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Discrete-time Stochastic Extremum Seeking

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Abstract: We present discrete-time stochastic extremum seeking algorithms and prove their convergence using stochastic averaging theory that we recently developed. First, we provide a discrete stochastic extremum seeking algorithm for a static map, in which measurement noise is considered and an ergodic discrete-time stochastic process is used as the excitation signal. Second, for discrete-time nonlinear dynamical systems, in which the output equilibrium map has an extremum, we present a discrete-time stochastic extremum seeking scheme and, with a singular perturbation reduction, we prove the stability of the reduced system. Compared with classical stochastic approximation methods, while the convergence that we prove is in a weaker sense, the conditions of the algorithm are easy to verify and no requirements (e.g., boundedness) are imposed on the algorithm itself.

Keywords: Stochastic averaging, extremum seeking, stochastic perturbation

1. INTRODUCTION

Extremum seeking is a real-time optimization tool and also a method of adaptive control. Since the first proof of the convergence of extremum seeking Krstic and Wang [2000], the research on extremum seeking has triggered considerable interest in the theoretical control community (Teel and Popovic [2001], Choi et al. [2002], Tan et el. [2006], Stankovic and Stipanovic [2009], Moase et al. [2010], Stankovic and Stipanovic [2010]) and in applied communities (Ou et al. [2007], Popovic et al. [2006]).

In Liu and Krstic [2010], we establish a framework of continuous-time stochastic extremum seeking algorithms by developing general stochastic averaging theory in continuous time. However, there exists a need to consider stochastic extremum seeking in discrete time due to computer implementation. Discrete-time extremum seeking with stochastic perturbation is investigated without measurement noise in Manzie and Krstic [2009], in which the convergence of the algorithm involves strong restrictions on the iteration process. In Stankovic and Stipanovic [2009] and Stankovic and Stipanovic [2010], discrete-time extremum seeking with sinusoidal perturbation is studied with measurement noise considered and the proof of the convergence is based on the classical idea of stochastic approximation method, in which the boundedness of iteration sequence is assumed to guarantee the convergence of the algorithm.

In this paper, we investigate general discrete-time stochastic extremum seeking with stochastic perturbation and measurement noise. We supply discrete-time stochastic extremum seeking algorithm for a static map and analyze stochastic extremum seeking scheme for nonlinear dynamical systems with output equilibrium map. With the help of our developed discrete-time stochastic averaging theory Liu and Krstic [2013], we prove the convergence of the algorithms. Unlike in the continuous-time case Liu and Krstic [2010], in this work we consider the measurement noise, which is assumed to be bounded. In the classical stochastic approximation method, boundedness condition or other restrictions are imposed on the iteration algorithm itself to achieve the convergence. In our stochastic discrete-time algorithm, the convergence condition is only imposed on the cost function or considered systems and is easy to verify, but as a consequence, we obtain a weaker form of convergence.

The remainder of the paper is organized as follows. In Section 2 we present stochastic extremum algorithms for a static map. In Section 3, we give stochastic extremum seeking scheme for dynamical systems and its stability analysis. In Section 4 we offer some concluding remarks. The discrete-time stochastic averaging results are listed in Appendix.

2. DISCRETE-TIME STOCHASTIC EXTREMUM SEEKING ALGORITHM FOR STATIC MAP

Consider the quadratic function

$$\varphi(x) = \varphi^* + \frac{\varphi''}{2}(x - x^*)^2, \tag{1}$$

where $x^* \in \mathbb{R}$, $\varphi^* \in \mathbb{R}$, and φ'' are unknown. Any $\mathbb{C}^2$ function $\varphi(\cdot)$ with an extremum at $x = x^*$ and with $\varphi'' \neq 0$ can be locally approximated by (1). Without loss of generality, we assume that $\varphi'' > 0$. In this section, we design an algorithm to make $|x_k - x^*|$ as small as possible, so that the output $y = \varphi(x_k)$ is driven to its minimum φ^*. The only available information is the output with measurement noise.

Denote $\hat{x}_k$ as the k step estimate of the unknown optimal input x^*. Design iteration algorithm as

$$\hat{x}_{k+1} = \hat{x}_k - \varepsilon \sin(v_{k+1}) y_{k+1}, \quad k = 0, 1, \ldots, \tag{2}$$

where $y_{k+1} = \varphi(x_k) + W_{k+1}$ is the measurement output, $\{v_k, k = 1, 2, \ldots,\}$ is an ergodic stochastic process with invariant measure μ and living space S_Y, and $\{W_k, k = 1, 2, \ldots,\}$ is measurement noise, which is assumed to be bounded with a bound $M > 0$ and ergodic with invariant measure ν and living space

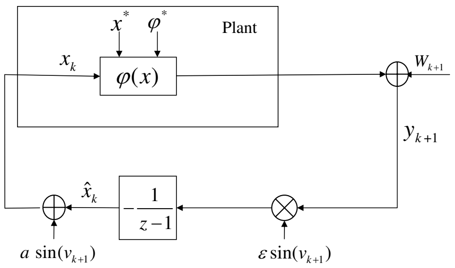

Fig. 1. Discrete-time stochastic extremum seeking scheme for a static map.

S_W. $\varepsilon \in (0, \varepsilon_0)$ is a positive small parameter for some constant $\varepsilon_0 > 0$. The perturbation process $\{v_k, k = 1, 2, \ldots, \}$ is independent of the measurement noise $\{W_k, k = 1, 2, \ldots, \}$.

Define $x_k = \hat{x}_k + a\sin(v_{k+1})$, $a > 0$ and the estimate error $\tilde{x}_k = \hat{x}_k - x^*$. Then we have

$$\tilde{x}_{k+1} = \tilde{x}_k - \varepsilon \sin(v_{k+1}) \left[\varphi^* + \frac{\varphi''}{2} (\tilde{x}_k + a\sin(v_{k+1}))^2 + W_{k+1} \right]. \quad (3)$$

To analyze the solution property of the error dynamics (3), we use stochastic averaging theory provided in Appendix. First, to calculate the average system, we assume that the excitation process $\{v_k, k = 1, 2, \ldots, \}$ is i.i.d. gaussian random variable sequence with invariant distribution $\mu(dy) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{y^2}{2\sigma^2}} dy$ and that the measurement noise process $\{W_k, k = 1, 2, \ldots, \}$ is any bounded ergodic process.

