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Backstepping Observers for Periodic Quasi-Linear Parabolic PDEs

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Abstract: The extension of the backstepping-based state observer design is considered for general periodic quasi-linear parabolic PDEs in one space dimension. Here, the extended linearization combined with a suitable coordinate and state transformation is employed. This allows for the application of the backstepping method for the determination of an extended Luenberger observer with observer gains entering the PDE and the boundary condition which ensures an exponential decay of the linearized observer error dynamics. This is confirmed analytically, while the convergence of the quasi-linear observer error system is studied in numerical simulations.

1. INTRODUCTION

In technical applications, quasi-linear partial differential equations (PDEs) arise in mathematical modeling of, e.g., chemical reactions (Jakobsen [2008]), semiconductor devices (Jüngel [2009]), or thermal processes in steel production (Speicher et al. [2012]). Thereby, the solution of the state estimation problem plays a crucial role for the design of state feedback control or monitoring purposes.

An extended distributed-parameter Luenberger observer is proposed by Zeitz [1977] for a semi-linear model of a chemical fixed-bed reactor. Here, correction terms are designed by following a heuristic and physics based approach. Similar observer design methods are applied by Hua et al. [1998] and Kreuzinger et al. [2008]. State observers based on optimal estimation are addressed, e.g., by Mangold et al. [2009], where an unscented Kalman filter is developed for a population balance model, and by Speicher et al. [2013], where an extended Kalman filter is proposed for the estimation of the spatio-temporal plate temperature evolution governed by a quasi-linear PDE. The backstepping-based state estimation for a non-linear Navier-Stokes PDE can be found in Vazquez et al. [2008] and for semi-linear parabolic PDEs in Meurer [2013] and Jadachowski et al. [2013].

This contribution presents a generalization of the results outlined by Meurer [2013] and Jadachowski et al. [2013] by addressing the distributed-parameter extended Luenberger observer design for the class of periodic quasi-linear PDEs in one space dimension. For the determination of the observer gains, the extended linearization is applied to the quasi-linear observer error dynamics. This results in a linear diffusion-convection-reaction equation (DCRE) with spatially and time-varying parameters governing the observer error dynamics. Followed by successive evaluation of the Hopf-Cole transformation (see, e.g., Hopf [1950]), the observer gains are determined by extending the classical time-invariant backstepping method (Smyshlyaev and Krstic [2005]) to the considered time-varying problem. For numerical computation of the observer gains, an efficient solution approach proposed by Jadachowski et al. [2012] is suitably modified.

The paper is organized as follows: In Section 2, the state estimation problem for the class of periodic quasi-linear parabolic PDEs is considered. Moreover, the extended Luenberger observer is formulated and observer gains are determined. Section 3 provides analytical results on the convergence of the linearized observer error dynamics. Simulation results of an exemplary set-up are presented in Section 4. Final remarks close the paper.

Notation: Wherever it is clear from the context, arguments of functions are omitted. Moreover, ∂_x, ∂_{x_z} and $\partial_{x_{zz}}$ denote partial derivatives with respect to $x(z,t)$, $\partial_z x(z,t)$ and $\partial_z^2 x(z,t)$, respectively.

2. STATE ESTIMATION PROBLEM

Let us consider the quasi-linear parabolic PDE

$$\partial_t x(z,t) = a(z,t,x)\partial_z^2 x(z,t) + f(z,t,x,\partial_z x) \tag{1}$$

defined on $(z,t) \in (0,L) \times \mathbb{R}_{t_0}^+$, $\mathbb{R}_{t_0}^+ = \{t \in \mathbb{R}^+ \mid t > t_0\}$, with boundary conditions (BCs)

$$\partial_z x(0,t) + p(t)x(0,t) = 0, \quad t \in \mathbb{R}_{t_0}^+ \tag{2a}$$

$$\partial_z x(L,t) + q(t)x(L,t) = 0, \quad t \in \mathbb{R}_{t_0}^+, \tag{2b}$$

and the consistent initial condition (IC) according to

$$x(z,t_0) = x_0(z) \quad z \in [0,L]. \tag{3}$$

Assumption 1. For (1)–(3) it is assumed that:

(i) $a(z,t,x)$ is $\mathcal{C}^2$ in z, t, x and periodic in t with the period T. Moreover, let $0 < a_l \leq a(z,t,x) \leq a_r < \infty$ with positive constants a_l and a_r.
(ii) $f(z,t,x,\partial_z x)$ is $\mathcal{C}^2$ in z, t, x, $\partial_z x$ and periodic in t with the same period T.
(iii) $p(t)$ and $q(t)$ are $\mathcal{C}^2$-functions and T-periodic.

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For results on the existence and uniqueness of classical solutions for such PDEs, the interested reader is referred to, e.g., Brunovský et al. [1992] and the references therein.

The system output restricted to the boundary $z = 0$ is given by

$$y(t) = \partial_z x(0,t), \quad t \in \mathbb{R}^+_{t_0}. \tag{4}$$

Remark 2. It is required that the T-periodic function $p(t) \neq 0$ for any period T to guarantee the observability of the system (1)–(4). This avoids that (4) vanishes identically according to (2a). However, the proposed observer design approach is not restricted to systems with BCs (2) and is in principle identical for other types of BCs.

2.1 Extended Luenberger observer

Extending the results from Meurer [2013] and Jadachowski et al. [2013], a distributed-parameter Luenberger-type state observer in the observer state $\hat{x}(z,t)$ is set-up as

$$\begin{aligned}\partial_t \hat{x}(z,t) &= a(z,t,\hat{x})\partial_z^2 \hat{x}(z,t) + f(z,t,\hat{x},\partial_z \hat{x}) \\ &\quad + l(z,t)\tilde{y}(t), \quad (z,t) \in (0,L) \times \mathbb{R}^+_{t_0}\end{aligned} \tag{5}$$

with $\tilde{y}(t) = y(t) - \hat{y}(t)$. The BCs with an IC are chosen as

$$\partial_z \hat{x}(0,t) + p(t)\hat{x}(0,t) = l_0(t)\tilde{y}(t), \quad t \in \mathbb{R}^+_{t_0} \tag{6a}$$

$$\partial_z \hat{x}(L,t) + q(t)\hat{x}(L,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{6b}$$

