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Nonlinear Luenberger observer design via invariant manifold computation

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Abstract: A framework to extend Luenberger observer to nonlinear systems is proposed. The theory of invariant manifolds plays a central role in the framework. It is shown that the invariant manifolds for observer design are often non-standard ones and this makes their computations challenging. The proposed theory successfully removes the condition imposed on the system to be observed in the previous research in nonlinear Luenberger observer. Numerical examples show that the design methods proposed produce more effective observers compared with linear observers.

1. INTRODUCTION

In practice, measurements of states of dynamical systems are often limited. It is then natural to pose a problem of reconstruction of states from the limited number of measurements, which is called the observer design problem. The first approach to this problem is attributed to Luenberger Luenberger [1964] who was the first to solve this problem.However, attempts to the problem of state reconstruction of nonlinear systems took longer to appear. The output injection method (Krener and Isidori [1983]), high-gain observer (Khalil [1999], Atassi and Khalil [1999]), and H_∞ approach (Pertew et al. [2006]), to name a few. The basis of Luenberger observer is the invariance relation between system to be observed and observer. Kazantsis and Kravaris (Kazantzis and Kravaris [1998]) derived the nonlinear counterpart for the Sylvester equation in Luenberger [1964]. However, the paper does not deal with practical methods for solving this pde except for Taylor series expansion. Recent papers Andrieu and Praly [2006], Andrieu [2010] investigate further this pde, but, still computational aspects need to be studied.

The center, center-stable and stable manifolds are well-known objects. of great importance. A detailed description and proof of its existence are presented in Kelley [1967]. The difficulty of using the Taylor expansion-based approach has been long recognized, an explicit example is shown in Sakamoto [2013a]. Recently, a method for finding the stable manifold based on iterative solution of certain differential equations appeared. This method is suitable for solution of practical problems, optimal control of nonlinear systems among all. It is presented in Sakamoto and van der Schaft [2008], Sakamoto [2013a] and successfully extended to the construction of the center and center-stable manifolds in Sakamoto and Rehák [2011] with application to the optimal output regulation.

The aim of this paper is to propose a framework for computing the invariant manifold that defines the system-observer invariance relation based on the aforementioned method for computation of the center, unstable or center-unstable manifolds. It is equivalent to solve the pde derived in Kazantzis and Kravaris [1998]. This is an alternative approach compared to Andrieu and Praly [2006], Andrieu [2010] in the sense that observer design can be explicitly carried out. Also, the methods presented here enable to remove the requirement of Kazantzis and Kravaris [1998], namely that the linearization of the observed system must be either asymptotically stable or have all eigenvalues with positive real parts. Invariant manifolds that appear in the observer design theory are non-standard in the sense that the invariant manifold theory in Aulbach et al. [1986], Coddington and Levinson [1955], Chow and Hale [1982] cannot be applied. This comes from the fact that the observer states which have stronger stability must be dependent variables on the invariant manifold as well as from the special way of interconnection of the system and observer.

- Removing the assumption in Kazantzis and Kravaris [1998] about the location of eigenvalues of the observed system. This allows one to handle systems with linearization having eigenvalues with positive, negative and zero real parts.
- The proposed observer design method is more practical and viable than Andrieu and Praly [2006].

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- The computation method for invariant manifolds is based on the flow approach for the manifolds (see Sakamoto and van der Schaft [2008], Sakamoto and Rehák [2011], Sakamoto [2013a]), which has been already applied to many actual systems with experimental verifications (see Fujimoto and Sakamoto [2011], Sakamoto [2013b]) and proven to be superior to the Taylor method from the viewpoints of accuracy and computational complexity.

The organization of the paper is as follows. The next section contains the definition of the observer problem. §3 provides the framework for observer design theory based on invariant manifold theory. Next section proposes the computational methods In section 6, we show some illustrative numerical examples. Unfortunately, due to space limitations, no proof of the Theorem 3.6 can be given here. It will be presented in some future work.

2. NONLINEAR LUENBERGER OBSERVER THEORY AND MOTIVATION OF THE RESEARCH

Let us consider the nonlinear system

$$\dot{x} = \bar{f}(x) = Ax + f(x), \quad y = h(x) = Cx + \gamma(x), \qquad (1)$$

where A is an $n \times n$ matrix, C is a $1 \times n$ matrix and the functions $f : \mathbb{R}^n \to \mathbb{R}^n$ and $\gamma : \mathbb{R}^n \to \mathbb{R}$ (which satisfy $f(0) = 0$, $Df(0) = 0$, $\gamma(0) = 0$, $D\gamma(0) = 0$) consist of higher-order terms.