By (A.4), we have

$$\mathrm{Ave}\{\sin(v_{k+1})\} \triangleq \int_{S_Y} \sin(y)\mu(dy) = 0, \quad (4)$$

$$\mathrm{Ave}\{\sin^2(v_{k+1})\} \triangleq \int_{S_Y} \sin^2(y)\mu(dy) = \frac{1}{2} - \frac{1}{2} e^{-2\sigma^2}, \quad (5)$$

$$\begin{aligned}\mathrm{Ave}\{\sin(v_{k+1})W_{k+1}\} &\triangleq \int_{S_Y \times S_W} \sin(y)x\mu(dy) \times \nu(dx) \\ &= \int_{S_Y} \sin(y)\mu(dy) \times \int_{S_W} x\nu(dx) = 0. \quad (6)\end{aligned}$$

Thus, we obtain the average system of the error system (3)

$$\tilde{x}_{k+1}^{\mathrm{ave}} = \left(1 - \varepsilon \frac{a\varphi''(1 - e^{-2\sigma^2})}{2}\right) \tilde{x}_k^{\mathrm{ave}}. \quad (7)$$

Since $\varphi'' > 0$, there exists $\varepsilon^* = \frac{2}{a\varphi''(1 - e^{-2\sigma^2})}$ such that the average system (7) is globally exponentially stable for $\varepsilon \in (0, \varepsilon^*)$.

Thus by Theorem A.2, for the discrete-time stochastic extremum seeking algorithm in Fig. 1, we have the following theorem.

Theorem 2.1. Consider the static map (1) under iteration algorithm (2). Then there exist constants $c_\varepsilon > 0$ and $0 < \gamma_\varepsilon < 1$ such that for any initial condition $\tilde{x}_0 \in \mathbb{R}$ and any $\delta > 0$,

$$\lim_{\varepsilon \to 0} \inf \left\{ k \in \mathbb{N} : |\tilde{x}_k| > c_\varepsilon |\tilde{x}_0| \gamma_\varepsilon^k + \delta \right\} = +\infty \text{ a.s.} \quad (8)$$

and

$$\lim_{\varepsilon \to 0} P \left\{ |\tilde{x}_k| \le c_\varepsilon |\tilde{x}_0| \gamma_\varepsilon^k + \delta, \forall k = 0, 1, \ldots, [N/\varepsilon] \right\} = 1. \quad (9)$$

These two results imply that the norm of the error vector $\tilde{x}_k$ exponentially converges, both almost surely and in probability, to below an arbitrarily small residual value δ, over an arbitrarily long time interval which tends to infinity as ε goes to zero. To quantify the output convergence to the extremum, for any $\varepsilon > 0$, define a stopping time

$$\tau_\varepsilon^\delta = \inf \left\{ k \in \mathbb{N} : |\tilde{x}_k| > c_\varepsilon |\tilde{x}_0| \gamma_\varepsilon^k + \delta \right\}.$$

Then by (8), we know that $\lim_{\varepsilon \to 0} \tau_\varepsilon^\delta = +\infty$, *a.s.* and

$$|\tilde{x}_k| \le c_\varepsilon |\tilde{x}_0| \gamma_\varepsilon^k + \delta, \quad \forall k \le \tau_\varepsilon^\delta. \quad (10)$$

Since $y_{k+1} = \varphi(x^* + \tilde{x}_k + a\sin(v_{k+1})) + W_{k+1}$ and $\varphi'(x^*) = 0$, we have

$$\begin{aligned} y_{k+1} - \varphi(x^*) = &\frac{\varphi''(x^*)}{2} (\tilde{x}_k + a\sin(v_{k+1}))^2 \\ &+ O\left((\tilde{x}_k + a\sin(v_{k+1}))^3\right) + W_{k+1}. \quad (11)\end{aligned}$$

Thus by (10), it holds that $\forall k \le \tau_\varepsilon^\delta$

$$|y_{k+1} - \varphi(x^*)| \le O(a^2) + O(\delta^2) + C_\varepsilon |\tilde{x}_0|^2 \gamma_\varepsilon^{2k} + M, \quad (12)$$

for some positive constant C_ε. Similarly, by (9),

$$\begin{aligned}\lim_{\varepsilon \to 0} P \Big\{ &|y_{k+1} - \varphi(x^*)| \le O(a^2) + O(\delta^2) + C_\varepsilon |\tilde{x}_0|^2 \gamma_\varepsilon^{2k} \\ &+ M, \ \forall k = 0, 1, \ldots, [N/\varepsilon] \Big\} = 1, \quad (13)\end{aligned}$$

Remark 2.1. As an optimization method, besides the different derivative estimation methods, there are some other differences between stochastic extremum seeking (SES) and stochastic approximation (SA)(Ljung [1977], Spall [2003], Stankovic and Stipanovic [2010]). First, in the iteration, the gain coefficients in SA is changing with the iteration step, but for SES, the gain coefficient is a small constant and denotes the amplitude of the excitation signal; Second, stochastic approximation may consider more kinds of measurement noise (i.e., martingale difference sequence, some kind of infinite correlated sequence), but here we assume the measurement noise as bounded ergodic stochastic sequence; Third, to prove the convergence of the algorithm ($P\{\lim_{k \to \infty} x_k = x^*\} = 1$) , SA algorithm requires some restrictions on the cost function or the iteration sequence, while the conditions of SES algorithm are simple and easy to verify.

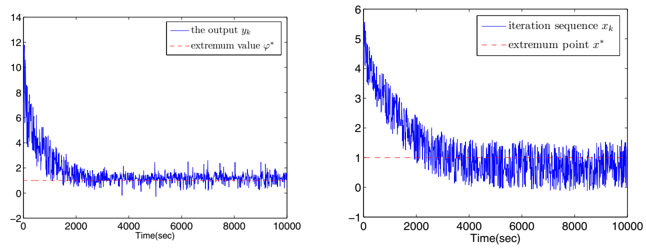

Fig. 2. Discrete-time stochastic ES with independent variables with the same gaussian distribution as the stochastic perturbation.