$$\hat{x}(z,t_0) = \hat{x}_0(z), \quad z \in [0,L]. \tag{6c}$$

The output estimate evaluated with $\hat{x}(z,t)$ is given by

$$\hat{y}(t) = \partial_z \hat{x}(0,t), \tag{7}$$

while $l(z,t)$ and $l_0(t)$ in (5) and (6a) denote the observer gains to be determined. Introducing the observer error $e(z,t) = x(z,t) - \hat{x}(z,t)$, it follows directly that the observer error dynamics satisfies

$$\begin{aligned}\partial_t e(z,t) &= F(z,t,x,\partial_z x,\partial_z^2 x) - F(z,t,\hat{x},\partial_z \hat{x},\partial_z^2 \hat{x}) \\ &\quad - l(z,t)\partial_z e(0,t)\end{aligned} \tag{8}$$

defined on $(z,t) \in (0,L) \times \mathbb{R}^+_{t_0}$ with $F(z,t,x,\partial_z x,\partial_z^2 x) = a(z,t,x)\partial_z^2 x(z,t) + f(z,t,x,\partial_z x)$. The BCs follow as

$$\big(1 + l_0(t)\big)\partial_z e(0,t) + p(t)e(0,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{9a}$$

$$\partial_z e(L,t) + q(t)e(L,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{9b}$$

with the IC

$$e(z,t_0) = e_0(z), \quad z \in [0,L]. \tag{10}$$

With this, the determination the observer gains $l(z,t)$ and $l_0(t)$ is based on the linearized observer error dynamics with respect to the current state estimate $\hat{x}(z,t)$ and its spatial derivatives. Considering that $x(z,t) = \hat{x}(z,t) + e(z,t)$ this yields

$$\begin{aligned}&F(z,t,x,\partial_z x,\partial_z^2 x) = \partial_{x_{zz}} F(z,t,\hat{x},\partial_z \hat{x},\partial_z^2 \hat{x})\partial_z^2 e(z,t) \\ &+ \partial_{x_z} F(z,t,\hat{x},\partial_z \hat{x},\partial_z^2 \hat{x})\partial_z e(z,t) \\ &+ \partial_x F(z,t,\hat{x},\partial_z \hat{x},\partial_z^2 \hat{x})e(z,t) + F(z,t,\hat{x},\partial_z \hat{x},\partial_z^2 \hat{x}).\end{aligned} \tag{11}$$

Substituting (11) into (8) results in the linearized observer error dynamics

$$\begin{aligned}\partial_t \tilde{e}(z,t) &= \tilde{a}(z,t)\partial_z^2 \tilde{e}(z,t) + \tilde{b}(z,t)\partial_z \tilde{e}(z,t) \\ &\quad + \tilde{c}(z,t)\tilde{e}(z,t) - l(z,t)\partial_z \tilde{e}(0,t)\end{aligned} \tag{12}$$

with

$$\begin{aligned}&\tilde{a}(z,t) = a(z,t,\hat{x}) \qquad \tilde{b}(z,t) = \partial_{x_z} f(z,t,\hat{x},\partial_z \hat{x}) \\ &\tilde{c}(z,t) = \partial_x a(z,t,\hat{x})\partial_z^2 \hat{x}(z,t) + \partial_x f(z,t,\hat{x},\partial_z \hat{x}).\end{aligned} \tag{13}$$

Obviously, the BCs and the IC remain unchanged, i.e.,

$$\big(1 + l_0(t)\big)\partial_z \tilde{e}(0,t) + p(t)\tilde{e}(0,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{14a}$$

$$\partial_z \tilde{e}(L,t) + q(t)\tilde{e}(L,t) = 0, \quad t \in \mathbb{R}^+_{t_0} \tag{14b}$$

$$\tilde{e}(z,t_0) = \tilde{e}_0(z), \quad z \in [0,L]. \tag{14c}$$

Note that contrary to Jadachowski et al. [2013] (12) is characterized by a spatially and time-varying diffusion parameter $\tilde{a}(z,t)$. This has to be explicitly taken into account in the subsequent design of the observer gains.

Remark 3. Although the presented linearization can be formally applied to every quasi-linear parabolic PDE, in general it is not ensured that the stability of the linearized observer error system (12)–(14) implies stability of the original nonlinear system (8)–(10). For the considered case of periodic quasi-linear parabolic PDEs, see Assumption 1, Brunovský et al. [1992] shows that the observer error $e(z,t)$ of (8)–(10) can be locally approximated by a classical solution of (12)–(14) for $l(z,t) = l_0(t) = 0$.

2.2 Transformation into a normalized form

For the determination of the observer gains $l(z,t)$ and $l_0(t)$, the linearized observer error dynamics is transformed into a simpler equivalent form by normalization of the diffusion parameter $\tilde{a}(z,t)$ and the spatial coordinate z to unity. For this, the coordinate transformation

$$\zeta = \bar{z}(z,t) := \frac{1}{\bar{z}_L(t)} \int_0^z \tilde{a}^{-\frac{1}{2}}(s,t)\mathrm{d}s \tag{15a}$$

$$\tau = \bar{t}(t) := \int_{t_0}^t \bar{z}_L^{-2}(s)\mathrm{d}s \tag{15b}$$

with $\bar{z}_L(t) = \int_0^L \tilde{a}^{-\frac{1}{2}}(s,t)\mathrm{d}s$ followed by the Hopf-Cole state transformation (see, e.g., Hopf [1950])

$$\breve{e}(\zeta,\tau) = \tilde{e}(z,t)\,\mathrm{e}^{\int_0^\zeta \chi(s,\tau)\mathrm{d}s} \tag{16a}$$

$$\begin{aligned}\chi(\zeta,\tau) &= \frac{\bar{z}_L^2(t)}{2}\big[\tilde{a}(z,t)\partial_z^2 \bar{z}(z,t) + \tilde{b}(z,t)\partial_z \bar{z}(z,t) \\ &\quad - \partial_t \bar{z}(z,t)\big]\end{aligned} \tag{16b}$$

evaluated for

$$z = \bar{z}^{-1}(\zeta,\tau), \qquad t = \bar{t}^{-1}(\tau) \tag{17}$$

are applied to (12)–(14). As a result, the following normalized diffusion-reaction PDE is obtained

