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Synchronized model matching: a novel approach to cooperative control of nonlinear multi-agent systems ⋆

Georg S. Seyboth *, **Frank Allgöwer** *

* *Institute for Systems Theory and Automatic Control, University of Stuttgart, Pfaffenwaldring 9, 70550 Stuttgart, Germany.*

Abstract: We present a novel hierarchical approach to cooperative control of multi-agent systems. The agents are modeled as non-identical nonlinear single-input single-output systems. The control strategy achieves synchronization of the agents to a common output trajectory of a desired type. It is based on synchronization of reference models on the network level and asymptotic model matching control on the agent level. In order to cooperatively attenuate disturbances acting on individual agents, we establish feedback from the agent level to the network level and introduce integral action on the network level. The approach is illustrated by a simulation example with four magnetic levitation systems.

1. INTRODUCTION

The research area of cooperative and distributed control of multi-agent systems has made impressive advances over the last decade. The goal is to develop analysis and control design methods for distributed large-scale dynamical systems which allow to synthesize a desired cooperative behavior. Typical cooperative control problems arise in vehicle coordination, formation flight, coordination of robotic manipulators, as well as smart grids. A core problem in this area is output synchronization (or consensus). Since the seminal papers by Fax and Murray [2004], Olfati-Saber and Murray [2004], and Ren and Beard [2005], there have been significant advances in that direction. Before we discuss recent developments and the contributions of our paper with respect to prior work, we briefly describe the problem setup and our solution approach.

Problem Description and Solution Approach: We consider a group of N dynamical agents, each described by an input-affine nonlinear differential equation of the form

$$\dot{x}_k = f_k(x_k) + g_k(x_k)u_k \tag{1a}$$

$$y_k = h_k(x_k), \tag{1b}$$

where $x_k \in X_k \subset \mathbb{R}^{n_k}$ is the state, $u_k \in \mathbb{R}$ is the input, $y_k \in \mathbb{R}$ is the output of agent k, f_k and g_k are smooth vector fields defined on X_k, and h_k is a smooth mapping, for all k in the index set $\mathcal{N} = \{1, \ldots, N\}$. Hence, each agent is a single-input single-output (SISO) system. We aim at synthesizing a cooperative behavior of the group. The output trajectories of all agents shall agree upon and converge to some common output trajectory and show a desired behavior. For all agents $k, j \in \mathcal{N}$, we require that

$$y_k(t) - y_j(t) \to 0 \quad \text{as} \quad t \to \infty.$$

We propose a novel hierarchical control scheme consisting of two levels: the network level and the agent level as illustrated in Fig. 1. On the agent level, each agent is equipped with a local controller that achieves asymptotic tracking of a given reference trajectory. On the network level, a synchronization mechanism achieves agreement of the reference signals. This approach is appealing since the local tracking control problems and the synchronization problem on the network level are decoupled. The main challenge will be to establish feedback from the agents to the network level such that the group can react cooperatively on disturbances acting on individual agents.

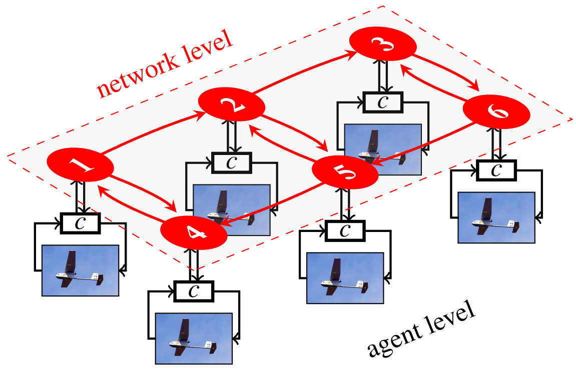


Fig. 1. Hierarchical approach to cooperative control tasks. [1]

Related Work: The idea of separating the tracking control from the coordination scheme on a higher planning level has already been presented by Egerstedt and Hu [2001] for nonlinear multi-agent systems. On the planning level, a so-called virtual leader is generated, which is tracked by locally controlled mobile robots. The coordination control strategy can thus be designed independently of the agent models. A limitation of this setup is that there is no feedback from the agents to the planning level. Hence the group is not able to react cooperatively on external disturbances acting on individual agents. In the present paper we show how such feedback can be introduced and use recent distributed control design methods on the planning level.

Consensus problems with reference models for single and double-integrator networks have been addressed by Ren and Beard [2008]. All agents shall track the same reference signal generated by a single external model. In order to establish feedback from the agents to the reference state, it is suggested

⋆ Corresp. author: G.S.Seyboth (`georg.seyboth@ist.uni-stuttgart.de`). The authors thank the German Research Foundation (DFG) for financial support of the project within the Cluster of Excellence in Simulation Technology (EXC 310/1) at the University of Stuttgart and within the Priority Programme 1305 "Control Theory of Digitally Networked Dynamical Systems". Moreover, the authors gratefully thank C. Scherer and V. Ugrinovskii for fruitful discussions.

[1] Photos: Raven UAV (U.S. Air Force photo/Dennis Rogers).

to stretch the reference trajectory in time depending on the disagreement of the group.

A synchronization problem for robots modeled as Lagrangian systems has been studied by Chung and Slotine [2009]. The control objective is synchronization and tracking of a desired trajectory, where synchronization of the robots is motivated by disturbances that shall be attenuated cooperatively. The solution approach is to realize synchronization and trajectory tracking on different time-scales instead of different hierarchical levels.