Fig.2 displays the simulation results with $\varphi^* = 1, \varphi'' = 1, x^* = 1$, in the static map (1) and $a = 0.8, \varepsilon = 0.002$ in the parameter update law (2) and initial condition $\hat{x}_0 = 5$. The excitation signal $\{v_k, k = 1, 2, \ldots, \}$ is taken as i.i.d. gaussian random variables with distribution $N(0, 4)$ and the measurement noise is taken as truncated i.i.d. gaussian random variables with distribution $N(0, 0.2)$.

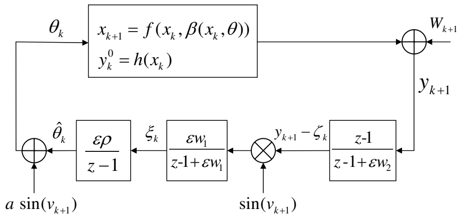

Fig. 3. Discrete-time stochastic extremum seeking scheme for nonlinear dynamics

3. DISCRETE-TIME STOCHASTIC EXTREMUM SEEKING FOR DYNAMIC SYSTEMS

Consider a general nonlinear model

$$x_{k+1} = f(x_k, u_k), \tag{14}$$
$$y_k^0 = h(x_k), \ \ k = 0, 1, 2, \dots, \tag{15}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}$ is the input, $y_k^0 \in \mathbb{R}$ is the nominal output, and $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $h : \mathbb{R}^n \to \mathbb{R}$ are smooth functions. Suppose that we know a smooth control law

$$u_k = \beta(x_k, \theta) \tag{16}$$

parameterized by a scalar parameter θ. Then the closed-loop system

$$x_{k+1} = f(x_k, \beta(x_k, \theta)) \tag{17}$$

has equilibria parameterized by θ. We make the following assumptions about the closed-loop system.

Assumption 3.1. There exists a smooth function $l : \mathbb{R} \to \mathbb{R}^n$ such that

$$f(x_k, \beta(x_k, \theta)) = 0 \ \text{ if and only if } \ x_k = l(\theta). \tag{18}$$

Assumption 3.2. There exists $\theta^* \in \mathbb{R}$ such that

$$(h \circ l)'(\theta^*) = 0, \tag{19}$$
$$(h \circ l)''(\theta^*) < 0. \tag{20}$$

Thus, we assume that the output equilibrium map $y = h(l(\theta))$ has a local maximum at $\theta = \theta^*$.

Our objective is to develop a feedback mechanism which makes the output equilibrium map $y = (h(l(\theta)))$ as close as possible to the maximum $y^* = h(l(\theta^*))$ but without requiring the knowledge of either θ^* or the functions h and l. The only available information is the measurement output with measurement noise.

As discrete-time stochastic extremum seeking scheme in Fig. 3, we choose the parameter update law

$$\hat\theta_{k+1} = \hat\theta_k + \varepsilon\rho\xi_k, \tag{21}$$
$$\xi_{k+1} = \xi_k - \varepsilon w_1\xi_k + \varepsilon w_1(y_{k+1} - \zeta_k)\sin(v_{k+1}), \tag{22}$$
$$\zeta_{k+1} = \zeta_k - \varepsilon w_2\zeta_k + \varepsilon w_2 y_{k+1}, \tag{23}$$
$$y_{k+1} = y_k^0 + W_{k+1}, \tag{24}$$

where $\rho > 0, w_1 > 0, w_2 > 0, \varepsilon > 0$ are design parameters and $\{v_k, k = 1, 2, \dots,\}$ is assumed to be a i.i.d. gaussian random variable sequence with distribution $\mu(dx) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{x^2}{2\sigma^2}}dx$. $W_k = (-M) \vee Z_k \wedge M$ is measurement noise, where $\{Z_k, k = 1, 2, \dots,\}$ is i.i.d. gaussian random variable sequence with distribution $\nu(dx) = \frac{1}{\sqrt{2\pi}\sigma_1}e^{-\frac{x^2}{2\sigma_1^2}}dx$. We assume that the probing signal $\{v_k, k = 1, 2, \dots,\}$ is independent of the measure noise $\{W_k, k = 1, 2, \dots,\}$. It is easy to verify that $\{W_k, k = 1, 2, \dots,\}$ is a bounded and ergodic process with invariant distribution $\nu_1(A) = \nu(A \wedge (-M, M)) + q_1 + q_2$, where $q_1 = \nu([M, +\infty))$ if $M \in A$, else $q_1 = 0$ and $q_2 = \nu((-\infty, -M])$ if $-M \in A$, else $q_2 = 0$.

Define $\theta_k = \hat\theta_k + a\sin(v_{k+1})$. Then we obtain the closed-loop system as

$$x_{k+1} = f(x_k, \beta(x_k, \hat\theta_k + a\sin(v_{k+1}))), \tag{25}$$
$$\hat\theta_{k+1} = \hat\theta_k + \varepsilon\rho\xi_k, \tag{26}$$
$$\xi_{k+1} = \xi_k - \varepsilon w_1\xi_k + \varepsilon w_1(y_k^0 + W_{k+1} - \zeta_k)\sin(v_{k+1}), \tag{27}$$
$$\zeta_{k+1} = \zeta_k - \varepsilon w_2\zeta_k + \varepsilon w_2(y_k^0 + W_{k+1}). \tag{28}$$

With the error variable

$$\tilde\theta_k = \hat\theta_k - \theta^*, \tag{29}$$
$$\tilde\zeta_k = \zeta_k - h \circ l(\theta^*), \tag{30}$$

the closed-loop system is rewritten as

$$x_{k+1} = f(x_k, \beta(x_k, \hat\theta_k + a\sin(v_{k+1}))), \tag{31}$$
$$\tilde\theta_{k+1} = \tilde\theta_k + \varepsilon\rho\xi_k, \tag{32}$$
$$\xi_{k+1} = \xi_k - \varepsilon w_1\xi_k + \varepsilon w_1(h(x_k) - h \circ l(\theta^*) - \tilde\zeta_k + W_{k+1}) \times \sin(v_{k+1}), \tag{33}$$
$$\tilde\zeta_{k+1} = \tilde\zeta_k - \varepsilon w_2\tilde\zeta_k + \varepsilon w_2\left(h(x_k) - h \circ l(\theta^*) + W_{k+1}\right). \tag{34}$$