$$\begin{aligned}\partial_\tau \breve{e}(\zeta,\tau) &= \partial_\zeta^2 \breve{e}(\zeta,\tau) + \breve{c}(\zeta,\tau)\breve{e}(\zeta,\tau) \\ &\quad - \breve{l}(\zeta,\tau)\left(\partial_\zeta \breve{e}(0,\tau) - \chi(0,\tau)\breve{e}(0,\tau)\right)\end{aligned} \tag{18}$$

defined on $(\zeta,\tau) \in (0,1) \times \mathbb{R}^+_0$, $\mathbb{R}^+_0 = \{\tau \in \mathbb{R} \mid \tau > 0\}$ with

$$\begin{aligned}&\big(1 + \breve{l}_0(\tau)\big)\partial_\zeta \breve{e}(0,\tau) \\ &+ \big(\breve{p}(\tau) - \breve{l}_0(\tau)\chi(0,\tau)\big)\breve{e}(0,\tau) = 0, \quad \tau \in \mathbb{R}^+_0\end{aligned} \tag{19a}$$

$$\partial_\zeta \breve{e}(1,\tau) + \breve{q}(\tau)\breve{e}(1,\tau) = 0, \quad \tau \in \mathbb{R}^+_0 \tag{19b}$$

$$\breve{e}(\zeta,0) = \breve{e}_0(\zeta), \quad \zeta \in [0,1]. \tag{19c}$$

In (18), the transformed reaction parameter is given by $\breve{c}(\zeta,\tau) = \int_0^\zeta \partial_\tau \chi(s,\tau)\mathrm{d}s - \chi^2(\zeta,\tau) - \partial_\zeta \chi(\zeta,\tau) + \bar{c}(\zeta,\tau)$ with $\bar{c}(\zeta,\tau) = \bar{z}_L^2(t)\tilde{c}(z,t)$ evaluated according to (17). With (17) the mapping of the observer gains yields

$$l(z,t) \mapsto \breve{l}(\zeta,\tau) := l(z,t)\frac{\bar{z}_L(t)}{\sqrt{\tilde{a}(0,t)}}\,\mathrm{e}^{\int_0^\zeta \chi(s,\tau)\mathrm{d}s} \tag{20a}$$

$$l_0(t) \mapsto \breve{l}_0(\tau) := l_0(t), \tag{20b}$$

while the boundary parameters are defined by

$$p(t) \mapsto \breve{p}(\tau) := p(t)\bar{z}_L(t)\sqrt{\tilde{a}(0,t)} - \chi(0,\tau) \tag{21a}$$

$$q(t) \mapsto \breve{q}(\tau) := q(t)\bar{z}_L(t)\sqrt{\tilde{a}(L,t)} - \chi(1,\tau). \tag{21b}$$

Subsequently, the backstepping method is considered for the computation of the observer gains $\breve{l}(\zeta,\tau)$ and $\breve{l}_0(\tau)$ to stabilize (18)–(19).

2.3 Stabilization by means of the backstepping method

The main idea of the backstepping-based observer design relies on the specification of the desired observer error decay by an appropriate selection and mapping to a desired target dynamics. Proceeding along the lines of the classical backstepping approach (Smyshlyaev and Krstic [2005]), the following steps are required for the observer synthesis in presence of spatially and time-varying parameters.

Selection of the target dynamics The desired dynamics for the linearized observer error $\breve{e}(\zeta,\tau)$ is chosen to mimic the behavior of the system

$$\partial_\tau w(\zeta,\tau) = \partial_\zeta^2 w(\zeta,\tau) - \mu(\tau)w(\zeta,\tau) \tag{22}$$

defined on $(\zeta,\tau) \in (0,1) \times \mathbb{R}_0^+$ with corresponding BCs

$$\partial_\zeta w(0,\tau) + p^w w(0,\tau) = 0, \quad \tau \in \mathbb{R}_0^+ \tag{23a}$$

$$\partial_\zeta w(1,\tau) + q^w w(1,\tau) = 0, \quad \tau \in \mathbb{R}_0^+ \tag{23b}$$

and the IC

$$w(\zeta,0) = w_0(\zeta), \quad \zeta \in [0,1]. \tag{24}$$

Thereby, a suitable choice of a design parameter $\mu(\tau)$ ensures the exponential stability of the target system. In Meurer and Kugi [2009], the exponential stability of the PDE (22)–(24) is proven in the L^2-norm, if the inequality $\mu(\tau) + \underline{\lambda} \geq \varepsilon$, $\forall \tau \geq 0$ is satisfied for some $\varepsilon > 0$. Here, $\underline{\lambda}$ is the smallest eigenvalue of the Sturm-Liouville problem $\partial_\zeta^2 w(\zeta,\tau) + \lambda w(\zeta,\tau) = 0$ with BCs (23). This implies that

$$\|w(\zeta,\tau)\|_{L^2} \leq \exp(-\varepsilon\tau)\|w_0(\zeta)\|_{L^2}. \tag{25}$$

In the following, the observer gains are determined by mapping the target dynamics to the linearized observer error PDE by means of the backstepping transformation.

Determination of the observer gains The main step of the backstepping approach is based on the use of the Volterra integral transformation

$$\breve{e}(\zeta,\tau) = w(\zeta,\tau) - \int_0^\zeta g(\zeta,s,\tau)w(s,\tau)\mathrm{d}s \tag{26}$$

with the integral kernel $g(\zeta,s,\tau)$ to map the target system (22)–(24) to the observer error dynamics (18)–(19). For this, expressions for the observer error (26) and its partial derivatives are substituted into (18)–(19). Thereby, differentiation of (26) with respect to ζ yields

$$\begin{aligned}\partial_\zeta \breve{e}(\zeta,\tau) &= \partial_\zeta w(\zeta,\tau) - g(\zeta,\zeta,\tau)w(\zeta,\tau)\\ &\quad - \int_0^\zeta \partial_\zeta g(\zeta,s,\tau)w(s,\tau)\mathrm{d}s\\ \partial_\zeta^2 \breve{e}(\zeta,\tau) &= \partial_\zeta^2 w(\zeta,\tau) - \mathrm{d}_\zeta g(\zeta,\zeta,\tau)w(\zeta,\tau)\\ &\quad - g(\zeta,\zeta,\tau)\partial_\zeta w(\zeta,\tau) - \partial_\zeta g(\zeta,\zeta,\tau)w(\zeta,\tau)\\ &\quad - \int_0^\zeta \partial_\zeta^2 g(\zeta,s,\tau)w(s,\tau))\mathrm{d}s\end{aligned}$$