Output synchronization problems in networks of autonomous dynamical systems without external references or disturbances have been studied extensively. A typical problem setup consists of a group of linear agents which shall agree upon and converge to a common output trajectory. In networks of identical agents, this problem can be solved by static diffusive couplings, Tuna [2008], Wieland et al. [2011a]. In heterogeneous networks, dynamic diffusive couplings are favorable, Wieland et al. [2011b]. The proposed distributed control law in fact realizes a hierarchical control scheme. The controller of each agent embeds a copy of the common internal model and these *reference generators* synchronize asymptotically while each agent locally solves an output regulation problem with respect to its reference generator. However, no external disturbances are considered and there is no information flow from the agent to the network level.

A closely related problem setup appears in the literature under the terms *synchronized output regulation* and *cooperative output regulation*. The setup, as studied by Xiang et al. [2009], Huang [2011], Su and Huang [2012], consists of an autonomous linear exosystem $\dot{w} = Sw$ and a group of linear agents which are affected by w and shall solve local output regulation problems with respect to the exosystem. This setup captures tracking and disturbance rejection problems, where both external disturbances and reference signals are modeled by the exosystem. Communication among the agents is necessary since not all agents can measure the state w by assumption. The solution is based on distributed estimation of w and classical output regulation at each node with respect to the local estimate of w. In the present paper, we deal with local disturbances which may act on any agent in the group, but there is no need to estimate the full vector of all disturbances at each node.

An hierarchical approach has also been presented by De Campos et al. [2012]. It consists of single-integrators on the network level and local asymptotic output tracking controllers, but there is no feedback from the agents to the consensus network.

Contribution: Motivated by the discussion above, we propose a novel hierarchical approach to cooperative control of multi-agent systems. We address an output synchronization problem for heterogeneous groups of nonlinear SISO agents (1). In a first step, we decouple the coordination problem on the network level and the local tracking problems on the agent level by a suitable choice of control methods on both levels. Second, we show how information feedback from the agents to the network level can be realized in this setup. Third, we consider external disturbances acting on the agents and introduce integral action on the network level, which allows to cooperatively react on and attenuate disturbances. The synchronization mechanism with integral action on the network level is of independent interest and may be useful in other cooperative control scenarios.

Outline: The novel hierarchical approach is introduced in Section 2. In Section 3, the control method is modified such that the group reacts cooperatively on external disturbances; the modifications are feedback from the agent to the network level and integral action on the network level. An illustrative example is presented in Section 4 and Section 5 concludes the paper.

2. THE HIERACHICAL APPROACH

2.1 Agent Level: Asymptotic Model Matching

Each agent is equipped with a controller that achieves asymptotic output tracking for some reference output $y_k^*(t)$. We resort to the *asymptotic model matching* control technique based on exact input-output linearization of the plant, which is described in Isidori [1995] and summarized in this section. We adopt the notation from Isidori [1995] and write $\mathrm{L}_f\lambda(x)$ for the derivative of a real-valued function λ along a vector field f. System (1) is said to have a well-defined relative degree r_k at point $x_k^\circ \in X_k$, if

i) $\mathrm{L}_{g_k}\mathrm{L}_{f_k}^i h_k(x_k) = 0$ in a neighborhood of x_k° for all $i < r_k - 1$,

ii) $\mathrm{L}_{g_k}\mathrm{L}_{f_k}^{r_k-1} h_k(x_k^\circ) \neq 0$.

Assumption 1. Each agent (1) has a well-defined relative degree r_k at point $x_k^\circ \in X_k$, $k \in \mathcal{N}$.

Under Assumption 1, system (1) can locally be transformed to Byrnes Isidori normal form. That is, there exists a diffeomorphism Φ_k such that $[\xi_k^\top\ \eta_k^\top]^\top = \Phi_k(x_k)$, where the states $\xi_k \in \mathbb{R}^{r_k}$ are given by $\xi_{k,1} = h_k(x_k)$, $\xi_{k,2} = \mathrm{L}_{f_k}h_k(x_k)$, ..., $\xi_{k,r_k} = \mathrm{L}_{f_k}^{r_k-1}h_k(x_k)$, and $\eta_k \in \mathbb{R}^{n_k-r_k}$ are the states of the internal dynamics of (1), i.e., $\dot{\eta}_k = q_k(\xi_k,\eta_k)$ for some smooth $q_k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_k-r_k}$. The input-output behavior of (1) can be linearized via

$$u_k = \frac{1}{\mathrm{L}_{g_k}\mathrm{L}_{f_k}^{r_k-1}h_k(x_k)}\left(-\mathrm{L}_{f_k}^{r_k}h_k(x_k) + v_k\right), \tag{2}$$

where $v_k \in \mathbb{R}$ is a novel control input. The input-output behavior of (1) with (2) is that of an integrator chain of length r_k. Let $y_k^*(t)$ be a reference signal for $y_k(t)$ which is at least r_k times continuously differentiable. Then, v_k can be used in order to achieve asymptotic output tracking by setting

$$v_k = y_k^{*(r_k)} - \sum_{i=1}^{r_k} c_{i-1}\left(\mathrm{L}_{f_k}^{i-1}h_k(x_k) - y_k^{*(i-1)}\right). \tag{3}$$

The tracking error $e_k = y_k - y_k^*$ of the closed loop (1), (2), (3) is governed by the linear differential equation

$$e_k{}^{(r_k)} + c_{r_k-1}e_k{}^{(r_k-1)} + \cdots c_1\dot{e}_k + c_0 e_k = 0. \tag{4}$$

By a proper choice of the coefficients c_{i-1}, $i = 1,...,r_k$ in (3), any desired asymptotic behavior of the tracking error can be achieved. Note that the coefficients can be chosen differently for each agent k. An additional index k is omitted for brevity.