We employ a singular perturbation reduction, freeze x_k in (31) at its quasi-steady state value as $x_k = l(\theta^* + \tilde\theta_k + a\sin(v_{k+1}))$ and substitute it into (32)-(34), and then get the reduced system

$$\tilde\theta_{k+1}^{\mathrm{r}} = \tilde\theta_k^{\mathrm{r}} + \varepsilon\rho\xi_k^{\mathrm{r}}, \tag{35}$$
$$\xi_{k+1}^{\mathrm{r}} = \xi_k^{\mathrm{r}} - \varepsilon w_1\xi_k^{\mathrm{r}} + \varepsilon w_1(\varsigma(\tilde\theta_k^{\mathrm{r}} + a\sin(v_{k+1})) - \tilde\zeta_k^{\mathrm{r}} + W_{k+1}) \times \sin(v_{k+1}), \tag{36}$$
$$\tilde\zeta_{k+1}^{\mathrm{r}} = \tilde\zeta_k^{\mathrm{r}} - \varepsilon w_2\tilde\zeta_k^{\mathrm{r}} + \varepsilon w_2\left(\varsigma(\tilde\theta_k^{\mathrm{r}} + a\sin(v_{k+1})) + W_{k+1}\right). \tag{37}$$

where $\varsigma(\tilde\theta_k^{\mathrm{r}} + a\sin(v_{k+1})) \triangleq h\left(l(\theta^* + \tilde\theta_k^{\mathrm{r}} + a\sin(v_{k+1}))\right) - h \circ l(\theta^*)$. With Assumption 3.2, we have

$$\varsigma(0) = 0, \tag{38}$$
$$\varsigma'(0) = (h \circ l)'(\theta^*) = 0, \tag{39}$$
$$\varsigma''(0) = (h \circ l)''(\theta^*) < 0. \tag{40}$$

Now we use our stochastic averaging theorems to analyze system (35)-(37). According to (A.4), we obtain that the average system of (35)-(37) is

$$\begin{bmatrix} \tilde\theta_{k+1}^{\mathrm{r,ave}} - \tilde\theta_k^{\mathrm{r,ave}} \\ \xi_{k+1}^{\mathrm{r,ave}} - \xi_k^{\mathrm{r,ave}} \\ \tilde\zeta_{k+1}^{\mathrm{r,ave}} - \tilde\zeta_k^{\mathrm{r,ave}} \end{bmatrix} = \varepsilon \begin{bmatrix} \rho\xi_k^{\mathrm{r,ave}} \\ -w_1\xi_k^{\mathrm{r,ave}} + w_1\int_{S_Y}\varsigma(\tilde\theta_k^{\mathrm{r,ave}} + a\sin(y))\sin(y)\mu(dy) \\ -w_2\tilde\zeta_k^{\mathrm{r,ave}} + w_2\int_{S_Y}\varsigma(\tilde\theta_k^{\mathrm{r,ave}} + a\sin(y))\mu(dy) \end{bmatrix}, \tag{41}$$

where we use the following facts: $\int_{S_W} x\nu_1(dx) = 0, \int_{S_W \times S_Y} x\sin(y)\nu_1(dx) \times \mu(dy) = 0$.

Now, we determine the average equilibrium $(\tilde\theta^{\mathrm{a,e}}, \xi^{\mathrm{a,e}}, \tilde\zeta^{\mathrm{a,e}})$ which satisfies

$$\xi^{\mathrm{a,e}} = 0, \quad (42)$$

$$-w_1\xi^{\mathrm{a,e}} + w_1 \int_{S_Y} \varsigma(\tilde{\theta}^{\mathrm{a,e}} + a\sin(y))\sin(y)\mu(dy) = 0, \quad (43)$$

$$-w_2\tilde{\zeta}^{\mathrm{a,e}} + w_2 \int_{S_Y} \varsigma(\tilde{\theta}^{\mathrm{a,e}} + a\sin(y))\mu(dy) = 0. \quad (44)$$

We assume that $\tilde{\theta}^{\mathrm{a,e}}$ has the form

$$\tilde{\theta}^{\mathrm{a,e}} = b_1 a + b_2 a^2 + O(a^3). \quad (45)$$

By (38) and (39), define

$$\varsigma(x) = \frac{\varsigma''(0)}{2}x^2 + \frac{\varsigma'''(0)}{3!}x^3 + O(x^4). \quad (46)$$

Then substituting (45) and (46) into (43), we have

$$\int_{-\infty}^{+\infty} \varsigma(b_1 a + b_2 a^2 + O(a^3) + a\sin(y))\sin(y)\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{y^2}{2\sigma^2}}dy$$
$$= O(a^4) + \varsigma''(0)b_1\left(\frac{1}{2} - \frac{1}{2}e^{-2\sigma^2}\right)a^2 +$$
$$\left[\left(b_2\varsigma''(0) + \frac{\varsigma'''(0)}{2}b_1^2\right)\left(\frac{1}{2} - \frac{1}{2}e^{-2\sigma^2}\right) + \frac{\varsigma'''(0)}{6}\left(\frac{3}{8} - \frac{1}{2}e^{-2\sigma^2} + \frac{1}{8}e^{-8\sigma^2}\right)\right]a^3 = 0, \quad (47)$$

where the following facts are used: $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\sin^{2k+1}(y)e^{-\frac{y^2}{2\sigma^2}}dy = 0, k = 0,1,2,\ldots$, $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\sin^2(y)e^{-\frac{y^2}{2\sigma^2}}dy = \frac{1}{2} - \frac{1}{2}e^{-2\sigma^2}$, $\frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\sin^4(y)e^{-\frac{y^2}{2\sigma^2}}dy = \frac{3}{8} - \frac{1}{2}e^{-2\sigma^2} + \frac{1}{8}e^{-8\sigma^2}$. Comparing the coefficients of the powers of a on the right-hand and left-hand sides of (47), we have $b_1 = 0$, $b_2 = -\frac{\varsigma'''(0)(3-4e^{-2\sigma^2}+e^{-8\sigma^2})}{24\varsigma''(0)(1-e^{-2\sigma^2})}$, and thus by (45), we have

$$\tilde{\theta}^{\mathrm{a,e}} = -\frac{\varsigma'''(0)(3-4e^{-2\sigma^2}+e^{-8\sigma^2})}{24\varsigma''(0)(1-e^{-2\sigma^2})}a^2 + O(a^3). \quad (48)$$

