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Zeros of transfer functions in networked control with higher-order dynamics

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Abstract: This paper presents some results regarding location of transfer function zeros in general network control systems with dynamics of arbitrary order. The numerator polynomial of the transfer function is derived as a function of single agent dynamics and a Laplacian matrix. The results already known in literature are extended from single integrator and bidirectional formations to general dynamics and general interconnection structures. The location of zeros is related to poles of a slightly modified structure. Therefore, in some cases the zeros must follow the same root-locus-like rules as the poles do and they interlace.

1. INTRODUCTION

Networked control systems have become very intensive field of research. Numerous results for control of highway platoons, robot formations or squads of helicopters were published. It has a close relation to consensus, in which convergence to common value is desired. Formation control adds additional requirements of exogenous control input. Then, tracking of a desired value or formation leader's position becomes important. In this paper we consider transfer functions in general network control systems or formation control. The terms networked control systems and formation control are used interchangeably.

Transfer functions can be used in analysis of performance or norms of the networked system. One example is the paper by Li et al. (2011) where $\mathcal{H}_\infty$ norm of the system is derived using transfer functions. A platoon transfer function is used in a proof of harmonic instability of asymmetric vehicular platoon in the paper Herman et al. (2014). Even non-rational transfer function proves to be useful in reflection absorbing, as shown in Martinec et al. (2013). Other interesting results related to norms of formations are those of Zelazo and Mesbahi (2011). They indicate that $\mathcal{H}_2$ norm of the overall system does not depend directly on the algebraic connectivity, while $\mathcal{H}_\infty$ norm does. Algebraic connectivity is related to the second smallest eigenvalue of Laplacian as shown in Olfati-Saber et al. (2007).

In any formation control, stability is a crucial issue. For the ease of analysis, only linear and identical models are often considered in the formations. It was shown in the paper by Fax and Murray (2004) that the overall formation of identical vehicles is stable if and only if it is stable for all eigenvalues of Laplacian matrix. Nyquist criterion was used to test stability of SISO systems for all eigenvalues of Laplacian. It is also shown in Herman et al. (2013) that the eigenvalues of Laplacian matrix act as a gain in the feedback loop of individual vehicle model. If the exchange of information between any two vehicles is symmetric, then the poles of whole formation are determined to lie on a root-locus-like plot. Numerous results are available for stabilization of consensus networks. One approach, which is used in Fradkov and Junussov (2011); Zhang et al. (2011), is based on changing the gains when the graph topology changes.

Input-output behavior of a linear system is given by the poles and zeros of the transfer function matrix. While the location of poles of formation model is now well understood, much lower attention was paid to the location of zeros. Transfer functions and mainly results on location of zeros in consensus based algorithms were derived in the paper by Briegel et al. (2011). Only single integrator model of one vehicle and symmetric communication structure were considered. Conditions on relative order of the transfer function and requirements for minimal-phase system are given. Their results are extended in our paper to arbitrary open loop dynamics and arbitrary communication topology.

We study the location of zeros in a system where one agent acts as a controlling node (it has some known input) and the other node serves as an output. The object of interest is a transfer function from the controlling node to the observing node.

This paper is structured as follows. In the next section we present some necessary results from the graph theory. The third section presents the models used and block diagonalizes the system. In the fourth section the numerator of transfer function is derived and its relationship the graph Laplacian is shown. Fifth section discusses an example of a transfer function in an undirected graph. Finally, the paper is concluded and we shed some light on the future work.

Notation: We denote matrices with capital letters and particular element in matrix A is denoted as a_{ij}. Vectors are denoted with lowercase letters, their elements as g_i and constants by Greek letters. A canonical basis vector is $e_i = [0, \dots, 1, \dots, 0]^T$ with 1 on the ith position.

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2. GRAPH THEORY

The formation interconnection (sharing of information) can be viewed as a directed graph. The graph $\mathcal{G}$ has a vertex set $\mathcal{V}(\mathcal{G})$ and an arc set $\mathcal{E}(\mathcal{G})$. The arc $\epsilon(\nu_i, \nu_j)$ is oriented and means that the jth vehicle receives its information from the ith vehicle (the tail of the arc). A directed path $p_{i:j}$ from i to j of length $l(p_{i:j})$ is a sequence of vertices and arcs $\nu_1, \epsilon_1, \nu_2, \epsilon_2, \ldots \nu_{l+1}$, where each vertex and arc can be used only once. The length of the shortest path between i and j is called the distance $d_{i:j}$ of vertices. A cycle is a path with the first and last vertices identical.