with $\mathrm{d}_\zeta g(\zeta,\zeta,\tau) = [\partial_\zeta g(\zeta,s,\tau) + \partial_s g(\zeta,s,\tau)]|_{s=\zeta}$, while evaluating the derivative of (26) with respect to τ in view of (22) results after integration by parts in

$$\begin{aligned}&\partial_\tau \breve{e}(\zeta,\tau) = \partial_\zeta^2 w(\zeta,\tau) - \mu(\tau)w(\zeta,\tau)\\ &- \left[g(\zeta,s,\tau)\partial_s w(s,\tau) - \partial_s g(\zeta,s,\tau)w(s,\tau)\right]\Big|_{s=0}^{s=\zeta}\\ &- \int_0^\zeta \left[\partial_\tau g(\zeta,s,\tau) + \partial_s^2 g(\zeta,s,\tau) - \mu(\tau)g(\zeta,s,\tau)\right]w(s,\tau)\mathrm{d}s.\end{aligned}$$

After some intermediate calculations, the so-called kernel-PDE governing the evolution of the kernel $g(\zeta,s,\tau)$ can be deduced, i.e.,

$$\begin{aligned}\partial_\tau g(\zeta,s,\tau) = \partial_\zeta^2 g(\zeta,s,\tau) - \partial_s^2 g(\zeta,s,\tau)\\ + \gamma(\zeta,\tau)g(\zeta,s,\tau)\end{aligned} \tag{27a}$$

$$\mathrm{d}_\zeta g(\zeta,\zeta,\tau) = \frac{\gamma(\zeta,\tau)}{2} \tag{27b}$$

$$\partial_\zeta g(1,s,\tau) = -\breve{q}(\tau)g(1,s,\tau) \tag{27c}$$

$$g(1,1,\tau) = \breve{q}(\tau) - q^w \tag{27d}$$

$$g(\zeta,s,0) = g_0(\zeta,s) \tag{27e}$$

defined on $\Omega_g := \{(\zeta,s) \in \mathbb{R}^2 \mid s \in [0,1],\ \zeta \in [s,1]\}$ with $\gamma(\zeta,\tau) = \breve{c}(\zeta,\tau) + \mu(\tau)$ and $g_0(\zeta,s)$ being consistent with (27c) and (27d). Moreover, the observer gains $\breve{l}(\zeta,\tau)$ and $\breve{l}_0(\zeta)$ follow as

$$\breve{l}(\zeta,\tau) = -\frac{\partial_s g(\zeta,0,\tau) + p^w g(\zeta,0,\tau)}{\chi(0,\tau) + g(0,0,\tau) + p^w} \tag{28a}$$

$$\breve{l}_0(\tau) = \frac{\breve{p}(\tau) - g(0,0,\tau) - p^w}{\chi(0,\tau) + g(0,0,\tau) + p^w}. \tag{28b}$$

Herein, it is assumed that $\chi(0,\tau) + g(0,0,\tau) + p^w \neq 0$. Taking into account the inverse of (20) the observer gains in (z,t)-coordinates are given by

$$l(z,t) = \frac{\sqrt{\tilde{a}(0,t)}}{\bar{z}_L(t)}\left(\breve{l}(\zeta,\tau)\,\mathrm{e}^{-\int_0^\zeta \chi(s,\tau)\mathrm{d}s}\right)\Bigg|_{(17)} \tag{29a}$$

$$l_0(t) = \breve{l}_0(\tau)\big|_{\tau=\bar{t}(t)}. \tag{29b}$$

Since (28) is governed by the evolution of the integral kernel $g(\zeta,s,\tau)$, an explicit solution of the kernel-PDE (27) is necessary to determine $l(z,t)$ and $l_0(t)$.

2.4 Numerical computation of the observer gains

The numerical determination of the observer gains governed by the integral kernel $g(\zeta,s,\tau)$ according to (27) is computed by means of the solution procedure presented by Jadachowski et al. [2012]. In view of the considered case, a modification of the solution method is proposed providing an algebraic equation for the discretized integral kernel. Thereby, the remainder of this section relies on the existence of a classical solution to the kernel-PDE (27).

Assumption 4. There exists a bounded strong solution $g(\zeta,s,\tau) \in \mathcal{C}^2(\Omega_g) \cap \mathcal{C}^1(\mathbb{R}_0^+)$ to the kernel-PDE (27).

Remark 5. On the assumption that $\gamma(\zeta,\tau)$ is analytic in τ, the existence of a solution to the kernel-PDE (27) can be ensured, see Colton [1977]. In Meurer and Kugi [2009], a strong solution of the kernel-PDE is constructed by means of successive approximation of integral operators, presuming $\gamma(\zeta,\tau)$ to be $\mathcal{C}^0$ in ζ and of Gevrey class 2 in τ. At this point it is worth noting that Kannai [1990] shows the nonexistence of a solution of (27) if $\gamma(\zeta,\tau)$ is related to an infinitely differentiable positive function $\rho(\tau)$, whose time derivatives cannot be bounded by $|\partial^n\rho(\tau)/\partial\tau^n| \leq C^{n+2}n^{2n}$ for no values of C. However, the result of Kannai [1990] imposes a very strong restriction on the time behavior of $\gamma(\zeta,\tau)$ and thus does not contradict our assumption of the existence of a bounded strong solution to (27).

Kernel integral equation The subsequent solution procedure is based on the trapezoidal quadrature of the kernel integral formulation followed by a direct numerical time integration (NTI) of the resulting ordinary differential equation. Therefore, formal integration preceded by a coordinate transformation $\xi = 2-\zeta-s$, $\eta = \zeta - s$ with $g(\zeta,s,\tau) = \bar{g}(\xi,\eta,\tau)$ is applied to (27). This yields an implicit integral equation of the PDE (27), i.e.,

$$\bar{g}(\xi,\eta,\tau) = \mathcal{A}(\xi,\eta,\tau) + \int_\eta^\xi \mathcal{B}(\beta,\eta,\tau)\mathrm{d}\beta + \int_0^\eta \mathcal{C}(\beta,\tau)\mathrm{d}\beta \tag{30}$$