Suppose that the reference signal $y_k^*(t)$ is generated by a linear dynamical reference model of the form

$$\dot{z}_k = Az_k + Bw_k \tag{5a}$$
$$y_k^* = Cz_k, \tag{5b}$$

where $z_k \in \mathbb{R}^n$, $y_k^* \in \mathbb{R}$ and $w_k \in \mathbb{R}$, $k \in \mathcal{N}$. Then, the asymptotic output tracking control problem turns into an *asymptotic model matching* problem. The system (1) is controlled such that it asymptotically tracks the reference output (5b), i.e., such that it matches the reference model (5). The reference model (5) will be part of the cooperative control design later on. It is chosen such that it generates references of a desired form. Suppose that we have chosen (5) such that its relative degree $r \leq n$ satisfies $r > \max_k r_k$, i.e., the relative degree of (5) is larger than the largest relative degree of all agents in the group. Then, the derivatives of $y_k^*(t)$ in (3) can be expressed with (5), and (2) with (3) takes the form

$$u_k = \frac{1}{\mathrm{L}_{g_k}\mathrm{L}_{f_k}^{r_k-1}h_k(x_k)}\Bigg(- \mathrm{L}_{f_k}^{r_k}h_k(x_k) + CA^{r_k}z_k - \sum_{i=1}^{r_k} c_{i-1}\left(\mathrm{L}_{f_k}^{i-1}h_k(x_k) - CA^{i-1}z_k\right)\Bigg). \tag{6}$$

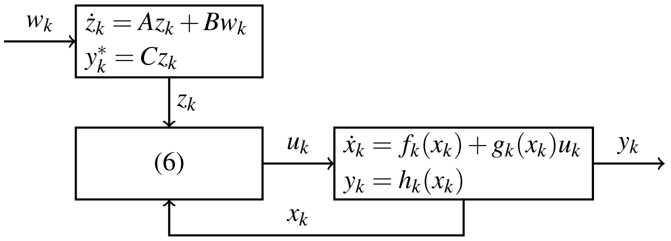


Fig. 2. Asymptotic model matching control setup. [2]

The control law (6) is independent of the input w_k of the reference model and its derivatives, since $CB = CAB = \cdots = CA^{r-2}B = CA^{r_k-1}B = 0$. For arbitrary input signals w_k to the reference model, control law (6) guarantees that $y_k(t)$ asymptotically tracks the reference output $y_k^*(t)$. A block diagram of the control setup is shown in Fig. 2. Besides Assumption 1, a second condition has to be satisfied in order to guarantee that all states of the closed loop (1), (5), (6) remain bounded for bounded references $z_k(t)$.

Assumption 2. For each agent k, the response $\eta_k(t)$ of the internal dynamics $\dot{\eta}_k = q_k\left(\xi_k^*, \eta_k\right)$ to the reference $\xi_k^*(t)$ consisting of $y_k^*(t)$ and its first $r_k - 1$ derivatives is bounded.

For further details the reader is refered to Byrnes et al. [1988]. The fact that the control law (6) is independent of the input w_k is the key feature that allows to decouple the tracking control problems on the agent level and the synchronization problem on the network level, which we will exploit next.

2.2 Network Level: Reference Synchronization

The coordination problem on the network level reduces to an output synchronization problem of the linear reference models (5). Our goal is to find a distributed control law w_k, which guarantees that $\forall k, j \in \mathcal{N}$: $y_k^*(t) - y_j^*(t) \to 0$ as $t \to \infty$. The models (5) are identical for all agents and each agent has access to the full state of its model since it is part of the controller. Hence, the synchronization problem on the network level can be solved via static diffusive couplings of the form

$$w_k = K \sum_{j=1}^{N} a_{kj}(z_j - z_k), \tag{7}$$

where $K \in \mathbb{R}^{1\times n}$ is a coupling gain matrix, and $a_{kj} \geq 0$ are the entries of the adjacency matrix $A_{\mathcal{G}}$ of the communication graph $\mathcal{G}$ on the network level. It it shown by Wieland et al. [2011a] that there exists a gain matrix K such that (7) solves the synchronization problem if and only if the underlying graph $\mathcal{G}$ is connected, i.e., $\mathcal{G}$ contains a directed spanning tree, given the pair (A,B) is stabilizable. Furthermore, Wieland et al. [2011a] present an LMI-based design method for suitable gains K and (7) in fact guarantees state synchronization $z_k(t) - z_j(t) \to 0$ as $t \to \infty$. Since (5) is part of the cooperative control design, it can be chosen such that (A,B) is stabilizable. Concluding, if the graph $\mathcal{G}$ is connected, the synchronization problem on the network level can easily be solved.

2.3 Synchronized Model Matching

The novel hierarchical approach to cooperative control design may be termed *synchronized model matching* since it consists of synchronization of reference models on the network level and asymptotic model matching on the agent level. Each agent (1) in the group is equipped with a copy of the reference model (5). These models are coupled through (7), which guarantees that $y_k^*(t) - y_j^*(t) \to 0$ as $t \to \infty$ for all $k, j \in \mathcal{N}$. Furthermore, each agent has a local asymptotic model matching controller (6) which guarantees that $y_k(t) - y_k^*(t) \to 0$ as $t \to \infty$. Hence, the output synchronization problem for the agents (1) is solved and the behavior of the group matches the reference model (5).

This hierarchical approach has the following limitation: there is no feedback from the individual agents to the coordinating network level. Assume that one of the agents is influenced by an external disturbance and the output deviates from its reference. Then, the group can not react cooperatively on this disturbance (i.e., keep the synchronization error small) since neither the agent's reference model nor any other agent in the group will notice the disturbance. If, for instance, in a vehicle platoon, one vehicle gets slowed down due to an external influence, it would be desirable that the reference signals of the other vehicles adapt to this situation such that the formation can be maintained without crashes. The rest of this paper addresses this limitation.