Assumption 2.1. The pair (C, A) is observable.

Luenberger's notion of observer is based on invariant manifold (invariant subspace in the case of linear systems) of closed loop system (see, Luenberger [1964]).

Definition 2.2. A dynamical system $\dot{\hat{x}} = \beta(\hat{x}, y)$ with $\beta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is called an observer for (1) if there exists a locally invertible (around the origin) map $T : \mathbb{R}^n \to \mathbb{R}^n$ with $T(0) = 0$ such that $\hat{x} = T(x)$ is an positively invariant manifold for the composite system (1)-(2.2) around $(x, \hat{x}) = (0, 0)$ and that the manifold is locally positively attracting for the composite system.

An observer in the sense of Definition 2.2 is proposed by Kazantsis and Kravaris under a restrictive assumption.

Theorem 2.3. (Kazantzis and Kravaris [1998], Theorem 1, 2) Assume that all eigenvalues of A lie in the left or right open half complex plane. Then, there exists a locally invertible analytic map T satisfying $\frac{\partial T}{\partial x}(x)\bar{f}(x) = \tilde{A}T(x) + bh(x)$, where $\tilde{A}$ is a suitably chosen Hurwitz matrix and a vector $b \in \mathbb{R}^n$ is such that the pair $(\tilde{A}, b)$ is controllable. Using $T(x)$, the following system is an observer for (1):

$$\dot{\hat{x}} = \bar{f}(\hat{x}) + \left(\frac{\partial T}{\partial x}(\hat{x})\right)^{-1} b(y - h(\hat{x})) \qquad (2)$$

Remark 2.4. The key assumption in Kazantzis and Kravaris [1998] is that the linearization of the system whose states are reconstructed is either asymptotically stable or has all eigenvalues with positive real parts. This is because they rely on the so-called Lyapunov's Auxiliary Theorem and compute $T(x)$ by the Taylor expansion approach. The present paper aims at removing their assumptions and present a computational framework for $T(x)$ based on invariant manifold theory. To motivate our approach better, let us consider a simple example of computing invariant manifolds around the origin for $\dot{x} = ax$, $\quad \dot{w} = bw$, $ab \neq 0$. Although this system of equations is uncoupled and looks simple, the topological properties around the origin of the equations in this study such as (8) can be well understood with this example since the effect of higher order terms is negligible. The invariant manifold $w = T(x)$ satisfies $\frac{dT}{dx} \times ax = bT(x)$, which is readily solved as $T(x) = ce^{\frac{b}{a}x}$, $\quad c :$ arbitrary constant. The differentiability and uniqueness of this solution can be summarized in Table 1. We note that when $ab < 0$, which is included in the unique C^ω case, the invariant manifold is a stable or unstable manifold. The assumptions in Kazantzis and Kravaris [1998] correspond to the unique C^ω solution case that only allows the observed system to have an asymptotic or a totally unstable linear part. The C^ω smoothness is necessary since they use the Taylor series approach. For the purpose of observer design, however, C^1 smoothness is sufficient and, as was mentioned in Introduction, the Taylor approach has serious drawbacks for solving nonlinear pdes (Sakamoto [2013a]). Generalizing their results requires to handle the cases where non-unique C^r $(r \geqslant 1)$ solutions exist. The non-uniqueness causes a challenge in the computation of invariant manifolds since most theories of invariant manifolds rely on uniqueness.

	unique	not unique
C^r	empty	$r < \frac{b}{a} < r+1$ $T(x) = ce^{\frac{b}{a}x}$
C^ω	$b \neq na$ for any $n \in \mathbb{N}$ $T(x) = 0$	$b = na$ for some $n \in \mathbb{N}$ $T(x) = ce^{\frac{b}{a}x}$

Table 1. The differentiability and uniqueness of an invariant manifold

3. INVARIANT MANIFOLD THEORY FOR OBSERVER DESIGN

In this section, we show that the construction of map T for observer mentioned in the previous section. The construction procedure depends on the stability type of the system to be observed. Let us consider the system of ordinary differential equations of the following form.

$$\begin{cases} \dot{x} = A_1 x + X(x, y, z) \\ \dot{y} = A_2 y + Y(x, y, z) \\ \dot{z} = A_3 z + Z(x, y, z) \end{cases} \qquad (3)$$

where $\dim x = n_1$, $\dim y = n_2$, $\dim z = n_3$, with $n = n_1 + n_2 + n_3$, $X : \mathbb{R}^n \to \mathbb{R}^{n_1}$, $Y : \mathbb{R}^n \to \mathbb{R}^{n_2}$ and $Z : \mathbb{R}^n \to \mathbb{R}^{n_3}$ are C^k functions $(k \geqslant 1)$ with $(X, Y, Z)(0, 0, 0) = 0$, $D(X, Y, Z)(0, 0, 0) = 0$.