with $\mathcal{A}(\xi,\eta,\tau) = -\frac{1}{2}\int_0^\eta \gamma\big(1-\frac{\beta}{2},\tau\big)\mathrm{d}\beta - \frac{1}{4}\int_\eta^\xi \gamma\big(1-\frac{\beta}{2},\tau\big)\mathrm{d}\beta + \check{q}(\tau) - q^w$, $\mathcal{B}(\beta,\eta,\tau) = \frac{1}{4}\int_0^\eta \big[-\partial_\tau\bar{g}(\beta,\alpha,\tau) + \gamma\big(1-\frac{\beta-\alpha}{2},\tau\big)\bar{g}(\beta,\alpha,\tau)\big]\mathrm{d}\alpha$ and $\mathcal{C}(\beta,\tau) = -\check{q}(\tau)\bar{g}(\beta,\beta,\tau) + \frac{1}{2}\int_0^\beta \big[-\partial_\tau\bar{g}(\beta,\alpha,\tau) + \gamma\big(1-\frac{\beta-\alpha}{2},\tau\big)\bar{g}(\beta,\alpha,\tau)\big]\mathrm{d}\alpha$, defined on $\Omega_{\bar{g}} := \{(\xi,\eta)\in\mathbb{R}^2 \mid \eta\in[0,1],\ \xi\in[\eta,2-\eta]\}$. Discretization of $\Omega_{\bar{g}}$ and suitable approximation of (30) by means of the composite trapezoidal rule yields the formulation in terms of an explicit-in-time ODE, see, Jadachowski et al. [2012]. However, a significant computational cost due to the continuous update of the extended linearization followed by successive evaluation of the transformation (15)–(17) motivates to apply a sample-and-hold solution approach to (30). This results in an algebraic formulation for the discretized kernel $\bar{g}(\xi,\eta,\tau)$ solved in every sampling interval providing an efficient solution procedure.

Sample-and-hold solution of (30) In the following, it is assumed that the output (4) and the linearization are updated only at $t_k = kT_a + t_0$, $k\in\mathbb{N}_0$ with a sampling time T_a. Consequently, the observer gains $l(z,t)$ and $l_0(t)$ are computed for $t = t_{k+1}$ depending on the previous result $\hat{x}_k(z,t) = \hat{x}(z,t_k)$ and the output $y_k = y(t_k)$ and are held constant for $t\in[t_{k+1},t_{k+2})$. In view of (13) evaluated at $t = t_k$, let $\tilde{a}_k(z) := \tilde{a}(z,t_k)$, $\tilde{b}_k(z) := \tilde{b}(z,t_k)$ and $\tilde{c}_k(z) := \tilde{c}(z,t_k)$. With this, (30) implies

$$\bar{g}_k(\xi,\eta) = \mathcal{A}_k(\xi,\eta) + \mathcal{B}_k(\xi,\eta) + \mathcal{C}_k(\eta) \tag{31}$$

with $\mathcal{A}_k(\xi,\eta) = \mathcal{A}(\xi,\eta,\tau_k)$, $\mathcal{B}_k(\xi,\eta) = \int_\eta^\xi \mathcal{B}(\beta,\eta,\tau_k)\mathrm{d}\beta$ and $\mathcal{C}_k(\eta) = \int_0^\eta \mathcal{C}(\beta,\tau_k)\mathrm{d}\beta$. Thereby, the time dependency of $\bar{g}(\xi,\eta,\tau)$ in each time interval $[\tau_k,\tau_{k+1})$ is approximately neglected, which allows to discard the differentiation with respect to τ in (30).

Remark 6. Note that (31) requires the reformulation of the coordinate transformation (15a) according to $\zeta = \bar{z}_k(z) := \bar{z}_{L,k}^{-1}\int_0^z \tilde{a}_k^{-\frac{1}{2}}(s)\mathrm{d}s$, while the time scaling (15b) is computed by means of the trapezoidal quadrature with $\bar{z}_{L,k} = \int_0^L \tilde{a}_k^{-\frac{1}{2}}(s)\mathrm{d}s$. Hence, the Hopf-Cole transformation can be evaluated at every sampling step t_k according to

$$\breve{e}(\zeta,\tau_k) = \breve{e}_k(\zeta) = \tilde{e}\big(\bar{z}_k^{-1}(\zeta),t_k\big)\,\mathrm{e}^{\int_0^\zeta \chi_k(s)\mathrm{d}s} \tag{32a}$$

$$\chi(\zeta,\tau_k) = \chi_k(\zeta) = \frac{\bar{z}_{L,k}^2}{2}\big[\tilde{a}_k(z)\partial_z^2\bar{z}_k(z) + \tilde{b}_k(z)\partial_z\bar{z}_k(z) - D_t\circ\bar{z}_k(z)\big]\big|_{z=\bar{z}_k^{-1}(\zeta)}. \tag{32b}$$

Here, $D_t\circ\bar{z}_k(z)$ denotes numerical differentiation with respect to time t obtained from a one-sided difference quotient (Stoer [2002]). Similarly, numerical differentiation is used for the computation of $\check{c}_k(\zeta) = \int_0^\zeta D_\tau\circ\chi_k(s)\mathrm{d}s - \chi_k^2(\zeta) - \partial_\zeta\chi_k(\zeta) + \bar{z}_{L,k}^2\tilde{c}_k\big(\bar{z}_k^{-1}(\zeta)\big)$. The inverse $\bar{z}_k^{-1}(\zeta)$ is computed by means of cubic interpolation.

Subsequently, trapezoidal quadrature is applied to (31) with spatially discretized kernel $\bar{g}_k^{i,j} = \bar{g}_k(\xi_i,\eta_j)$ and discretized function $\gamma_k^{i,j} = \gamma\big(1-\frac{\xi_i-\eta_j}{2},\tau_k\big)$. With $\delta = 1/(N_\delta-1)$ the (ξ,η)-domain is divided into equidistant intervals with $\xi_i = \delta(i-1)$, $i = 1,\ldots,2N_\delta-1$ and $\eta_j = \delta(j-1)$, $j = 1,\ldots,N_\delta$. This yields a pointwise approximation of the time discrete equation (31), i.e.,

$$\bar{g}_k^{i,j} = \bar{\mathcal{A}}_k^{i,j} + \bar{\mathcal{B}}_k^{i,j} \tag{33}$$