An adjacency matrix is defined as $A = (a_{ij})$. Its entries a_{ij} are either zero if there is no arc between ν_i and ν_j or a positive number called weight if the arc is present. We also define the weight of the path as $w(p_{i:j})$ and it equals $w(p_{i:j}) = \prod_{k,m \in p_{i:j}} a_{km}$. It is the product of weights of all arcs in the path. Similarly, we can define the weight of a subset $\mathcal{G}^{'}$ of a graph as

$$w(\mathcal{G}^{'}) = \prod_{\epsilon(k,m) \in \mathcal{E}(\mathcal{G}^{'})} a_{km}. \tag{1}$$

A directed tree is a connected subset of a graph without cycles. A diverging directed tree always has a path from the particular node called root to every other node in the tree. There is no path going to the root in the diverging tree and each vertex in a tree has in-degree one. A forest is a set of mutually disjoint trees. A spanning forest is a forest on all vertices of the graph. We denote (following the notation in Chebotarev and Agaev (2002)) $\mathcal{F}_k^{i \to j}$ a set of all spanning forests with k arcs, which contains a tree with node j which diverge from the root i. The weight of this set is

$$w(\mathcal{F}_k^{i \to j}) = \sum w(\bar{\mathcal{F}}_k^{i \to j}), \tag{2}$$

where the sum is taken over all forests $\bar{\mathcal{F}}_k^{i \to j}$ in the set.

Let us denote as $D = d(\nu_i)$ the diagonal matrix of sums of weights of the arcs incident to the vertex i. Then the Laplacian matrix of a directed graph is defined as

$$L = D - A. \tag{3}$$

The Laplacian has the following properties

(1) the vector $\mathbf{1}$ of all ones is always a right eigenvector of L with a corresponding eigenvalue 0, i. e. $L\mathbf{1} = 0$. The reason for this is that the sum in rows equals zero by definition of L.
(2) the real parts of all eigenvalues λ_i are nonnegative. $Lv = \lambda_i v$ and $\Re\{\lambda\} \geq 0$.

Since the graph is generally directed, Laplacian matrix does not have to be symmetric and then the eigenvalues λ_i can be complex.

3. SYSTEM MODEL AND DIAGONALIZATION

We assume a formation consisting of N vehicles or agents. All are modelled as a SISO system, where a dynamic controller is used. Each vehicle is governed locally, so no central controller is used. The vehicle model is given as a transfer function

$$G(s) = \frac{b(s)}{a(s)}. \tag{4}$$

The output of the ith agent is usually position and is denoted as x_i. The agent model is fed from an output of the controller, which is again given in a transfer function form

$$R(s) = \frac{q(s)}{p(s)}. \tag{5}$$

As the plant and controller are connected in series, the open-loop model is given as

$$M(s) = G(s)R(s) = \frac{b(s)q(s)}{a(s)p(s)}. \tag{6}$$

The order of denominator of open loop is n, the order of numerator m. The open loop model can be written in a state space form with $x_i \in \mathbb{R}^{n \times 1}$ as a state variable as

$$\dot{x_i} = Ax_i + Bu_i \tag{7}$$
$$x_i = Cx_i. \tag{8}$$

The dimensions of the matrices are $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$.

The goal of the formation is to keep preset distances to the neighboring vehicles. The neighbor of a vehicle i is defined as a vehicle j, from which vehicle i can obtain information about the output, that is, there exists an arc $\epsilon(\nu_i, \nu_j)$ in the graph. The vehicle should drive the relative spacing error to zero. This input to the ith vehicle is defined as

$$\tilde{e}_i = \sum_{j \in \mathcal{N}(i)} (x_i - x_j - \delta_{\mathrm{ref}i,j}), \tag{9}$$

where $\mathcal{N}(i)$ denotes the set of neighbors of ith vehicle. This can be written in a more compact form as

$$\tilde{e} = Lx + u \tag{10}$$

with $\tilde{e} = [\tilde{e}_1, \ldots, \tilde{e}_N]^T$, $x = [x_1, \ldots, x_N]^T$ and $u = [\sum_{j \in \mathcal{N}(1)} \delta_{\mathrm{ref}1,j}, \ldots, \sum_{j \in \mathcal{N}(N)} \delta_{\mathrm{ref}N,j}]^T$.