Assumption 3.1. Real parts of all the eigenvalues of A_1, A_2 and A_3 are negative, zero and positive, respectively.

Assumption 3.2. System (3) is forward and backward complete.

To avoid the non-uniqueness of center manifolds a cut-off function for the y-component is used (Sijbrand [1985]).

$$\tilde{X}(x, y, z) = X\left(x, y\psi\left(\frac{|y|}{\delta}\right), z\right)$$

$$\tilde{Y}(x, y, z) = Y\left(x, y\psi\left(\frac{|y|}{\delta}\right), z\right)$$

$$\tilde{Z}(x,y,z) = Z\left(x, y\psi\left(\frac{|y|}{\delta}\right), z\right),$$

$\delta > 0$ is a constant and $\psi : [0,\infty) \to [0,1]$ is a C^∞ function satisfying $\psi(s) = 1$ on $[0,1]$ and $\psi(s) = 0$ on $[2,\infty)$. Hereafter, we re-write $\tilde{X}$, $\tilde{Y}$ and $\tilde{Z}$ as X, Y and Z.

The goal is to find an invariant manifold $w = \eta(x,y,z)$ in a neighborhood of the origin for (3) and a system

$$\dot{w} = \tilde{A}w + W(x,y,z), \tag{4}$$

where $\dim w = n$, $W : \mathbb{R}^n \to \mathbb{R}^n$ is C^k function ($k \geqslant 1$) with $W(0,0,0) = 0$, $DW(0,0,0) = 0$. For system (4), we assume the following.

Assumption 3.3. $\tilde{A} \in \mathbb{R}^{n\times n}$ is an asymptotically stable matrix and it satisfies $\operatorname{Re}\lambda_i(\tilde{A}) < \operatorname{Re}\lambda_j(A_1)$ for $i = 1,\dots,n$, $j = 1,\dots,n_1$, where λ_i denotes the i-th eigenvalues of a matrix.

The following lemma (Kelley [1967]) plays an crucial role.

Lemma 3.4. For (3), there exist unique invariant manifolds

(i) C^k stable manifold: $y = \mathrm{s}_1(x)$, $z = \mathrm{s}_2(x)$
(ii) C^{k-1} center manifold: $x = \mathrm{c}_1(y)$, $z = \mathrm{c}_2(y)$
(iii) C^k unstable manifold: $x = \mathrm{u}_1(z)$, $y = \mathrm{u}_2(z)$
(iv) C^{k-1} center-stable manifold: $z = \mathrm{cs}(x,y)$
(v) C^{k-1} center-unstable manifold: $x = \mathrm{cu}(y,z)$

These manifolds are defined in neighborhoods of the origin in appropriate subspaces.

We first consider the special case below.

Proposition 3.5. Suppose that $n_1 = 0$. Then, there exists a center-unstable manifold $w = \eta(y,z)$ of (3) and (4), which is unique in a neighborhood of the origin. It can be computed by

$$\eta(y,z) = \int_{-\infty}^{0} e^{-\tilde{A}s} W(\varphi(s,y,z))\, ds, \tag{5}$$

where $\varphi(t,y,z)$ denotes the solution of (3) starting from $(y,z) \in U$ at $t = 0$ with U being the domain of attraction for the center-unstable manifold in the negative time direction.

Note that the case in Proposition 3.5 includes $n_1 = n_2 = 0$, in which an unstable manifold is computed and the case $n_1 = n_3 = 0$, in which a center manifold is computed. Note also that formula (5) is a special case of the algorithm in Sakamoto and Rehák [2011] due to the one-way interaction structure between (3) and (4).

Theorem 3.6. When $n_3 = 0$, an invariant manifold $w = \eta(x,y)$ exists in a neighborhood of $(x,y) = (0,0)$. This manifold is C^1 but not unique.

The proof of Theorem 3.6 is omitted due to space limitations.

Theorem 3.7. When n_1, $n_3 > 1$, there exists a C^1 invariant manifold $w = \eta(x,y,z)$ in a neighborhood of the origin. This manifold is not unique, but, it is unique and C^{k-1} on the center-unstable manifold $x = \mathrm{cu}(y,z)$ in Lemma 3.4.

