$ is such that for any initial state $x_0 \in \mathscr{R}$ and $\theta_0 \in \Theta$, we have $\mathbf{E}[x(t)\,] \in \mathscr{R}_0$, $\forall\ t \geq 0$ and $\lim_{t \to \infty} \mathbf{E}[\, \|x\| \,] = 0$, with $\mathscr{R}_0$ being as defined in (29).

Remark 1. Notice that since $\mathscr{R}$ is a subset of $\mathscr{R}_0$, it follows that $\mathscr{R}$ is not necessarily a contractive and positive invariant set (in the mean square sense). Nevertheless, any state trajectory starting with x_0 inside $\mathscr{R}$ and for any $\theta_0 \in \Theta$ satisfies $\lim_{t \to \infty} \mathbf{E}[\, \|x(t)\|^2 \,] = 0$.

4. AN EXAMPLE

Consider a Markov jump nonlinear quadratic system as in (1) with two operating modes and the following matrices:

$$A_1(x) = \begin{bmatrix} 0.2x_1 & 0 \\ 0 & 1 + 0.2x_2 \end{bmatrix},$$

$$A_2(x) = \begin{bmatrix} -1 + 0.2x_1 & -1 \\ 0 & -0.9 + 0.2x_2 \end{bmatrix},$$

$$B_1(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_2(x) = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} -3 & 3 \\ 6 & -6 \end{bmatrix}.$$

Note that as the following Markov jump linear system:

$$\dot{x} = A_{\theta_t}(0)\, x$$

with $A_i(\cdot)$, $i = 1, 2$ as above is not mean square stable, it turns out that the equilibrium point $x = 0$ of the nonlinear system under consideration is not locally mean square stable. On the other hand, it can be readily verified that the Markov jump linear system as below is mean square stabilizable

$$\dot{x} = A_{\theta_t}(0)\,x + B_{\theta_t}(0)\,u\,.$$

In this example, the objective is to design a nonlinear quadratic control law as in (4) which guarantees the local exponential mean square stability of the closed-loop system while maximizing the stability region $\mathscr{R}$ of the equilibrium point $x=0$ for a given polytope $\mathscr{X}$ in the state-space containing the origin. To this end, we consider $\mathscr{X}$ to be a square, that is:

$$\mathscr{X} = \{\,x \in \mathbb{R}^2 : |x_i| \le \rho,\; i = 1,2\,\}\,, \tag{35}$$

where ρ is a positive scalar defining the size of $\mathscr{X}$.

In the following, we apply Theorem 2 to derive a stabilizing quadratic state feedback controller for the given system with $\rho = 2$ in (35), which has led to the following results:

$$P_1 = P = \begin{bmatrix} 0.341 & 0.000 \\ 0.000 & 0.251 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 0.250 & 0.000 \\ 0.000 & 0.250 \end{bmatrix},$$

$$K_1(x) = \begin{bmatrix} 0.000 \\ -57.56 \end{bmatrix}' + x_1 \begin{bmatrix} 0.137 \\ 0.000 \end{bmatrix}' \times 10^{-7} - x_2 \begin{bmatrix} 0.000 \\ 0.200 \end{bmatrix}',$$

$$K_2(x) = \begin{bmatrix} -57.70 \\ 1.000 \end{bmatrix}' - x_1 \begin{bmatrix} 0.162 \\ 0.000 \end{bmatrix}' + x_2 \begin{bmatrix} 0.037 \\ 0.000 \end{bmatrix}'.$$

The stability region $\mathscr{R}$ along with the sets $\mathscr{R}_1$ (both in solid line) and $\mathscr{R}_2$ (in dotted line) as defined in (28) and (30) are shown in Fig. 1. Note in this case that $\mathscr{R}$ coincides with $\mathscr{R}_1 = \mathscr{R}_0 = \mathscr{R}_1 \cap \mathscr{R}_2$.

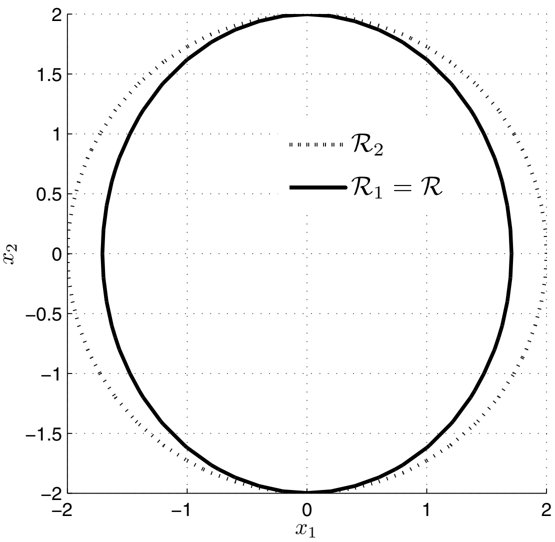

Fig. 1. Stability region $\mathscr{R}$ and the sets $\mathscr{R}_1$ and $\mathscr{R}_2$ for $\rho = 2$.

Notice that we can iteratively apply Theorem 2 aiming to enlarge the size of $\mathscr{X}$ and consequently the size of $\mathscr{R}$. More specifically, we can increase ρ until the conditions in (21), (32) and (33) are no longer feasible to maximize the size of $\mathscr{X}$. In light of that, we have obtained the results shown in Fig. 2 for a maximal $\rho = 7$.

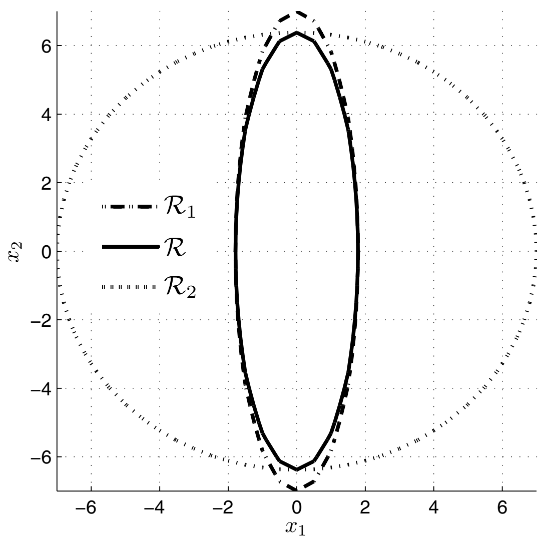

Fig. 2. Stability region $\mathscr{R}$ and the sets $\mathscr{R}_1$ and $\mathscr{R}_2$ for $\rho = 7$.

5. CONCLUDING REMARKS

This paper has investigated the problem of state feedback local stabilization of open-loop unstable Markov jump nonlinear quadratic systems. Specifically, we have derived a condition for local exponential mean square stabilization in terms of state-dependent LMIs which are required to be satisfied at the vertices of a given polytopic region of the state-space containing the zero equilibrium point of the closed-loop system. In addition, a convex optimization procedure in terms of LMIs has been proposed for designing a local stabilizing nonlinear quadratic control law while ensuring a maximized stability region for the closed-loop system inside the given polytopic region.

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