with $\bar{\mathcal{A}}_k^{i,j} = \mathcal{A}_k(\xi_i,\eta_j)$ and $\bar{\mathcal{B}}_k^{i,j} = \mathcal{B}_k(\xi_i,\eta_j) + \mathcal{C}_k(\eta_j)$ collected in Fig. 1, where $\varkappa_k^{i,j} = \gamma_k^{i,j}\bar{g}_k^{i,j}$ and $\sum_n^{*,m} = \sum_n^m$ if $n-m<1$, $\sum_n^{*,m} = 0$ if $n-m=1$ and $\sum_n^{*,m} = -\sum_{m+1}^{n-1}$ if $n-m>1$. An appropriate indexing of (33) with $\bar{\mathcal{B}}_k^n = \bar{\mathcal{B}}_k^{i,j}$ explicitly depending on $\bar{g}_k^{i,j}$, $j>1$, $\bar{\mathcal{A}}_k^n = \bar{\mathcal{A}}_k^{i,j}$, and $\tilde{\mathcal{A}}_k^n = \tilde{\mathcal{A}}_k^{i,j}$, where $\tilde{\mathcal{A}}_k^{i,j}$ combines the last line of $\bar{\mathcal{B}}_k^{i,j}$, allows to obtain an algebraic expression for the vector $\bar{\mathbf{g}}_k = [\bar{g}_k^n]_{n=1,\ldots,\mathcal{N}}$, $\mathcal{N} = (N_\delta-1)^2$, i.e.,

$$\bar{\mathbf{g}}_k = -\left(\mathbf{D}_k - \mathbf{I}_\mathcal{N}\right)^{-1}\mathbf{b}_k. \tag{34}$$

Here, $\mathbf{I}_\mathcal{N}$ denotes the identity matrix, $\mathbf{D}_k = [D_k^{n,m}]$ is given elementwise by $D_k^{n,m} = \partial\bar{\mathcal{B}}_k^n/\partial\bar{g}_k^m$ and $\mathbf{b}_k = [b_k^n]$ is defined as $b_k^n = \bar{\mathcal{A}}_k^n + \tilde{\mathcal{A}}_k^n$. By using (34) in view of $g_k(\zeta,s) = \bar{g}_k(2-\zeta-s,\zeta-s)$ for the determination of $\breve{l}_k(\zeta)$, $\breve{l}_k$ from (28), the observer gains $l_k(z)$, $l_{0,k}$ at every sampling step t_k follow as

$$l_k(z) = \frac{\sqrt{\tilde{a}_k(0)}}{\bar{z}_{L,k}}\breve{l}_k(\zeta)\,\mathrm{e}^{-\int_0^\zeta \chi_k(s)\mathrm{d}s} \qquad l_{0,k} = \breve{l}_{0,k}. \tag{35}$$

Remark 7. The direct NTI provides a significant reduction of computing time as outlined in Jadachowski et al. [2012], which is of major importance in view of the successive evaluation in every sampling interval.

3. CONVERGENCE OF THE OBSERVER ERROR

The investigation of the stability of the nonlinear observer error dynamics (8)–(10) is highly challenging and beyond the scope of this paper. For semi-linear PDEs with locally and uniformly Lipschitz continuous non-linearities $f(z,t,x,\partial_z x) \equiv f(z,t,x)$, the exponential convergence of the semi-linear observer error dynamics is presented in Meurer [2013]. In the following, the stability investigations are restricted to the linearized observer error dynamics (12)–(14), which, for periodic quasi-linear PDEs, is justified by the results of Brunovský et al. [1992], see also Remark 3.

The exponential stability of (12)–(14) can be analyzed by taking into account the inverse of (26), i.e.,

$$w(\zeta,\tau) = \breve{e}(\zeta,\tau) + \int_0^\zeta m(\zeta,s,\tau)\breve{e}(s,\tau)\mathrm{d}s \tag{36}$$

with the inverse kernel $m(\zeta,s,\tau)$ mapping (18)–(19) to (22)–(24). The resulting kernel-PDE for $m(\zeta,s,\tau)$ is similar to (27) with the coefficient $\gamma(\zeta,\tau)$ in (27a) replaced

$$
\begin{aligned}
\bar{\mathcal{A}}_k^{i,j} &= -\tfrac{\delta}{2}\Big[\tfrac{1}{2}\big(\gamma_{1,1}^k+\gamma_{j,1}^k\big)+\textstyle\sum_{\Bbbk=2}^{*,j-1}\gamma_{\Bbbk,1}^k\Big]-\tfrac{\delta}{4}\Big[\tfrac{1}{2}\big(\gamma_{j,1}^k+\gamma_{i,1}^k\big)+\textstyle\sum_{\Bbbk=j+1}^{*,i-1}\gamma_{\Bbbk,1}^k\Big]+\breve{q}_k-q^w\\
\bar{\mathcal{B}}_k^{i,j} &= \tfrac{\delta^2}{4}\Big[\tfrac{1}{4}\big(\varkappa_k^{j,j}+\varkappa_k^{i,j}\big)+\tfrac{1}{2}\textstyle\sum_{\Bbbk=2}^{*,j-1}\big(\varkappa_k^{j,\Bbbk}+\varkappa_k^{i,\Bbbk}\big)+\tfrac{1}{2}\sum_{\Bbbh=j+1}^{*,i-1}\varkappa_k^{\Bbbh,j}+\sum_{\Bbbh=j+1}^{i-1}\sum_{\Bbbk=2}^{*,j-1}\varkappa_k^{\Bbbh,\Bbbk}\Big]-\delta\breve{q}_k\Big[\tfrac{1}{2}\bar{g}_k^{j,j}+\textstyle\sum_{\Bbbk=2}^{*,j-1}\bar{g}_k^{\Bbbk,\Bbbk}\Big]\\
&\quad+\tfrac{\delta^2}{2}\Big[\tfrac{1}{4}\varkappa_k^{j,j}+\tfrac{1}{2}\textstyle\sum_{\Bbbk=2}^{*,j-1}\varkappa_k^{j,\Bbbk}+\tfrac{1}{2}\sum_{\Bbbh=2}^{*,j-1}\varkappa_k^{\Bbbh,\Bbbh}+\sum_{\Bbbh=2}^{*,j-1}\sum_{\Bbbk=2}^{*,\Bbbh-1}\varkappa_k^{\Bbbh,\Bbbk}\Big]\\
&\quad+\tfrac{\delta^2}{4}\Big[\tfrac{1}{4}\big(\varkappa_k^{j,1}+\varkappa_k^{i,1}\big)+\tfrac{1}{2}\textstyle\sum_{\Bbbh=j+1}^{*,i-1}\varkappa_k^{\Bbbh,1}\Big]+\tfrac{\delta^2}{2}\Big[\tfrac{1}{4}\varkappa_k^{j,1}+\tfrac{1}{2}\textstyle\sum_{\Bbbh=2}^{*,j-1}\varkappa_k^{\Bbbh,1}\Big]-\tfrac{\delta}{2}\breve{q}_k\bar{g}_k^{1,1}
\end{aligned}
$$