3.1 Formation description

As the input to each vehicle is a function of the states of its neighbors, we combine (10) and (7) to obtain

$$\dot{x} = (I_N \otimes A)x - (L \otimes BC)x - (I_N \otimes B)u \tag{11}$$
$$x = (I_N \otimes C)x, \tag{12}$$

where $x = [x_1^T, \ldots, x_N^T]^T$ is a stacked vector of states of individual vehicles and $u = [u_1, \ldots, u_N]^T$. A Kronecker product is denoted as $\otimes$. The input u_i can be a sum of reference distances as above or any other value.

To obtain the poles, a diagonalization approach is proposed in the paper by Fax and Murray (2004). In the paper Schur decomposition is used, while here we use the transformation to a Jordan form. The state transform is given by

$$\hat{x} = (V \otimes I_n)^{-1} x \tag{13}$$

with $L = VJV^{-1}$. The matrix $V = [v_1, \ldots, v_N]$ consists of column eigenvectors or generalized eigenvectors v_i. Certain element of V is denoted as v_{ij}. Matrix J is a Jordan form of L. After this transformation, a block diagonal system is obtained

$$\dot{\hat{x}} = [I_N \otimes A - J \otimes BC]\hat{x} + (V^{-1} \otimes B)u \tag{14}$$
$$x = (V \otimes C)\hat{x}. \tag{15}$$

Let us have a closer look on a particular diagonal block. If it corresponds to a Jordan block of size one, then it has a form

$$\dot{\hat{x}}_i = [A - \lambda BC]\hat{x}_i + B(e_i V^{-1} u). \tag{16}$$

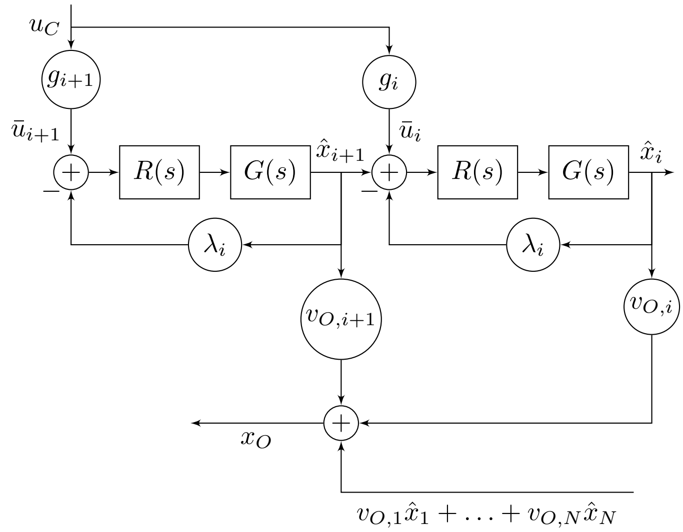

Fig. 1. One diagonal block for the case of Jordan block of size 2. The eigenvalue λ_i acts as a gain in the feedback. Only one closed loop is present if the diagonal subsystem corresponds only to Jordan block of size one.

The input to the ith block will be denoted $\bar{u}_i = e_i V^{-1} u$. The equation (16) describes an output feedback system with gain λ_i, so the diagonal block can be written in a transfer function form as

$$T_i(s) = \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)}. \quad (17)$$

If, on the other hand, the block in (16) corresponds to a Jordan block of size 2, then it can be written

$$\dot{\hat{x}}_i = (A - \lambda_i BC)\hat{x}_i + BC\hat{x}_{i+1} + B(e_i V^{-1} u) \quad (18)$$

$$\dot{\hat{x}}_{i+1} = (A - \lambda_i BC)\hat{x}_{i+1} + B(e_{i+1} V^{-1} u) \quad (19)$$

This easily generalizes to larger Jordan blocks. Clearly, this is a series connection of two identical blocks. The structure is shown in Fig. 1. This figure already shows the case with input at one controlling node and the output at the other one.