Remark 3.8. Unlike the one in Proposition 3.5, the invariant manifolds in Theorems 3.6, 3.7 are non-standard ones in the sense that the standard theory in Lemma 3.4 or computations in Sakamoto and Rehák [2011] cannot be applied. In Theorem 3.6, if one seeks an invariant manifold of the form $(x,y) = \zeta(w)$, where the variable with stronger stability is chosen as an independent variable, it uniquely exists and the theory in Aulbach et al. [1986], Chow and Hale [1982], Coddington and Levinson [1955] can be used. But, then, due to the one-way interaction between (3) and (4), the trivial manifold $\zeta = 0$ is obtained and no other manifold is found because of the uniqueness. Thus, the attempt to use the standard theory and converts the relation $(x,y) = \zeta(w)$ to get $w = \eta(x,y)$ necessarily fails.

Proof of Theorem 3.7: First of all, the center-unstable manifold $x = \mathrm{cu}(y,z)$ for (3) exists and it is unique. From Proposition 3.5, there exists a unique center-unstable manifold $w = \eta_{\mathrm{cu}}(y,z)$ for the system

$$\begin{cases} \dot{y} = A_2 y + Y(\mathrm{cu}(y,z),y,z) \\ \dot{z} = A_3 z + Z(\mathrm{cu}(y,z),y,z) \\ \dot{w} = \tilde{A}w + W(\mathrm{cu}(y,z),y,z). \end{cases} \tag{6}$$

The invariant manifold $w = \eta(x,y,z)$ we seek must agree with $w = \eta_{\mathrm{cu}}(y,z)$ on $x = \mathrm{cu}(y,z)$, namely, $\eta(\mathrm{cu}(y,z),y,z) = \eta_{\mathrm{cu}}(y,z)$. Next, we compute an invariant manifold $w = \eta_{\mathrm{cs}}(x,y)$ for

$$\begin{cases} \dot{x} = A_1 x + X(x,y,\mathrm{cs}(x,y)) \\ \dot{y} = A_2 y + Y(x,y,\mathrm{cs}(x,y)) \\ \dot{w} = \tilde{A}w + W(x,y,\mathrm{cs}(x,y)). \end{cases}$$

using Theorem 3.6, where $z = \mathrm{cs}(x,y)$ is the center-stable manifold for (3). The invariant manifold $w = \eta_{\mathrm{cs}}(x,y)$ is not unique, but, it is possible to prove that there exists an invariant manifold $w = \eta(x,y,z)$ satisfying $\eta(\mathrm{cu}(y,z),y,z) = \eta_{\mathrm{cu}}(y,z)$, $\eta(x,y,\mathrm{cs}(x,y)) = \eta_{\mathrm{cs}}(x,y)$ by connecting these two manifolds with the flows of the original ode (3). □

4. COMPUTATIONAL METHOD

4.1 General case

Here, we describe the computation for the case in Theorem 3.7. To be more specific, it will be shown how the manifolds from the previous section are obtained. Step 1: Let $M_{\mathrm{cu}} = \{(x,y,z) \,|\, x = \mathrm{cu}(y,z),\ (y,z) \in V\}$ be the center-unstable manifold in (3). To compute this, the algorithm in Sakamoto and Rehák [2011] is applied with $|y_0|$, $|z_0|$ sufficiently small so that the algorithm converges.

The set $M_{\mathrm{cu}}^{(k)} := \big\{(x_k(t,y_0,z_0), y_k(t,y_0,z_0), z_k(t,y_0,z_0)) \,|\, (y_0,z_0) \in V,\ t \leqslant 0\big\}$

is an approximation of M_{cu} in the sense that on $M_{\mathrm{cu}}^{(k)}$, it approximately holds that
$x_k(t,y_0,z_0) = \mathrm{cu}(y_k(t,y_0,z_0), z_k(t,y_0,z_0))$.

Step 2: Set

$$w_k(t,y_0,z_0) = \int_{-\infty}^{t} e^{\tilde{A}(t-s)} W(x_k(s,y_0,z_0), y_k(s,y_0,z_0), z_k(s,y_0,z_0))\, ds.$$

The convergence of this integral and the fact that $x_k(t,y_0,z_0)$, $y_k(t,y_0,z_0)$, $z_k(t,y_0,z_0)$, $w_k(t,y_0,z_0) \to 0$ as $t \to -\infty$ can be shown as in Sakamoto and Rehák [2011]. Note that the latter property will eventually guarantee that $\eta(0,0,0) = 0$.