Remark 3. In principle, the hierarchical control scheme can also be realized by a different choice of the output tracking control method on the agent level, as well as a different choice of the reference synchronization method on the network level. Nevertheless, the limitation described above is inherent in the hierarchical structure and has to be addressed.

3. COOPERATIVE REACTION ON DISTURBANCES

In the following, we consider external disturbances acting on individual agents. The objective is output synchronization despite these disturbances. A natural approach to solve this problem is to design the local controllers on the agent level such that the disturbances are attenuated locally. Here, we propose an alternative strategy: We leave the local controllers unchanged and solve the disturbance attenuation problem on the network level. Such a strategy is favorable when the attenuation of synchronization errors has the highest priority while effects of the disturbances on the synchronous motion are tolerable.

For this purpose, we introduce feedback from the agent to the network level and establish integral action on the network level such that the agents synchronize exactly even in presence of constant persistent tracking errors in the local control loops.

3.1 Feedback from Agent to Network Level

In the present setup, the distributed control law (7) is based on the states of neighboring reference models in the network. Our idea is to use the output of the physical agent instead in order to establish feedback from the agent to the network level. For this purpose, we choose reference models (5) of the form

$$\dot{z}_k = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & & & \\ -a_0 & -a_1 & \cdots & -a_{n-1} \end{bmatrix} z_k + \begin{bmatrix} 0 \\ \cdots \\ 0 \\ b \end{bmatrix} w_k, \tag{8a}$$

$$y_k^* = [1\ 0\ \cdots\ 0]\, z_k, \tag{8b}$$

where $b \neq 0$ and the system order n is larger than the maximal relative degree r_k of all agents. System (8) is controllable and observable and has relative degree $r = n$ by construction. The state z_k is a vector consisting of the reference output y_k^* and its first $n-1$ derivatives. In order to establish feedback from the agent to the network level, we replace z_k in the couplings (7) by the vector $\hat{z}_k$ consisting of the real physical output y_k of agent k and its derivatives, i.e., we replace z_k by

$$\hat{z}_k = \left[y_k\ \dot{y}_k\ \cdots\ y_k^{(n-1)}\right]^\mathsf{T}. \tag{9}$$

The first r_k derivatives of y_k can be computed as

[2] Illustration modified from Isidori [1995].

$$\dot{y}_k = \mathrm{L}_{f_k} h_k(x_k), \quad \ldots, \quad y_k^{(r_k-1)} = \mathrm{L}_{f_k}^{r_k-1} h_k(x_k),$$
$$y_k^{(r_k)} = \mathrm{L}_{f_k}^{r_k} h_k(x_k) + \mathrm{L}_{g_k} \mathrm{L}_{f_k}^{r_k-1} h_k(x_k) u_k.$$

Since the order n of the reference model is larger than $\max_k r_k$, we furthermore need the derivatives of y_k up to order $n-1$. Since the r_k-th derivative depends explicitly on u_k, the higher order derivatives will depend on derivatives of u_k, up to order $n-r_k-1$. However, the model (8) satisfies $CA^iB = 0$ for $i < n-1$ and $CA^{n-1}B \neq 0$. Therefore the derivatives of u_k as in (6) up to order $n-r_k-1$ are independent of w_k. Hence, the vector $\hat{z}_k$ in (9) is independent of w_k as well. This allows us to replace (7) by the couplings based on the physical outputs of the agents,

$$w_k = K \sum_{j=1}^{N} a_{kj} (\hat{z}_j - \hat{z}_k). \tag{10}$$

The closed loop with couplings (7) achieves output synchronization as discussed in Section 2.3. The following theorem states that output synchronization is still achieved if (7) is replaced by (10), which will be beneficial in the following.

Theorem 4. Consider a group of N nonlinear SISO agents (1), each equipped with a reference model (8), local control law (6), and with couplings (10). Suppose that Assumptions 1 and 2 are satisfied. Then, for all $k, j \in \mathcal{N}$,

$$y_k(t) - y_j(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore, there exists $\tilde{z}_0 \in \mathbb{R}^n$ such that for all agents $k \in \mathcal{N}$, $y_k(t) - s(t) \to 0$ as $t \to \infty$, where $s(t) = C\tilde{z}(t)$ and $\tilde{z}(t)$ is the solution of $\dot{\tilde{z}} = A\tilde{z}$ with A as in (8) and $\tilde{z}(0) = \tilde{z}_0$.

Proof. Let ε_k be the stack vector of e_k and its derivatives, i.e.,

$$\varepsilon_k = \begin{bmatrix} e_k & \dot{e}_k & \cdots & e_k^{(r_k-1)} \end{bmatrix}^\mathsf{T}.$$

Recall that the tracking error $e_k = y_k - y_k^*$ is governed by (4). In particular, with the model matching control law (6), the dynamics of the tracking error are $\dot{\varepsilon}_k = E_k \varepsilon_k$, where the matrix

$$E_k = \begin{bmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & & & \\ -c_0 & -c_1 & \cdots & -c_{r_k-1} \end{bmatrix}$$

is Hurwitz. The derivatives of e_k of order higher than $r_k - 1$ can be expressed as linear combinations of the derivatives of order up to $r_k - 1$ according to (4). The r_k-th derivative is the last element of $E_k \varepsilon_k$, the (r_k+1)-st derivative is the last element of $E_k^2 \varepsilon_k$, etc. Hence, we can construct $H_k \in \mathbb{R}^{n \times r_k}$ such that

$$\begin{bmatrix} e_k & \dot{e}_k & \cdots & e_k^{(n-1)} \end{bmatrix}^\mathsf{T} = H_k \varepsilon_k.$$