From this equation, together with (44), we have $\tilde{\zeta}^{\mathrm{a,e}} = \frac{\varsigma''(0)(1-e^{-2\sigma^2})}{4}a^2 + O(a^3)$. Thus the equilibrium of the average system (41) is

$$\begin{bmatrix}\tilde{\theta}^{\mathrm{a,e}}\\ \xi^{\mathrm{a,e}}\\ \tilde{\zeta}^{\mathrm{a,e}}\end{bmatrix} = \begin{bmatrix} -\frac{\varsigma'''(0)(3-4e^{-2\sigma^2}+e^{-8\sigma^2})}{24\varsigma''(0)(1-e^{-2\sigma^2})}a^2 + O(a^3)\\ 0\\ \frac{\varsigma''(0)(1-e^{-2\sigma^2})}{4}a^2 + O(a^3)\end{bmatrix}. \quad (49)$$

The Jacobian matrix of the average system (41) at the equilibrium $(\tilde{\theta}^{\mathrm{a,e}}, \xi^{\mathrm{a,e}}, \tilde{\zeta}^{\mathrm{a,e}})$ is

$$J_{\mathrm{r}}^{\mathrm{a}} = \begin{bmatrix} 1 & \varepsilon\rho & 0\\ \varepsilon J_{\mathrm{r21}}^{\mathrm{a}} & 1-\varepsilon w_1 & 0\\ \varepsilon J_{\mathrm{r31}}^{\mathrm{a}} & 0 & 1-\varepsilon w_2\end{bmatrix}, \quad (50)$$

where $J_{\mathrm{r21}}^{\mathrm{a}} = \frac{w_1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\varsigma'\left(\tilde{\theta}^{\mathrm{a,e}} + a\sin(y)\right)\sin(y)e^{-\frac{y^2}{2\sigma^2}}dy$, $J_{\mathrm{r31}}^{\mathrm{a}} = \frac{w_2}{\sqrt{2\pi}\sigma}\int_{-\infty}^{+\infty}\varsigma'(\tilde{\theta}^{\mathrm{a,e}} + a\sin(y))e^{-\frac{y^2}{2\sigma^2}}dy$. Thus we have

$$det(\lambda I - J_{\mathrm{r}}^{\mathrm{a}}) = (\lambda - 1 + \varepsilon w_2) \times \left((\lambda-1)^2 + \varepsilon w_1(\lambda - 1) - \varepsilon^2\rho J_{\mathrm{r21}}^{\mathrm{a}}\right). \quad (51)$$

With Taylor expansion and by calculating the integral, we get

$$\int_{-\infty}^{+\infty}\varsigma'\left(\tilde{\theta}^{\mathrm{a,e}} + a\sin(y)\right)\sin(y)e^{-\frac{y^2}{2\sigma^2}}dy = a\sqrt{2\pi}\sigma\varsigma''(0)\left(\frac{1}{2} - \frac{1}{2}e^{-2\sigma^2}\right) + O(a^2). \quad (52)$$

By substituting (52) into (51) we get

$$det(\lambda I - J_{\mathrm{r}}^{\mathrm{a}}) = (\lambda - 1 + \varepsilon w_2)(\lambda - 1 - \varepsilon\Phi_1)(\lambda - 1 - \varepsilon\Phi_2), \quad (53)$$

where $\Phi_1 = \frac{1}{2}(-w_1 + \mathrm{sqrt}(w_1^2 + 2\rho w_1 a\varsigma''(0)(1-e^{-2\sigma^2}) + \frac{4\rho w_1}{\sqrt{2\pi}\sigma}O(a^2)))$, $\Phi_2 = \frac{1}{2}(-w_1 - \mathrm{sqrt}(w_1^2 + 2\rho w_1 a\varsigma''(0)(1-e^{-2\sigma^2}) + \frac{4\rho w_1}{\sqrt{2\pi}\sigma}O(a^2)))$. Since $\varsigma''(0) < 0$, for sufficiently small a, $\mathrm{sqrt}(w_1^2 + 2\rho w_1 a\varsigma''(0)(1-e^{-2\sigma^2}) + \frac{4\rho w_1}{\sqrt{2\pi}\sigma}O(a^2))$ can be smaller than w_1. Thus there exist $\varepsilon_1^* > 0$, such that for $\varepsilon \in (0, \varepsilon_1^*)$, the eigenvalues of the Jacobian matrix of the average system (41) are in the unit disc, and thus the equilibrium of the average system is exponentially stable. Then according to Theorem A.2, we have the following result for stochastic extremum seeking algorithm in Fig. 3.

Theorem 3.1. Consider the reduced system (35)-(36)-(37) under Assumption 3.2. Then there exists a constant $a^* > 0$ such that for any $0 < a < a^*$ there exist constants $r > 0, c_\varepsilon > 0$, and $0 < \gamma_\varepsilon < 1$ such that for any initial condition $\left|\Delta_0^\varepsilon\right| < r$, and any $\delta > 0$,

$$\lim_{\varepsilon\to 0}\inf\left\{k \in \mathbb{N} : |\Delta_k^\varepsilon| > c_\varepsilon|\Delta_0^\varepsilon||\gamma_\varepsilon|^k + \delta\right\} = +\infty, \text{ a.s.} \quad (54)$$

and

$$\lim_{\varepsilon\to 0}P\left\{|\Delta_k^\varepsilon| \le c_\varepsilon|\Delta_0^\varepsilon||\gamma_\varepsilon|^k + \delta, \forall k = 0,1,\ldots,[N/\varepsilon]\right\} = 1 \quad \forall N \in \mathbb{N}, \quad (55)$$

where $\Delta_k^\varepsilon \triangleq (\tilde{\theta}_k^{\mathrm{r}}, \xi_k^{\mathrm{r}}, \tilde{\zeta}_k^{\mathrm{r}}) - \left(-\frac{\varsigma'''(0)(3-4e^{-2\sigma^2}+e^{-8\sigma^2})}{24\varsigma''(0)(1-e^{-2\sigma^2})}a^2 + O(a^3), 0, \frac{\varsigma''(0)(1-e^{-2\sigma^2})}{4}a^2 + O(a^3)\right)$.