Step 3: Define the set

$$\Omega'_{(k)} = \big\{(x_k(t,y_0,z_0), y_k(t,y_0,z_0), z_k(t,y_0,z_0), w_k(t,y_0,z_0)) \mid (y_0,z_0) \in V,\ t \leqslant 0\big\},$$

using the computation in Step 2. Note that this set is a parametrization of the center-unstable manifold for (6) in an approximate sense. Take a point $(x_0, y_0, z_0, w_0) \in \Omega'_{(k)}$ and a sufficiently large $t_f > 0$. Solve the boundary value problem:

$$P(x_0,y_0,z_0,w_0):\quad \begin{cases} \text{System (3) with } (x(t_f), y(t_f), z(t_f)) = (x_0,y_0,z_0), \\ \text{System (4) with } w(t_f) = w_0. \end{cases}$$

Define the set

$$\Omega_{(k)} = \big\{(x(t),y(t),z(t),w(t)) \,|\, (x(t),y(t),z(t),w(t)) \text{ solves } P(x_0,y_0,z_0,w_0),\ (x_0,y_0,z_0,w_0) \in \Omega'_{(k)},\ t \leqslant 0\big\}.$$

As $k \to \infty$, $\Omega_{(k)}$ approximates the invariant manifold of (3) and (4) around the origin, on which $w = \eta_{\rm cu}(y,z)$ holds.

Step 4: Compute a function $\eta_k(x,y,z)$ such that $w = \eta_k(x,y,z)$ on $\Omega_{(k)}$. This computation can be done using polynomial fitting or multi-dimensional interpolation.

4.2 Stable case

The computation of the invariant manifold in Theorem 3.6 (z-component missing) is not as complicated as that in Theorem 3.7. The initial value w_0 in (4) can be fixed arbitrarily. One just computes (3), (4) to get the set

$$\Omega = \{(x(t),y(t),w(t)) \,|\, (x(0),y(0)) \in U\},$$

where U is the domain of attraction for (3). For this, no iterative computation is necessary. Then, one computes $\eta(x,y)$ such that $w = \eta(x,y)$ on Ω. Theorem 3.6 guarantees that η is C^1, $\eta(0,0) = 0$ and $\partial\eta/\partial(x,y)(0,0) = 0$.

Remark 4.1. Numerical computation raises other issues that cannot be discussed into detail here. For instance, one of them is the influence of integration on finite intervals rather than on infinite ones or the number of points used for interpolation.

5. OBSERVER DESIGN

Conversion of the system into the form suitable for applying the above algorithm is described here. The observed system must be transformed into a block-diagonal form as in (3) and it must fit (4).

The equation for observer design is given by

$$\dot{\bar{w}} = \tilde{A}\bar{w} + bh(\xi) \tag{7}$$

$\tilde{A}$ introduced before, $\xi^T = (x^T, y^T, z^T)$, $b \in \mathbb{R}^n$ and $\frac{\partial h}{\partial \xi}(0) \neq 0$. Hence this equation is not in the form (4). Let us define $h(\xi) = C\xi + \gamma(\xi)$ where the smooth function γ vanishes at the origin together with its derivatives.

To remove the off-diagonal first-order terms, one uses a state transformation which will be defined below. Let $\Gamma = \left(\begin{smallmatrix} A & 0 \\ bC & \tilde{A}\end{smallmatrix}\right)$ with $A = \mathrm{diag}(A_1, A_2, A_3)$ The transformation matrix converting the matrix Γ into the block-diagonal form is $\left(\begin{smallmatrix} I & 0 \\ S & I\end{smallmatrix}\right)$, where S is the unique solution of $-SA + \tilde{A}S + bC = 0$. This equation has solution if $\lambda_i(-A) + \lambda_j(\tilde{A}) \neq 0$ for all $i, j = 1, \ldots, n$, which can be satisfied by choosing $\tilde{A}$ so that $\max(\mathrm{Re\,Eig}\tilde{A}) < \min(\mathrm{Re\,Eig}A)$. Note that the variable ξ remains unchanged. In the new coordinates $w = \bar{w} - S\xi$, the combined system reads

$$\begin{aligned} \dot{\xi} &= A\xi + \varphi(\xi) \\ \dot{w} &= \tilde{A}w + W(\xi), \end{aligned} \tag{8}$$

where W consists of higher order terms f, γ. Additionally, $\tilde{A}$ and b are chosen so that $(\tilde{A}, b)$ is controllable, which assures, with the observability of (C, A), that S is a nonsingular matrix.