Fig. 1. Trapezoidal quadrature approximation of the integral operators $\bar{\mathcal{A}}_k^{i,j}=\mathcal{A}_k(\xi_i,\eta_j)$ and $\bar{\mathcal{B}}_k^{i,j}=\mathcal{B}_k(\xi_i,\eta_j)+\mathcal{C}_k(\eta_j)$.

by $-\gamma(s,\tau)$ and q^w substituted for the time-varying coefficient $\breve{q}(\tau)$ in (27c). Hence, in view of Remark 5 the assumption below is justified for $m(\zeta,s,\tau)$.

Assumption 8. There exists a bounded strong solution $m(\zeta,s,\tau)\in\mathcal{C}^2(\Omega_g)\cap\mathcal{C}^1(\mathbb{R}_0^+)$ to the integral kernel in (36).

With Assumptions 4 and 8 as well as the Cauchy-Schwarz inequality let us consider the following estimates,

$$
\begin{aligned}
\Big\|\int_0^\zeta g(\zeta,s,t)w(s,t)\mathrm{d}s\Big\|_{L^2}^2 &\le \sup_{\zeta,s,\tau} g^2(\zeta,s,\tau)\|w(\zeta,\tau)\|_{L^2}^2\\
\Big\|\int_0^\zeta m(\zeta,s,0)\breve{e}(s,0)\mathrm{d}s\Big\|_{L^2}^2 &\le \sup_{\zeta,s} m_0^2(\zeta,s)\|\breve{e}_0(\zeta)\|_{L^2}^2.
\end{aligned}
$$

In view of (36) and the above estimates, an upper bound of $\|w_0(\zeta)\|_{L^2}$ is given by

$$\|w_0(\zeta)\|_{L^2}\le C_m\,\|\breve{e}_0(\zeta)\|_{L^2} \tag{37}$$

with $C_m=1+\sqrt{\sup_{\zeta,s}m_0^2(\zeta,s)}$. Evaluating the L^2-norm of (26) taking into account the stability property of the target system according to (25), it follows that

$$\|\breve{e}(\zeta,\tau)\|_{L^2}\le C_gC_m\exp(-\varepsilon\tau)\|\breve{e}_0(\zeta)\|_{L^2} \tag{38}$$

with $C_g=1+\sqrt{\sup_{\zeta,s}g_0^2(\zeta,s,\tau)}$. With this, the exponential decay of the linearized observer error dynamics (18)–(19) in the (ζ,τ)-coordinates with the observer gains $\breve{l}(\zeta,\tau)$ and $\breve{l}_0(\tau)$ from (28) is guaranteed in the L^2-norm. Taking the L^2-norm of (16a) and changing the limit of integration results, together with Assumption 1(i), in

$$
\begin{aligned}
\|\breve{e}(\zeta,\tau)\|_{L^2}^2 &= \int_0^1\tilde{e}^2\big(\bar{z}^{-1}(\zeta,\tau),\bar{t}^{-1}(\tau)\big)\,\mathrm{e}^{2\int_0^\zeta\chi(s,\tau)\mathrm{d}s}\mathrm{d}\zeta\\
&\ge \mathrm{e}^{2\inf_{\zeta,\tau}\int_0^\zeta\chi(s,\tau)\mathrm{d}s}\int_0^L\tilde{e}^2(z,t)\frac{1}{\bar{z}_L(t)}\tilde{a}^{-\frac{1}{2}}(z,t)\mathrm{d}z\\
&\ge C_{\inf}\,\|\tilde{e}(z,t)\|_{L^2}^2
\end{aligned} \tag{39}
$$

with $C_{\inf}=\exp\big(2\inf_{\zeta,\tau}\int_0^\zeta\chi(s,\tau)\mathrm{d}s\big)\sqrt{a_l/a_r}/L$. Similar computation for the upper bound of $\|\breve{e}_0\|_{L^2}^2$ yields

$$\|\breve{e}_0(\zeta)\|_{L^2}^2\le C_{\sup}\,\|\tilde{e}_0(z)\|_{L^2}^2 \tag{40}$$

with $C_{\sup}=\exp\big(2\sup_\zeta\int_0^\zeta\chi(s,0)\mathrm{d}s\big)\sqrt{a_r/a_l}/L$. Substituting (39), (40) into (38) taking into account that $\exp(-2\varepsilon\tau)\le\exp(-2\tilde{\varepsilon}(t-t_0))$ with $\tilde{\varepsilon}=\varepsilon a_l/L$ brings along

$$\|\tilde{e}(z,t)\|_{L^2}\le C\exp(-\tilde{\varepsilon}(t-t_0))\,\|\tilde{e}_0(z)\|_{L^2} \tag{41}$$

with $C=C_gC_m\sqrt{C_{\sup}}/\sqrt{C_{\inf}}$. This guarantees that the linearized observer error dynamics (12)–(14) in the (z,t)-coordinates with the observer gains $l(z,t)$, $l_0(t)$ according to (29) decays exponentially over time t in the L^2-norm.