The output of the node is given in (15), which can be written as

$$x_i(s) = \sum_{j=1}^{N} v_{ij} T_j(s) \bar{u}_j(s). \quad (20)$$

3.2 Observing and controlling node

In this paper a transfer function between two selected nodes in the graph is considered. We study the propagation of signals from the input of a controlling vehicle, denoted C, to the output of an observing vehicle, denoted O. Therefore, there is only one input and one output in the formation. The input to whole formation is given as $u = [0, \ldots, 0, u_C, 0, \ldots, 0]^T$. Then the input to the diagonal blocks in (17) equals

$$\bar{u}_i = e_i^T V^{-1} e_C u_C = g_i u_C \quad (21)$$

with $g_i = e_i^T V^{-1} e_C$. Thus, the input u_C enters to all diagonal blocks through the gains g_i. The output x_O of the observing node then using (20) reads

$$x_O(s) = \left[\sum_{i=1}^{N} v_{Oi} g_i T_i(s) \right] u_C = T_{CO}(s) u_C. \quad (22)$$

4. ZEROS OF THE GENERAL TRANSFER FUNCTIONS

This section is devoted to determining zeros of transfer functions in general network control systems. The poles of the transfer function are determined by poles of the diagonal blocks and are easy to calculate. On the other hand, as the dynamic behavior of any formation is given by poles as well as zeros, we also need to find the locations of zeros of the transfer function in the formation. This section extends the results of Briegel et al. (2011) from single integrator dynamics to dynamics of arbitrary high order. If not explicitly stated, all further results are valid for both undirected and directed case.

4.1 Numerator of the transfer function

Let us denote the numerator of the open loop in (6) as $\phi(s)$ and denominator as $\psi(s)$. Then the transfer function can be obtained from (22) as

$$T_{CO}(s) = \frac{N(s)}{D(s)} = \sum_{i=1}^{N} g_i v_{Oi} \frac{b(s)q(s)}{a(s)p(s) + \lambda_i b(s)q(s)} \quad (23)$$
$$= \frac{\sum_{i=1}^{N} g_i v_{Oi} \phi(s) \prod_{j=1, j\neq i}^{N} [\psi(s) + \lambda_j \phi(s)]}{\prod_{i=1}^{N} [\psi(s) + \lambda_i \phi(s)]}.$$

The product in the numerator $N(s)$ can be written as (argument (s) is omitted)

$$\prod_{j=1, j\neq i}^{N} [\psi + \lambda_j \phi] = \quad (24)$$
$$\psi^{N-1} + \left[\psi^{N-2} \phi \sum_{j=1, j\neq i}^{N} \lambda_j \right] + \left[\psi^{N-3} \phi^2 \sum_{j=1, k=1, k\neq i\neq j}^{N} \lambda_j \lambda_k \right]$$
$$+ \ldots + \left[\psi^1 \phi^{N-2} \sum_{j=1, j\neq i}^{N} \prod_{k=1, k\neq i\neq j}^{N} \lambda_k \right] + \left[\phi^{N-1} \prod_{j=1, j\neq i}^{N} \lambda_j \right].$$

From this formula it is hard to infer any properties, so is must be reformulated. First we will state the following technical lemma.

Lemma 1. The sums of products of coefficients $g_i v_{O,i}$ and eigenvalues are related to graph Laplacian as

$$\sum_{i=1}^{N} g_i v_{Oi} \lambda_i^k = (L^k)_{OC} \quad (25)$$

Proof. The coefficients $g_i v_{Oi}$ can be written using (21) as

$$\sum_{i=1}^{N} g_i v_{O,i} \lambda_i^k = (e_O^T V) J^k (V^{-1} e_C)$$
$$= e_O^T L^k e_C = (L^k)_{OC} \quad (26)$$

This holds also for Jordan blocks larger than one in J. □

For further development we will need also the characteristic polynomial of $-L$, the negative of Laplacian. The polynomial is given as

$$\det(sI_N + L) = s^N + c_{N-1} s^{N-1} + \ldots + c_1 s + c_0. \quad (27)$$

The coefficient $c_0 = 0$ since there is always a zero eigenvalue of $-L$. If the zero eigenvalue is simple, the coefficient c_1 is equal to the product of all nonzero eigenvalues of $-L$ and c_{N-1} is sum of all its eigenvalues. The other terms c_i are sums of n-products of eigenvalues.