Carrying out the computations from the prebious section, one obtains the manifold $w = \eta(\xi) = \eta(x,y,z)$ Using the inverse transformation, we get $\bar{w} = T(\xi) = \eta(\xi) + S\xi$. The function T is differentiable, moreover, its Jacobian is nonzero at the origin due to the nonsingularity of S and vanishing properties at the origin of the center-unstable manifold. Hence, the observer in the form (2) can be constructed with this function.

6. NUMERICAL EXAMPLES

6.1 Stable+unstable case 1

Consider a nonlinear system

$$\begin{pmatrix}\dot{x}_1 \\ \dot{x}_2\end{pmatrix} = \begin{pmatrix} x_1 - x_1^3 + x_2 \\ x_1 + x_1^2 x_2 - x_2 \end{pmatrix} \tag{9}$$

The matrix $A = \left(\begin{smallmatrix} 1 & 1 \\ 1 & -1\end{smallmatrix}\right)$ has eigenvalues $\pm\sqrt{2}$ and we apply Theorem 3.7 and its computation described in §4. Fig. 1

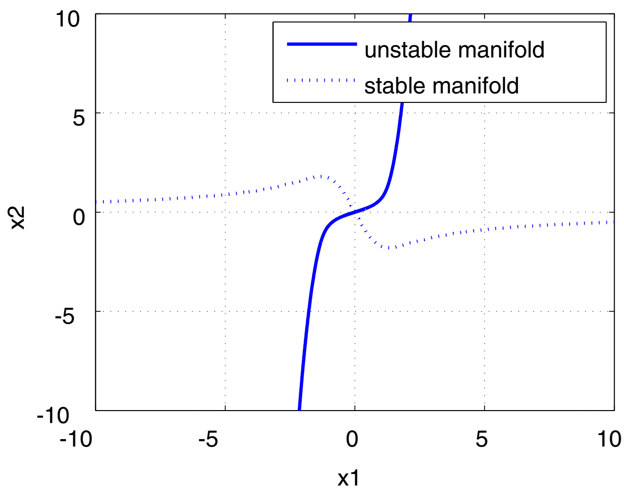

Fig. 1. Stable and unstable manifolds

shows the computed stable and unstable manifolds of (9). Let the observer candidate (in $\bar{w}$-space) be

$$\dot{\bar{w}} = \tilde{A}\bar{w} + by, \quad y = x_1, \tag{10}$$

where $\tilde{A} = \left(\begin{smallmatrix} -2.7 & 0 \\ 2 & -3\end{smallmatrix}\right)$, $b = (1,1)^T$.

The values of η on the unstable manifold, namely $\eta_{\rm cu}$ is first computed. To compute the values of η in a neighborhood of the origin, flows of (9) are computed, which are shown in Fig. 2. Integration of (9), (10) computes the

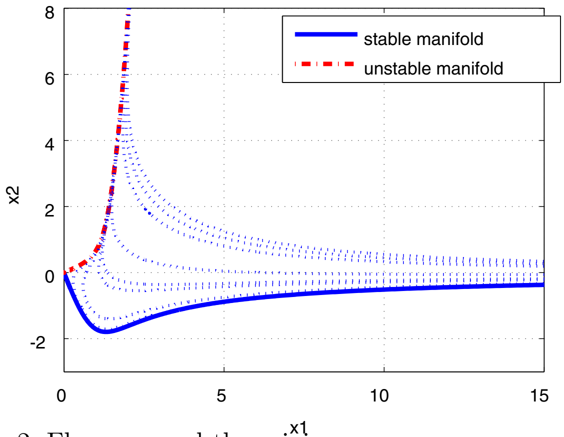

Fig. 2. Flows around the origin

values of w for each flow, obtaining the set $\Omega_{(k)}$, where k is the iteration number of the algorithm in Sakamoto and Rehák [2011]. Finally, $\eta(x_1, x_2)$ is approximated by 5-th order polynomials. Finally, $T(x_1, x_2) = \eta(x_1, x_2) + S\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$, with $S = \left(\begin{smallmatrix} 0.699 & 0.189 \\ 1.42 & 0.451 \end{smallmatrix}\right)$. The observer equation is

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 + \hat{x}_2 - \hat{x}_1^3 \\ \hat{x}_1 - \hat{x}_2 \hat{x}_1^2 \hat{x}_2 \end{pmatrix} + \left(\frac{\partial T}{\partial \hat{x}}(\hat{x})\right)^{-1} \begin{pmatrix} x_1 - \hat{x}_1 \\ x_1 - \hat{x}_1 \end{pmatrix}.$$

The simulation results are shown in Figs. 3, 4. The computation is performed in time duration $[0, 2]$.