These results imply that the norm of the error vector Δ_k^ε exponentially converges, both almost surely and in probability, to below an arbitrarily small residual value δ over an arbitrary large time interval as the perturbation parameter ε goes to zero. In particular, the $\tilde{\theta}_k^{\mathrm{r}}$-component of the error vector converges to below δ. To quantify the output convergence to the extremum, we define a stopping time

$$\tau_\varepsilon^\delta = \inf\left\{k \in \mathbb{N} : |\Delta_k^\varepsilon| > c_\varepsilon\left|\Delta_0^\varepsilon\right|\gamma_\varepsilon^k + \delta\right\}.$$

Then by (54) and the definition of Δ_k^ε, we know that $\lim_{\varepsilon\to 0}\tau_\varepsilon^\delta = +\infty$, *a.s.* and $\left|\tilde{\theta}_k^{\mathrm{r}} - \left(-\frac{v'''(0)(3-4e^{-q^2}+e^{-4q^2})}{24v''(0)(1-e^{-q^2})}a^2 + O(a^3)\right)\right| \le c_\varepsilon\left|\Delta_0^\varepsilon\right|\gamma_\varepsilon^k + \delta$, $\forall k \le \tau_\varepsilon^\delta$, which implies that

$$\left|\tilde{\theta}_k^{\mathrm{r}}\right| \le O(a^2) + c_\varepsilon\left|\Delta_0^\varepsilon\right|\gamma_\varepsilon^k + \delta, \quad \forall k \le \tau_\varepsilon^\delta. \quad (56)$$

Since the nominal output $y_k^0 = h(l(\theta^* + \tilde{\theta}_k^{\mathrm{r}} + a\sin(v_{k+1})))$ and $(h\circ l)'(\theta^*) = 0$, we have $y_k^0 - h\circ l(\theta^*) = \frac{(h\circ l)''(\theta^*)}{2}(\tilde{\theta}_k^{\mathrm{r}} + a\sin(v_{k+1}))^2 + O\left((\tilde{\theta}_k^{\mathrm{r}} + a\sin(v_{k+1}))^3\right)$. Thus by (56), it holds that

$$|y_k^0 - h\circ l(\theta^*)| \le O(a^2) + O(\delta^2) + C_\varepsilon\left|\Delta_0^\varepsilon\right|^2\gamma_\varepsilon^{2k}, \quad \forall k \le \tau_\varepsilon^\delta,$$

for some positive constant C_ε. Similarly, by (55)

$$\lim_{\varepsilon\to 0} P\left\{|y_k^0 - h\circ l(\theta^*)| \le O(a^2) + O(\delta^2) + C_\varepsilon |\Delta_0^\varepsilon|^2 \gamma_\varepsilon^{2k}, \quad \forall k = 0,1,\ldots,[N/\varepsilon]\right\} = 1.$$

With the measurement noise considered, we obtain that

$$|y_{k+1} - h\circ l(\theta^*)| \le O(a^2) + O(\delta^2) + C_\varepsilon |\Delta_0^\varepsilon|^2 \gamma_\varepsilon^{2k} + M, \quad \forall k \le \tau_\varepsilon^\delta,$$

for some positive constant C_ε, and moreover,

$$\lim_{\varepsilon\to 0} P\left\{|y_{k+1} - h\circ l(\theta^*)| \le O(a^2) + O(\delta^2) + C_\varepsilon |\Delta_0^\varepsilon|^2 \gamma_\varepsilon^{2k} + M, \quad \forall k = 0,1,\ldots,[N/\varepsilon]\right\} = 1.$$

4. CONCLUDING REMARKS

In this paper, we develop stochastic discrete-time extremum seeking algorithms. Compared with other stochastic optimization methods, e.g., stochastic approximation, simulated annealing method and genetic algorithm, the convergence conditions of discrete-time stochastic extremum seeking algorithm are easier to verify and clearer. Compared with continuous-time stochastic extremum seeking, in the discrete-time case, we consider the bounded measurement noise. In our results, we can only prove the weaker convergence than almost surely convergence of the classic stochastic approximation. Better convergence of algorithms and improved algorithms are our future work directions.

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Appendix A. DISCRETE-TIME STOCHASTIC AVERAGING

Consider system

$$X_{k+1} = X_k + \varepsilon f(X_k, Y_{k+1}),\ \ k = 0,1,2,\ldots, \tag{A.1}$$

where $X_k \in \mathbb{R}^n, k = 1,2,\ldots,$ are the states, $\{Y_k \in \mathbb{R}^m, k = 1,2,\ldots\}$ is a stochastic perturbation sequence defined on a complete probability space $(\Omega,\mathscr{F},P)$. Let $S_Y \subset \mathbb{R}^m$ be the living space of the perturbation process. $\varepsilon \in (0,\varepsilon_0)$ is a small parameter for some fixed positive constant ε_0.

The following assumptions will be considered.

Assumption A.1. The vector field $f(x,y)$ is a continuous function of (x,y), and for any $x \in \mathbb{R}^n$, it is a bounded function of y. Further it satisfies the locally Lipschitz condition in $x \in \mathbb{R}^n$ uniformly in $y \in S_Y$, i.e., for any compact subset $D \subset \mathbb{R}^n$, there is a constant k_D such that for all $x_1, x_2 \in D$ and all $y \in S_Y$,

$$|f(x_1,y) - f(x_2,y)| \le k_D\,|x_1 - x_2|.$$

Assumption A.2. The perturbation process $\{Y_k, k = 1,2,\ldots\}$ is ergodic with invariant distribution μ .

Under Assumption A.2, we define two classes of average system of system (A.1) as follows:

$$\text{Discrete average system:}\quad \bar{X}_{k+1}^{\mathrm{d}} = \bar{X}_k^{\mathrm{d}} + \varepsilon \bar{f}(\bar{X}_k^{\mathrm{d}}), \tag{A.2}$$

$$\text{Continuous average system:}\quad \frac{d\bar{X}^{\mathrm{c}}(t)}{dt} = \bar{f}(\bar{X}^{\mathrm{c}}(t)), \tag{A.3}$$

where $\bar{X}_0^{\mathrm{d}} = \bar{X}^{\mathrm{c}}(0) = X_0$ and

$$\bar{f}(x) \triangleq \int_{S_Y} f(x,y)\mu(dy) = \lim_{N\to\infty} \frac{1}{N+1}\sum_{k=0}^{N} f(x, Y_{k+1}) \quad \text{a.s.} \tag{A.4}$$

Here the definition of discrete average system is different from that in Solo and Kong [1995], where the average vector field is defined by $\bar{f}(x) \triangleq Ef(x,Y_{k+1})$ (there, the perturbation process $\{Y_{k+1}, k = 0,1,\ldots,\}$ is assumed to be strict stationary). Here we consider ergodic process as perturbation. It is easy to find discrete-time ergodic processes, e.g., (i) i.i.d random variables sequence; (ii) finite state irreducible and aperiodal Markov process; (iii) $\{Y_i, i = 0,1,\ldots,\}$ where $\{Y_t, t \ge 0\}$ is OU process. In fact, for any continuous-time ergodic process $\{Y_t, t \ge 0\}$, the subsequence $\{Y_i, i = 0,1,\ldots,\}$ is a discrete-time ergodic process.