It holds that $\hat{z}_k = z_k - H_k \varepsilon_k$ and we can rewrite (10) as

$$w_k = K \sum_{j=1}^{N} a_{kj}(z_j - z_k) - K \sum_{j=1}^{N} a_{kj}(H_j \varepsilon_j - H_k \varepsilon_k). \tag{11}$$

The first term is the original coupling (7) based on the reference model states z_k, $k \in \mathcal{N}$. The second term depends on the tracking errors. The reference model (8) with (11) yields

$$\dot{z}_k = A z_k + BK \sum_{j=1}^{N} a_{kj}(z_j - z_k) - BK \sum_{j=1}^{N} a_{kj}(H_j \varepsilon_j - H_k \varepsilon_k). \tag{12}$$

The solution of a linear system with exponentially decaying input converges exponentially to a solution of the autonomous linear system, cf., Wieland [2010]. Since $\varepsilon_k(t) \to 0$ exponentially as $t \to \infty$, we can hence conclude that the solutions $z_k(t)$ converge exponentially to solutions of the nominal network

$$\dot{\bar{z}}_k = A \bar{z}_k + BK \sum_{j=1}^{N} a_{kj}(\bar{z}_j - \bar{z}_k). \tag{13}$$

By construction of the coupling gain K, it holds that for all $k, j \in \mathcal{N}$, $\bar{z}_k(t) - \bar{z}_j(t) \to 0$ exponentially as $t \to \infty$ and therefore also $z_k(t) - z_j(t) \to 0$ and $y_k^*(t) - y_j^*(t) \to 0$ as $t \to \infty$. The local controllers (6) guarantee that $y_k(t) - y_k^*(t) \to 0$ as $t \to \infty$. Hence, we can conclude that

$$y_k(t) - y_j(t) \to 0 \quad \text{as} \quad t \to \infty.$$

The solutions of (12) converge to a solution of (13) and, by the same argument, the solutions of (13) converge to a solution of the autonomous system (8). We can conclude that there exists $\tilde{z}_0 \in \mathbb{R}^n$ such that for all $k \in \mathcal{N}$, $y_k(t) - s(t) \to 0$ as $t \to \infty$, where $s(t) = C\tilde{z}(t)$ and $\tilde{z}(t)$ is the solution of $\dot{\tilde{z}} = A\tilde{z}$ with initial condition $\tilde{z}(0) = \tilde{z}_0$ and A as in (8). In words, the synchronous output trajectory $s(t)$ is generated by the autonomous reference model (8). Moreover, if $z_k(t)$ is bounded, the internal states η_k of each agent remain bounded by Assumption 2. ■

Replacing z_k by $\hat{z}_k$ in (7) introduces a disturbance on the network level due to the local tracking errors, see (11). In order to attenuate this disturbance in case of persistent tracking errors, we introduce integral action.

3.2 Integral Action on the Network Level

On the network level, integral action can be included in order to guarantee exact synchronization of the reference trajectories under constant disturbances. Such disturbances may be caused by persistent tracking errors on the agent level when using the couplings (10) based on the agents' physical outputs. Consensus protocols with integral action have already been proposed by Yucelen and Egerstedt [2012] and Andreasson et al. [2012] for single and double-integrator agents. The procedure presented here is applicable to general linear agents.

Suppose that we want to synchronize the outputs of the systems

$$\dot{z}_k = A z_k + B z_k + P d_k \tag{14a}$$
$$y_k^* = C z_k \tag{14b}$$

despite constant disturbances $d_k \in \mathbb{R}^r$ acting through the input matrix $P \in \mathbb{R}^{n \times r}$. The following result is formulated for multi-input multi-output (MIMO) systems with $y_k^* \in \mathbb{R}^p$ and $u_k \in \mathbb{R}^q$, despite the fact that the rest of the paper deals with SISO systems ($p = q = 1$). We extend each system by integrator states $\xi_k \in \mathbb{R}^p$ that integrate the output y_k^*,

$$\dot{\xi}_k = y_k^*. \tag{15}$$

Systems (14) combined with (15) yield the extended model

$$\dot{z}_{\mathrm{e},k} = A_\mathrm{e} z_{\mathrm{e},k} + B_\mathrm{e} u_k + P_\mathrm{e} d_k, \tag{16a}$$
$$y_k^* = C_\mathrm{e} z_{\mathrm{e},k}, \tag{16b}$$

with

$$A_\mathrm{e} = \left[\begin{array}{c:c} 0 & C \\ \hdashline 0 & A \end{array}\right], \quad B_\mathrm{e} = \begin{bmatrix} 0 \\ B \end{bmatrix}, \quad P_\mathrm{e} = \begin{bmatrix} 0 \\ P \end{bmatrix}, \quad C_\mathrm{e} = \left[\begin{array}{c:c} 0 & C \end{array}\right],$$

where $z_{\mathrm{e},k} = [\xi_k^\mathsf{T} \; z_k^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^{n+p}$, for all $k \in \mathcal{N}$.

Theorem 5. Consider a group of N linear systems (16) with constant disturbances $d_k \in \mathbb{R}^r$, $k \in \mathcal{N}$. Suppose that $(A_\mathrm{e}, B_\mathrm{e})$ is stabilizable. Then, the following statements are equivalent:

i) The underlying graph $\mathcal{G}$ is connected.

ii) There exists $K_\mathrm{e} \in \mathbb{R}^{q \times (n+p)}$ such that the couplings

$$u_k = K_\mathrm{e} \sum_{j=1}^{N} a_{kj}(z_{\mathrm{e},j} - z_{\mathrm{e},k}) \tag{17}$$

guarantee $\forall k, j \in \mathcal{N}$: $y_k^*(t) - y_j^*(t) \to 0$ as $t \to \infty$.