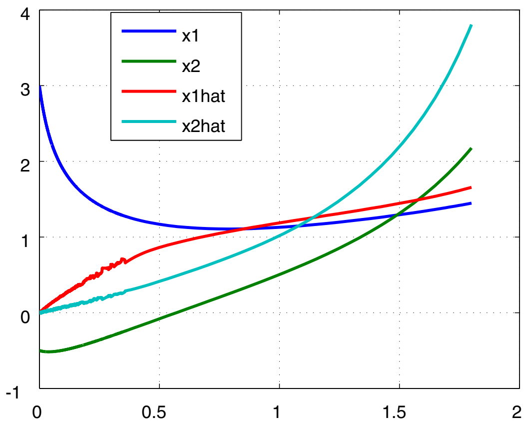

Fig. 3. States of system and nonlinear observer

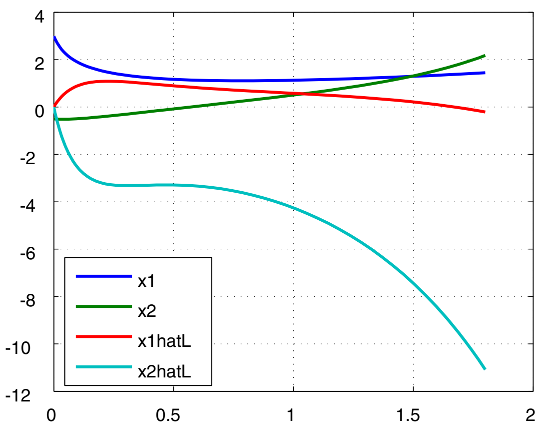

Fig. 4. States of system and linear observer

6.2 Stable+unstable case 2

Let $\varphi(x_1, x_2) = 0.1(x_1 + x_2)\exp(\frac{-16}{(x_1+x_2)^2})$ if $\|(x_1, x_2)\| \neq 0$, $\varphi(0, 0) = 0$. Next system is described by equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -x_1^3 \\ x_1^2 x_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \varphi(x_1, x_2)$$

Note that the Taylor series of the function φ does not converge at any neighborhood of the origin. Still, the method described here is applicable. As in the previous example, the approximation by fifth-order polynomials is used. The observer was defined by matrices $\tilde{A} = \left(\begin{smallmatrix} -3 & 0 \\ 2 & -3 \end{smallmatrix}\right)$ $b = (1, 1)^T$ the observable state being x_1 again. The observer attains the form

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_1 + \hat{x}_2 \\ \hat{x}_1 - \hat{x}_2 \end{pmatrix} + \frac{1}{5}\begin{pmatrix} -\hat{x}_1^3 \\ \hat{x}_1^2 \hat{x}_2 \end{pmatrix} + 0.1\varphi(\hat{x}_1, \hat{x}_2)$$
$$+ \left(\frac{\partial T}{\partial \hat{x}}(\hat{x})\right)^{-1} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (x_1 - \hat{x}_1). \tag{11}$$

This leads to $S = \left(\begin{smallmatrix} 0.321 & -0.189 \\ 0.523 & -0.451 \end{smallmatrix}\right)$. The Fig. 5 and 6 show the difference between the observer computed as described here and the Taylor polynomials-based one (using polynomials of degree 5).