By (A.1), we have

$$X_{k+1} = X_0 + \varepsilon\sum_{i=0}^{k} f(X_i, Y_{i+1}). \tag{A.5}$$

We introduce a new time $t_k = \varepsilon k$. Denote $m(t) = \max\{k : t_k \le t\}$ and define $X(t)$ as a piecewise constant version of X_k, i.e.,

$$X(t) = X_k,\ \text{as } t_k \le t < t_{k+1}, \tag{A.6}$$

and $Y(t)$ as a piecewise constant version of Y_n, i.e.,

$$Y(t) = Y_k,\ \text{as } t_k \le t < t_{k+1}. \tag{A.7}$$

Then we can write (A.1) in the following form:

$$X(t) = X_0 + \varepsilon\sum_{k=1}^{m(t)} f(X_{k-1}, Y_k) \tag{A.8}$$

or as the continuous-time version

$$X(t) = X_0 + \int_0^t f(X(s), Y(\varepsilon+s))ds - \int_{t_{m(t)}}^t f(X(s), Y(\varepsilon+s))ds. \tag{A.9}$$

Similarly, we can write the discrete average system (A.2) in the following continuous-time version

$$\bar{X}^{\mathrm{d}}(t) = X_0 + \int_0^t \bar{f}(\bar{X}^{\mathrm{d}}(s))ds - \int_{t_{m(t)}}^t \bar{f}(\bar{X}^{\mathrm{d}}(s))ds, \tag{A.10}$$

and write the continuous average system (A.3) by

$$\bar{X}^{\mathrm{c}}(t) = X_0 + \int_0^t \bar{f}(\bar{X}^{\mathrm{c}}(s))ds, \tag{A.11}$$

where $\bar{X}^{\mathrm{d}}(t)$ is a piecewise constant version of $\bar{X}_k^{\mathrm{d}}$, i.e., $\bar{X}^{\mathrm{d}}(t) = \bar{X}_k^{\mathrm{d}}$, as $t_k \le t < t_{k+1}$. We now rewrite the continuous-time version (A.9) of the original system (A.1) as two forms:

$$X(t) = X_0 + \int_0^t \bar{f}(X(s))ds - \int_{t_{m(t)}}^t \bar{f}(X(s))ds + R^{(1)}(t, X(\cdot), Y(\varepsilon+\cdot)), \tag{A.12}$$

$$X(t) = X_0 + \int_0^t \bar{f}(X(s))ds + R^{(2)}(t, X(\cdot), Y(\varepsilon+\cdot)), \tag{A.13}$$

where $R^{(1)}(t,X(\cdot),Y(\varepsilon+\cdot)) = \int_0^{t_{m(t)}} \big(f(X(s),Y(\varepsilon+s)) - \bar{f}(X(s))\big)ds$, $R^{(2)}(t,X(\cdot),Y(\varepsilon+\cdot)) = \int_0^t \big(f(X(s),Y(\varepsilon+s)) - \bar{f}(X(s))\big)\,ds - \int_{t_{m(t)}}^t f(X(s),Y(\varepsilon+s))ds$. Hence we consider system (A.12) as a random perturbation of the continuous-time version (A.10) of discrete average system (A.2) and consider system (A.13) as a random perturbation of the continuous average system (A.11).

To study the solution property of the original system (A.1), we develop discrete-time stochastic averaging principle, i.e., using average systems (A.2) or (A.3) to approximate the original system (A.1).

Remark A.1. Our developed averaging theory is also applicable to the following systems

$$X_{k+1} = X_k + \varepsilon \left(f(X_k, Y_{k+1}) + W_{k+1} \right), \quad k = 0, 1, 2, \ldots, \quad \text{(A.14)}$$

where $\{W_k \in \mathbb{R}^n, k = 1, 2, \ldots\}$ is bounded with a bound M and ergodic stochastic sequence, which is independent of the perturbation sequence $\{Y_k, k = 1, 2, \ldots\}$.

Take a function $g \in C_0(R)$ such that $g(x) = 1, \forall x \in B_M(0) = \{x \in \mathbb{R}^n | |x| \leq M\}$ and denote $F(X_k, Z_{k+1}) \triangleq f(X_k, Y_{k+1}) + g(W_{k+1})$. Then we obtain the following system

$$X_{k+1} = X_k + \varepsilon F(X_k, Z_{k+1}), \quad k = 0, 1, 2, \ldots. \quad \text{(A.15)}$$

Since $\{W_k \in \mathbb{R}^n, k = 1, 2, \ldots\}$ and $\{Y_k, k = 1, 2, \ldots\}$ are independent and ergodic, we can obtain the combination process $Z_k \triangleq \{(Y_k^T, W_k^T)^T, k = 1, 2, \ldots, \}$ is also ergodic.It is easy to check that the new system (A.15) satisfies Assumption A.1. Thus we know system (A.14) is included into our considered system (A.1).

Let $(X_k, k = 0, 1, 2, \ldots)$ and $(\bar{X}_k^{\mathrm{d}}, k = 0, 1, 2, \ldots)$ be the solutions of the original system (A.1), discrete average system (A.2), respectively. Rewrite system (A.1) as

$$X_{k+1} = X_k + \varepsilon \bar{f}(X_k) + R^{(3)}(X_k, Y_{k+1}), \quad k = 0, 1, 2, \ldots, \quad \text{(A.16)}$$

where $R^{(3)}(X_k, Y_{k+1}) = \varepsilon(f(X_k, Y_{k+1}) - \bar{f}(X_k))$. Hence we can consider system (A.16) (i.e. system (A.1)) as a random perturbation of discrete average system (A.2).