4. SIMULATION RESULTS

Motivated by the preceding results, the state estimation problem is considered for a T-periodic quasi-linear parabolic PDE (1)–(4) with non-linearities exemplarily given by

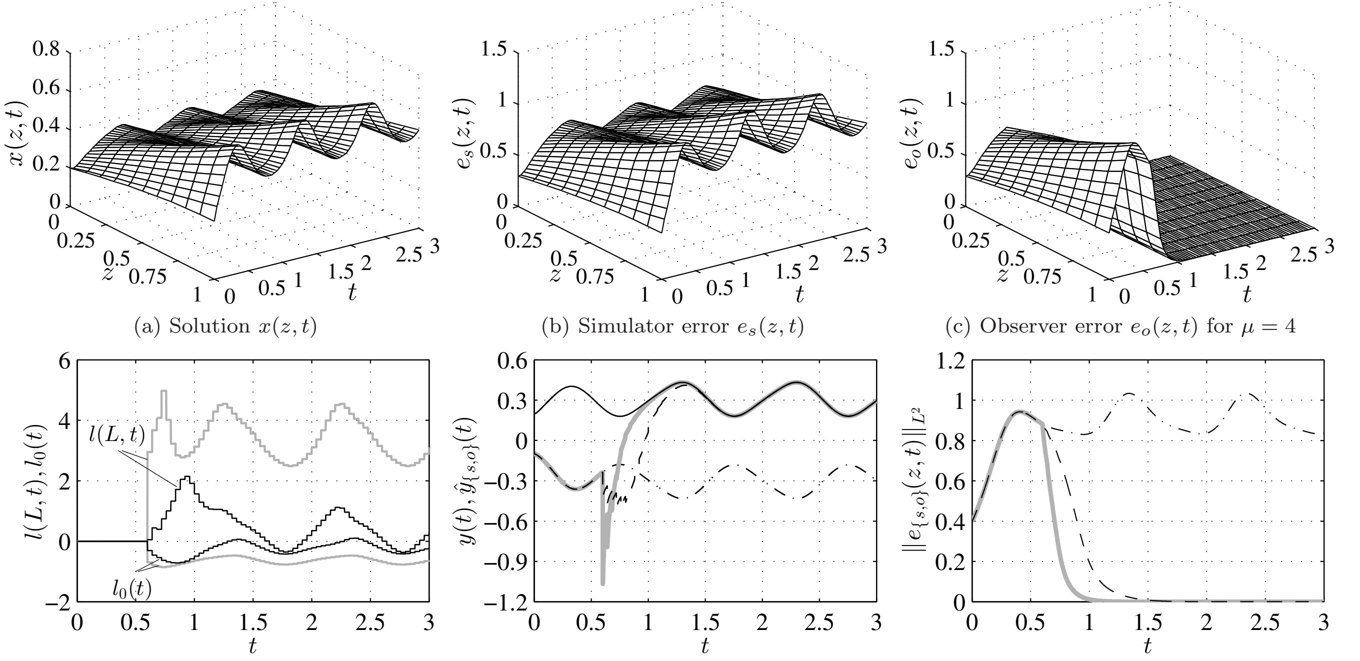
$$
\begin{aligned}
a(z,t,x) &= 4+2\cos(\pi x)+2z\sin(2\pi t)\\
f(z,t,x,\partial_z x) &= \sin(2\pi x)+10x^2(\partial_z x)^3\cos(2\pi t)
\end{aligned}
$$

and the time-varying parameters $p(t)=-1-0.5\sin(2\pi t)$, $q(t)=-0.4-0.3\sin(2\pi t)$ with $T=1$. Hence, both the BCs (2) are of mixed type. The IC (3) with $t_0=0$ is chosen as $x_0(z)=0.2+0.1(\sin(\pi/2z))$. The target system (22)–(24) is parameterized by $p^w=-1$, $q^w=1$ and $\mu(\tau)\equiv\mu\in\{1,4\}$. The numerical evaluation is carried out using the `pdepe`-algorithm of MATLAB. Thereby, the spatial dimension $z\in[0,L]$ with $L=1$ is divided into $N_\delta=10$ equidistant intervals of length $\Delta z=1/N_\delta=0.1$. This defines at the same time the dimension of the matrix $\mathbf{D}_k$ and vector $\mathbf{b}_k$ in (34).

Simulation results are presented in Fig. 2, where a simple simulator with $l(z,t)=l_0(t)=0$ is compared with the determined backstepping-based observer. Thereby, Fig. 2(a) presents the solution $x(z,t)$ of (1)–(3), while in Fig. 2(b) the evolution of the simulator error $e_s(z,t)$ with the IC $\hat{x}_0(z)=-0.5x_0(z)$ is shown. Due to the non-zero initial error and the lack of the output injection, the state evolution of the simulator diverges from the periodic equilibrium of the plant such that a significant oscillating observation error evolves. Fig. 2(c) shows estimation results of the proposed observer design approach with observer gains $l(z,t)$ and $l_0(t)$ according to (29) with $\mu=4$. The output injection is switched on for $t\ge0.6$ as addressed in Fig. 2(d), where the corresponding evolution of the observer gains $l(z,t)$ at $z=L$ and $l_0(t)$ with a sampling time $T_a=0.04$ is depicted for $\mu\in\{1,4\}$. Further results shown in Figs. 2(e)–(f) clearly confirm a good estimation performance of the proposed observer compared to the simple simulator. Fig. 2(e) depicts the system output (4) compared with the simulator $\hat{y}_s(t)$ and observer output $\hat{y}_o(t)$ for $\mu\in\{1,4\}$. The evolution of the observer error norm $\|e_o(z,t)\|_{L^2}$ and the simulator error norm $\|e_s(z,t)\|_{L^2}$ is addressed in Fig. 2(f). It becomes clear that higher values of the design parameter μ improve the estimation performance.

5. CONCLUSIONS

This contribution presents a backstepping-based solution of the state estimation problem for periodic quasi-linear parabolic PDEs. For this, an extended linearization is applied to the quasi-linear observer error dynamics assuming that the initial observer error is sufficiently small. The resulting linear time-varying observer error dynamics is



(a) Solution $x(z,t)$ (b) Simulator error $e_s(z,t)$ (c) Observer error $e_o(z,t)$ for $\mu = 4$

(d) Observer gains $l(L,t)$ and $l_0(t)$ for $\mu = 1$ (black) and $\mu = 4$ (gray) (e) Output $y(t)$ (solid), $\hat{y}_o(t)$ for $\mu = 1$ (dashed), $\mu = 4$ (gray), $\hat{y}_s(t)$ (dash-dotted) (f) Norm $\|e_o(z,t)\|_{L^2}$ for $\mu = 1$ (dashed), $\mu = 4$ (gray) and $\|e_s(z,t)\|_{L^2}$ (dash-dotted)

Fig. 2. Simulation results for the state estimation problem.

transformed into a normalized form, which serves as the basis for the design of the observer gains. A modification of the introduced numerical solution approach for the kernel-PDE results in an efficient computation of the observer gains. Convergence of the linearized observer error dynamics is verified analytically. The estimation performance of the proposed observer is illustrated by means of simulation results.

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