In this case all solutions of (16), (17) satisfy $y_k^*(t) - s(t) \to 0$ as $t \to \infty$, $k \in \mathcal{N}$, where $s(t) = C\tilde{z}_1(t)$ and $\tilde{z}_1(t)$ is the solution of

$$\dot{\tilde{z}}_1 = A\tilde{z}_1 + \sum_{k=1}^{N} p_k P d_k,$$

with initial condition $\tilde{z}_1(0) = (p^\mathsf{T} \otimes I_n)z(0)$, and where $p^\mathsf{T} = [p_1 \cdots p_N]$ satisfies $p^\mathsf{T} L = 0^\mathsf{T}$ and $p^\mathsf{T}\mathbf{1} = 1$.

Proof. From Wieland et al. [2011a], we know that if the pair $(A_\mathrm{e}, B_\mathrm{e})$ is stabilizable, connectedness of $\mathcal{G}$ is equivalent to the existence of a gain matrix K_e solving the synchronization problem without disturbances. Hence there exists a matrix K_e such that $y_k^*(t) - y_j^*(t) \to 0$ as $t \to \infty$ for $d_k = 0$, $k, j \in \mathcal{N}$, if and only if $\mathcal{G}$ is connected. In the following, we show that (17) with this matrix K_e also achieves exact output synchronization in presence of constant nonzero disturbances.

With stack vectors $z_\mathrm{e} = [z_{\mathrm{e},1}^\mathsf{T} \cdots z_{\mathrm{e},N}^\mathsf{T}]^\mathsf{T}$, $d = [d_1 \cdots d_N]^\mathsf{T}$, and Laplacian matrix L corresponding to the graph $\mathcal{G}$, the closed loop of (16) with couplings (17) can be written as

$$\dot{z}_\mathrm{e} = [(I_N \otimes A_\mathrm{e}) - (L \otimes B_\mathrm{e} K_\mathrm{e})]z_\mathrm{e} + (I_N \otimes P_\mathrm{e})d,$$

where $\otimes$ is the Kronecker product. We apply the state transformation $\tilde{z}_\mathrm{e} = (T^{-1} \otimes I_{n+p})z_\mathrm{e}$ as introduced by Fax and Murray [2004], where the transformation matrix T is chosen such that

i) $\Lambda = T^{-1}LT$ has Jordan canonical form,
ii) the first column of T is the vector of ones $\mathbf{1}$.
iii) The first row of T^{-1} is p^T, where $p^\mathsf{T} L = 0^\mathsf{T}$ and $p^\mathsf{T}\mathbf{1} = 1$.

Note that $\mathbf{1}$ is the right eigenvector and p^T is the normalized left eigenvector of L corresponding to the zero eigenvalue. This state transformation yields

$$\dot{\tilde{z}}_\mathrm{e} = [(I_N \otimes A_\mathrm{e}) - (\Lambda \otimes B_\mathrm{e} K_\mathrm{e})]\tilde{z}_\mathrm{e} + (T^{-1} \otimes P_\mathrm{e})d.$$

The matrix $[(I_N \otimes A_\mathrm{e}) - (\Lambda \otimes B_\mathrm{e} K_\mathrm{e})]$ is upper block triangular with blocks $A_\mathrm{e} - \lambda_k(L)B_\mathrm{e}K_\mathrm{e}$ on the diagonal, where $\lambda_k(L)$ are the eigenvalues of L. Since the graph $\mathcal{G}$ is connected and by construction of K_e, the matrix $A_\mathrm{e} - \lambda_k(L)B_\mathrm{e}K_\mathrm{e}$ is Hurwitz for $k = 2, \ldots, N$, i.e., for all nonzero eigenvalues of L. The transformated system has the structure

$$\dot{\tilde{z}}_\mathrm{e} = \left[\begin{array}{c|cccc} A_\mathrm{e} & 0 & \cdots & 0 \\ \hline 0 & A_\mathrm{e} - \lambda_2 B_\mathrm{e} K_\mathrm{e} & \star & 0 \\ \cdots & & \cdots & \star \\ 0 & 0 & & A_\mathrm{e} - \lambda_N B_\mathrm{e} K_\mathrm{e} \end{array}\right] \tilde{z}_\mathrm{e} + (T^{-1} \otimes P_\mathrm{e})d.$$

Let the transformed state be partitioned as $\tilde{z}_\mathrm{e}^\mathsf{T} = [\tilde{z}_{\mathrm{e},1}^\mathsf{T} \cdots \tilde{z}_{\mathrm{e},N}^\mathsf{T}]$ such that $\tilde{z}_{\mathrm{e},k} \in \mathbb{R}^{n+p}$, $k \in \mathcal{N}$. Since the lower right block of the transformed system matrix is Hurwitz, a constant disturbance d leads to a constant steady-states $\tilde{z}_{\mathrm{e},k}(t) \to \tilde{z}_{\mathrm{e},k}^\circ$ as $t \to \infty$ for $k = 2, \ldots, N$. By construction of (16), it follows that $C_\mathrm{e}\tilde{z}_{\mathrm{e},k}(t) \to 0$ as $t \to \infty$ for $k = 2, \ldots, N$ since the integrator states remain bounded and are not directly affected by the disturbance. Consequently,