We have the following approximation results. Own to the space limitation, the proof is omitted and referred to the case without measurement noise Liu and Krstic [2013].

Lemma A.1. Consider system (A.1) under Assumptions A.1 and A.2. Then for any $N \in \mathbb{N}$,

$$\lim_{\varepsilon \to 0} \sup_{0 \leq k \leq [N/\varepsilon]} |X_k - \bar{X}_k^{\mathrm{d}}| = 0 \ \text{ a.s.} \quad \text{(A.17)}$$

Theorem A.1. Consider system (A.1) under Assumptions A.1 and A.2. Then we have

(i) for any $\delta > 0$, $\lim_{\varepsilon \to 0} \inf\{k \in \mathbb{N} : |X_k - \bar{X}_k^{\mathrm{d}}| > \delta\} = +\infty$ a.s.;
(ii) for any $\delta > 0$ and any $N \in \mathbb{N}$,

$$\lim_{\varepsilon \to 0} P\left\{ \sup_{0 \leq k \leq [N/\varepsilon]} |X_k - \bar{X}_k^{\mathrm{d}}| > \delta \right\} = 0.$$

About the solution property of the original system (A.1) by analyzing the stability of discrete average systems (A.2), we have the following results.

Theorem A.2. Consider system (A.1) under Assumptions A.1 and A.2. Then if for any $\varepsilon \in (0, \varepsilon_0)$, the equilibrium $\bar{X}_k^{\mathrm{d}} \equiv 0$ of the discrete average system (A.2) is exponentially stable, then it is weakly exponentially stable under random perturbation $R^{(3)}(\cdot, Y_{k+1})$, i.e., there exist constants $r > 0$, $c_\varepsilon > 0$ and $\gamma_\varepsilon > 0$ such that for any initial condition $X_0 = x \in \{\check{x} \in \mathbb{R}^n : |\check{x}| < r\}$, and any $\delta > 0$, the solution of system (A.1) satisfies

$$\lim_{\varepsilon \to 0} \inf\left\{ k \in \mathbb{N} : |X_k| > c_\varepsilon |x| \gamma_\varepsilon^k + \delta \right\} = +\infty \ \text{ a.s.} \quad \text{(A.18)}$$

Moreover,

$$\lim_{\varepsilon \to 0} P\left\{ |X_k| \leq c_\varepsilon |x| \gamma_\varepsilon^k + \delta, \forall k = 0, 1, \ldots, [N/\varepsilon] \right\} = 1,$$

$$\text{for} \quad \forall N \in \mathbb{N}. \quad \text{(A.19)}$$

If the equilibrium $\bar{X}_k^{\mathrm{d}} \equiv 0$ of the discrete average system (A.2) is exponentially stable uniformly w.r.t. $\varepsilon \in (0, \varepsilon_0)$, then the above constants $c_\varepsilon > 0$ and γ_ε can be taken independent of ε. If the equilibrium $\bar{X}_k^{\mathrm{d}} \equiv 0$ of the discrete average system (A.2) is globally exponentially stable, then (A.18) and (A.19) hold for any initial condition $X_0 = x \in \mathbb{R}^n$.

REFERENCES

K. B. Ariyur and M. Krstic, *Real-Time Optimization by Extremum Seeking Control*, Hoboken, NJ: Wiley-Interscience, 2003.

J. -Y. Choi, M. Krstic, K. B. Ariyur, and J. S. Lee, Extremum seeking control for discrete time systems, *IEEE Translation on Automatic and Control*, vol. 47, no. 2, pp. 318-323, 2002.

M. Krstic and H. H. Wang, Stability of extremum seeking feedback for general nonlinear dynamic systems, *Automatica*, vol. 36, pp. 595-601, 2000.

S. -J. Liu and M. Krstic, Stochastic averaging in continuous time and its applications to extremum seeking, *IEEE Transactions on Automatic Control*, vol.55, no.10, pp.2235-2250, 2010.

S. -J. Liu and M. Krstic, Stochastic Averaging in Discrete Time, *Proc. of the 32st Chinese Control Conference*, July 26-28, pp. 1653-1658, Xi'an, 2013.

L. Ljung, Analysis of recursive stochastic algorithms, *IEEE Translation on Automatic and Control*,vol. 22, no. 4, pp.551-575,1977.

C. Manzie and M. Krstic, Extremum seeking with stochastic perturbations, *IEEE Transactions on Automatic Control*, vol. 54, pp. 580–585, 2009.

W. H. Moase, C. Manzie, and M. J. Brea, Newton-like extremum-seeking for the control of thermoacoustic instability, *IEEE Transactions on Automatic Control*, vol.55, pp. 2094-2105, 2010.

Y. Ou, C. Xu, E. Schuster, T. Luce, J.R.Ferron, and M. Walker, Exremum-seeking finite-time optimal control of plasma current profile at the DIII-D Tokamak, *Proceedings of the 2007 American Control Conference*, July 11-13, pp.4015-4020, 2007.

D. Popovic, M. Jankovic, S. Magner, and A. Teel, Extremum seeking methods for optimization of variable cam timing ergine operation, *IEEE Transactions on Control Systems Technology*, vol. 14, no.3, pp.398-407, 2006.

V. Solo and X. Kong, *Adaptive Signal Processing Algorithms: Stability and Performance*, Prentice Hall, 1995.

J. C. Spall, Introduction to Stochastic Search and Optimization: Estimation, Simulation, and Control, Wiley-Interscience, 2003.

M. S. Stankovic and D. M. Stipanovic, Discrete time extremum seeking by autonomous vehicles in stochastic environment, *Proceedings of the 48th IEEE Conference on Decision and Control*, Shanghai, China, Dec.16-18, pp. 4541-4546, 2009.

M. S. Stankovic and D. M. Stipanovic, Extremum seeking under stochastic noise and applications to mobile sensors, *Automatica*, vol. 46, pp. 1243-1251, 2010.

Y. Tan, D. Nesic and I. Mareels, On non-local stability properties of extremum seeking control,*Automatica*, vol. 42, no. 6, pp. 889-903, 2006.

A. R. Teel and D. Popovic, Solving smooth and nonsmooth multivariable extremum seeking problems by the methods of nonlinear programming. *Proceedings of American Control Conference*, vol. 3, pp. 2394-2399, 2001.