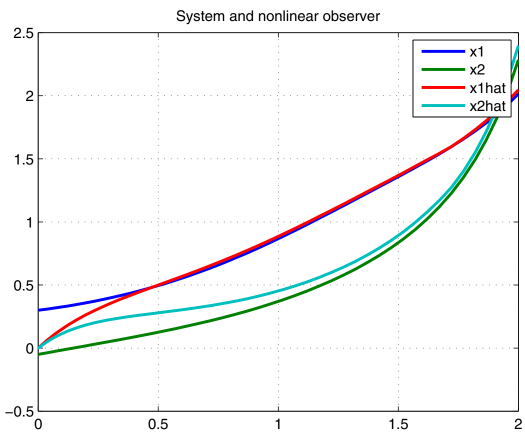

Fig. 5. States of system and nonlinear observer

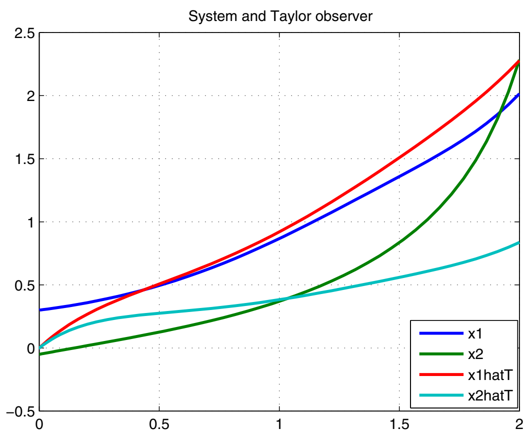

Fig. 6. States of system and observer constructed using Taylor expansions

6.3 Stable case

In this section, we consider a lightly damped nonlinear oscillator $\ddot{z} + 0.1\dot{z} + z^3 = 0$ with measurement $y = z$. The state space equation is

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -0.1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ -x_1^3 \end{pmatrix}, \; y = x_1.$$

The observer will be designed based on $\dot{\bar{w}} = \tilde{A}\bar{w} + by$, $\tilde{A} = \left(\begin{smallmatrix} -0.5 & 0 \\ 0 & -0.3 \end{smallmatrix}\right)$, $b = (1, -1)^T$. After block-diagonalizing the linear part of (6.3)-(6.3), the computation described in §4.2 is applied. $\eta(x_1, x_2)$ is approximated by 10-th order polynomials. Finally, $T(x_1, x_2) = \eta(x_1, x_2) + S\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$, with $S = \left(\begin{smallmatrix} 0.333 & -0.833 \\ -0.189 & 0.943 \end{smallmatrix}\right)$. The observer equation in $(\hat{x}_1, \hat{x}_2)$-space is

$$\begin{pmatrix} \dot{\hat{x}}_1 \\ \dot{\hat{x}}_2 \end{pmatrix} = \begin{pmatrix} \hat{x}_2 \\ -\hat{x}_1 - \hat{x}_1^3 - 0.1\hat{x}_2 \end{pmatrix} + \left(\frac{\partial T}{\partial \hat{x}}(\hat{x})\right)^{-1} \begin{pmatrix} x_1 - \hat{x}_1 \\ -(x_1 - \hat{x}_1) \end{pmatrix}.$$

Fig. 7 shows the states of system, nonlinear observer and linear observer. Fig. 8 shows the estimation errors of two observers. It can be seen that the errors in the nonlinear observer are smaller and converge faster than the linear observer.

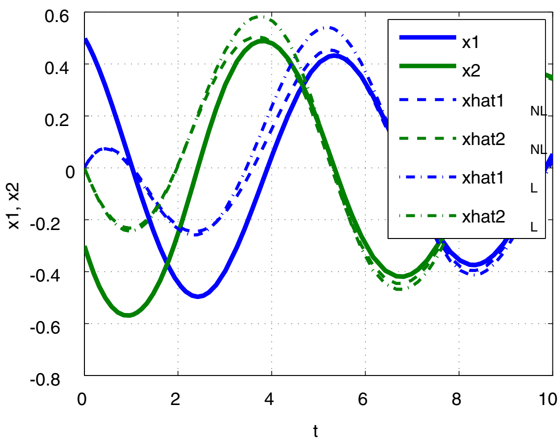

Fig. 7. States of system, nonlinear, linear observers

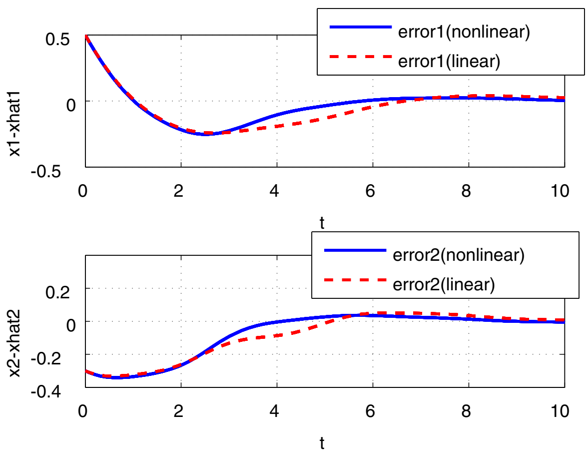

Fig. 8. Errors of nonlinear and linear observers

7. CONCLUDING REMARKS

A nonlinear extension of Luenberger observers based on computation of invariant manifolds is presented. These manifolds are rather nonstandard and their computation differs according to the stability of the observed system. Efficiency and capability of the algorithm was illustrated by an example. Issues concerning computation in higher dimensions as well as deriving error estimates remain an open problem for future.

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