$$y^*(t) = (I_N \otimes C_\mathrm{e})z_\mathrm{e}(t) = (T \otimes C_\mathrm{e})\tilde{z}_\mathrm{e}(t) \to (\mathbf{1} \otimes C_\mathrm{e})\tilde{z}_{\mathrm{e},1}(t),$$

or equivalently, $y_k^*(t) - C_\mathrm{e}\tilde{z}_{\mathrm{e},1}(t) \to 0$ as $t \to \infty$ for all $k \in \mathcal{N}$. The dynamics of $\tilde{z}_{\mathrm{e},1}$ are given by $\dot{\tilde{z}}_{\mathrm{e},1} = A_\mathrm{e}\tilde{z}_{\mathrm{e},1} + (p^\mathsf{T} \otimes P_\mathrm{e})d$ with initial condition $\tilde{z}_{\mathrm{e},1}(0) = (p^\mathsf{T} \otimes I_{n+p})z_\mathrm{e}(0)$. The synchronous output trajectory of the network is $s(t) = C_\mathrm{e}\tilde{z}_{\mathrm{e},1}(t)$. Let $\tilde{z}_{\mathrm{e},1}$ be partitioned as $\tilde{z}_{\mathrm{e},1}^\mathsf{T} = [\tilde{\xi}_1^\mathsf{T}\ \tilde{z}_1^\mathsf{T}]$ with $\tilde{\xi}_1 \in \mathbb{R}^p$. Then, it is easy to see that $\tilde{\xi}_1$ does not influence $\tilde{z}_1$ and $s(t) = C\tilde{z}_1(t)$, where $\dot{\tilde{z}}_1 = A\tilde{z}_1 + (p^\mathsf{T} \otimes P)d$ and $\tilde{z}_1(0) = (p^\mathsf{T} \otimes I_n)z(0)$. We refer to $s(t)$ as the synchronous (output) trajectory of the group. ■

Note that a feasible matrix K_e for the extended agents can be found with the LMI-based design method by Wieland et al. [2011a]. With Theorem 5 in place, we can establish integral action on the network level of the hierarchical control scheme. In Section 3.1, we have shown how feedback from the agent to the network level can be established. If the reference models (8) are extended according to (16), $\hat{z}_k$ has to be extended as well,

$$\hat{z}_{\mathrm{e},k} = \left[\textstyle\int y_k dt\ \ y_k\ \ \dot{y}_k \cdots y_k^{(n-1)}\right]^\mathsf{T}. \tag{18}$$

Suppose that agent (1) is affected by an external disturbance $d_k \in \mathbb{R}$ according to

$$\dot{x}_k = f_k(x_k) + g_k(x_k)u_k + p_k(x_k)d_k \tag{19a}$$
$$y_k = h_k(x_k), \tag{19b}$$

where p_k is a smooth vector field defined on X_k. As the following theorem shows, exact synchronization of the outputs y_k is guaranteed in the presence of constant local tracking errors $e_k = y_k - y_k^*$, which may be caused by external disturbances.

Theorem 6. Consider a group of N nonlinear SISO agents (19) with reference models (8) extended as in (16) and couplings

$$w_k = K_\mathrm{e} \sum_{j=1}^N a_{kj}(\hat{z}_{\mathrm{e},j} - \hat{z}_{\mathrm{e},k}). \tag{20}$$

Suppose that Assumptions 1 and 2 are satisfied and that the local controllers (6) achieve constant steady-state tracking errors $e_k(t) \to e_k^\circ$ as $t \to \infty$. Then, for all $k, j \in \mathcal{N}$,

$$y_k(t) - y_j(t) \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore, the synchronous output trajectory is generated by reference model (8).

Proof. With extended vector $\varepsilon_{\mathrm{e},k} = \left[\int e_k dt\ \ e_k\ \ \dot{e}_k \cdots e_k^{(r_k-1)}\right]^\mathsf{T}$, it holds that $\hat{z}_{\mathrm{e},k} = z_{\mathrm{e},k} - H_{\mathrm{e},k}\varepsilon_{\mathrm{e},k}$ for some $H_{\mathrm{e},k} \in \mathbb{R}^{(n+1)\times(r_k+1)}$ obtained from (4). The dynamics of $\varepsilon_{\mathrm{e},k}$ are $\dot{\varepsilon}_{\mathrm{e},k} = E_{\mathrm{e},k}\varepsilon_{\mathrm{e},k}$ with

$$E_{\mathrm{e},k} = \left[\begin{array}{c|ccc} 0 & 1 & 0 & \cdots \\ \hline 0 & & & \\ \cdots & & E_k & \\ 0 & & & \end{array}\right].$$

Hence, we have $\dot{\hat{z}}_{\mathrm{e},k} = \dot{z}_{\mathrm{e},k} - H_{\mathrm{e},k}\dot{\varepsilon}_{\mathrm{e},k}$ and with (16) and (20),

$$\dot{\hat{z}}_{\mathrm{e},k} = A_\mathrm{e}\hat{z}_{\mathrm{e},k} + B_\mathrm{e}K_\mathrm{e}\sum_{j=1}^N a_{kj}(\hat{z}_{\mathrm{e},j} - \hat{z}_{\mathrm{e},k}) + (A_\mathrm{e}H_{\mathrm{e},k} - H_{\mathrm{e},k}E_{\mathrm{e},k})\varepsilon_{\mathrm{e},k}. \tag{21}$$

By construction, the first row and the first column of $A_\mathrm{e}H_{\mathrm{e},k} - H_{\mathrm{e},k}E_{\mathrm{e},k}$ consist of zeros. Since by assumption $e_k(t)$ converges to a constant as $t \to \infty$, the coupled systems (21) are affected by constant disturbances and it follows from Theorem 5 that $C_\mathrm{e}\hat{z}_{\mathrm{e},k}(t) - C_\mathrm{e}\hat{z}_{\mathrm{e},k}(t) = y_k(t) - y_j(t) \to 0$ as $t \to \infty$. ■

Remark 7. In general, constant disturbances d_k in (19) do not necessarily lead to constant steady-state tracking errors under control law (6). Nevertheless, integral action on the network level according to Theorem 6 may be beneficial in order to attenuate the constant component of persistent tracking errors.

Remark 8. In order to use (18), the output y_k has to be integrated continuously. Depending on the desired output trajectory, this state ξ_k may grow without bound. However, since it is not a physical but a virtual state, an overflow may be avoided by suitable numerical measures.

4. EXAMPLE: MAGNETIC LEVITATION SYSTEMS

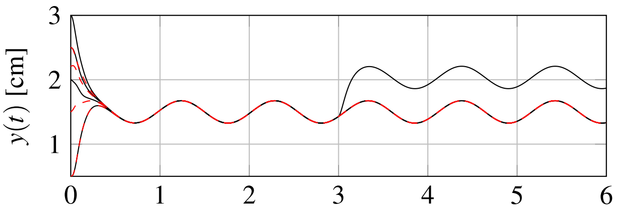
In order to illustrate the synchronized model matching approach, we apply it to the models of four magnetic levitation systems [3]. The dynamical behavior around the setpoint $y^\circ = 1.5$ cm is described by (19) where

$$f_k(x_k) = \begin{bmatrix} x_{k,2} \\ g - \alpha x_{k,2}/m_k - g(y^\circ + b)^4/(x_{k,1} + y^\circ + b)^4 \end{bmatrix},$$

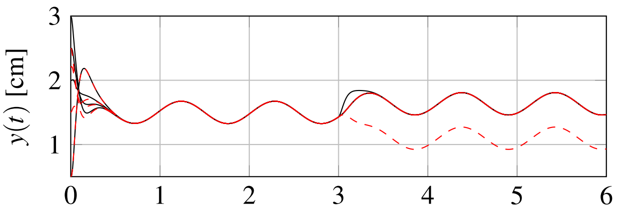
$$g_k(x_k) = \begin{bmatrix} 0 \\ -1/(am_k(x_{k,1} + y^\circ + b)^4) \end{bmatrix}, \quad p_k(x_k) = \begin{bmatrix} 0 \\ 1/m_k \end{bmatrix},$$

$$h_k(x_k) = x_{k,1},$$

[3] Educational Control Products (ECP) model 730: magnetic levitation system. `http://www.ecpsystems.com/controls_maglevit.htm`.



(a) Couplings (7) and control law (6).



(b) Couplings and integral action according to Theorem 6.

Fig. 3. Cooperative control of four magnetic levitation systems with disc positions y_k (——) and references y_k^* (– – –).

with force d_k acting as disturbance on the disc, input voltage u_k, disc velocity $x_{k,1}$, position $y_k = x_{k,2}$, and mass m_k. The parameters are $g = 981$ cm/s^2, $\alpha = 1.875$ kg/s, $a = 2.0736 \cdot 10^{-7}$ Vs^2/(kg cm^5), $b = 6.7434$ cm, $m_1 = 0.14$ kg, $m_2 = 0.12$ kg, $m_3 = 0.11$ kg, $m_4 = 0.19$ kg. The goal is that all four discs move synchronously around the setpoint. We choose a reference model (8) generating sinusoidal trajectories with frequency $\omega = 6$ rad/s and with relative degree $r = 3$, i.e.,

$$\dot{z}_k = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -108 & -36 & -3 \end{bmatrix} z_k + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} w_k \qquad y_k^* = [1\ 0\ 0]\, z_k.$$

The system satisfies Assumptions 1 and 2 with $r_k = 2$. The model matching controllers (6) are designed so that the error dynamics have two poles at -20. We apply a step disturbance $d_1(t) = 0.3$ N for $t \geq 3$s to disc one. Under control law (6), this disturbance leads to a persistent constant tracking error e_1°. The graph $\mathcal{G}$ on the network level is a directed cycle.

The simulation in Fig. 3(a) shows the result with couplings (7) and $K = 10^3 \cdot [1.5593, 0.209, 0.0077]$. The disc positions synchronize to a sinusoidal trajectory. However, the disturbance d_1 results in a persistent synchronization error. Therefore, in the second simulation shown in Fig. 3(b), we include integral action on the network level according to (16) and use the couplings (20) based on the real disc positions and $K_e = 10^4 \cdot [9.5294, 1.5528, 0.0861, 0.0018]$. As expected, the disturbance is attenuated cooperatively by the group while the synchronous motion is influenced by the disturbance.

5. CONCLUSIONS

We have presented a novel hierarchical approach to cooperative control of multi-agent systems termed *synchronized model matching* consisting of two levels: the network level on which linear reference models for each agent are synchronized, and the agent level on which each agent implements an asymptotic model matching controller in order to track the output of its reference model. The control scheme is applicable to groups of non-identical nonlinear SISO agents with well-defined relative degree and stable internal dynamics. We have shown how feedback from the agents to the network level can be established in this setup and how the distributed control law can be extended by integral action, which allows a cooperative attenuation of disturbances acting on individual agents. The extension to agent models with multiple inputs and outputs is currently under investigation.

The proposed procedure of establishing integral action on the network level is applicable to general homogeneous linear multi-agent systems. It is of independent interest since it shows how steady-state synchronization errors can be avoided despite constant disturbances.

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