Now let us consider the term in the numerator $N(s)$ in (23) corresponding to $\psi^{N-1}(s)\phi(s)$. It equals $\psi^{N-1}(s)\phi(s)\sum_{i=1}^{N} g_i v_{Oi}$. The sum using Lemma 1 equals

$$\sum_{i=1}^{N} g_i v_{Oi} = \begin{cases} 1 \text{ for } O = C \\ 0 \text{ for } O \neq C \end{cases}. \tag{28}$$

Second, the terms with $\psi^{N-2}(s)\phi^2(s)$ are

$$\begin{aligned} &\psi^{N-2}(s)\phi^2(s)\sum_{i=1}^{N} g_i v_{Oi} \sum_{j=1, j\neq i}^{N} \lambda_j \\ &= \psi^{N-2}(s)\phi^2(s)\sum_{i=1}^{N} g_i v_{Oi}(c_{N-1} - \lambda_i) \\ &= \psi^{N-2}(s)\phi^2(s)\left(c_{N-1}L_{OC}^0 - L_{OC}^1\right). \end{aligned} \tag{29}$$

Similarly, all power terms $\psi^m(s)\phi^n(s)$ are functions of characteristic polynomial and a power of Laplacian. Last term is the one with $\phi^N(s)$ given as

$$\begin{aligned} &\phi^N(s)\sum_{i=}^{N} g_i v_{Oi} \prod_{j=1, j\neq i}^{N} \lambda_j \\ &= \phi^N(s)\left[c_1 L_{O,C}^0 - c_2 L_{OC}^1 + \ldots + L_{OC}^{N-1}\right] \end{aligned} \tag{30}$$

Let us denote the constants h_i as

$$h_0 = L_{OC}^0 \tag{31}$$
$$h_1 = c_{N-1}L_{OC}^0 - L_{OC}^1 \tag{32}$$
$$h_2 = c_{N-2}L_{OC}^0 - c_{N-1}L_{OC}^1 + L_{OC}^2 \tag{33}$$
$$\vdots$$
$$h_{N-1} = c_1 L_{OC}^0 - c_2 L_{OC}^1 + \ldots + L_{OC}^{N-1}. \tag{34}$$

Finally, the development can be summarized as follows: the numerator $N(s)$ in (23) equals

$$\begin{aligned} N(s) = \phi(s)\Big(&h_0\psi(s)^{N-1} + h_1\psi^{N-2}(s)\phi(s) \\ &+ h_2\psi^{N-3}(s)\phi^2(s) + \ldots + h_{N-1}\phi^{N-1}(s)\Big). \end{aligned} \tag{35}$$

4.2 Relative degree

To obtain the relative degree of the transfer function, we use the Lemma 3.1 from paper by Briegel et al. (2011). We provide here a different proof, as the original proof is valid only for commuting symmetric matrices and for unweighted graphs only.

Lemma 2. Let L be the Laplacian matrix of the graph. Then for $l \leq d_{i:j}$,

$$(-L^l)_{ij} = \begin{cases} 0, & \text{for } l < d_{i:j} \\ p, & \text{for } l = d_{i:j} \end{cases}, \tag{36}$$

where p is the sum of weights of the shortest paths between nodes i, j and $d_{i:j}$ denotes the distance of the nodes i and j.

Proof. We will use Proposition 8 from the paper by Chebotarev and Agaev (2002). There it is shown that

$$(-L)^m = \sum_{k=0}^{m} \alpha_k^{'} Q_{m-k}. \tag{37}$$

Matrices Q_{m-k} are matrices of in-forests of $\mathcal{G}$ with $m-k$ arcs. The (i,j)th element q_{ij}^{m-k} of Q_{m-k} denotes the weight $w(\mathcal{F}_{m-k}^{i\to j})$ of the set of all in-forests $\mathcal{F}_{m-k}^{i\to j}$ with $m-k$ arcs containing the node j and diverging from the root i. Thus, the minimal number of arcs for such forest to exist is the distance $d(p_{j:i})$. So, for $m < d(p_{j:i})$, (i,j)th element of all Q_{m-k} is zero and therefore $(-L)_{ij}^m$ is also zero. For $m = d(p_{j:i})$ it follows that $(-L^l)_{ij}$ is the sum of weights of all shortest paths. □

Theorem 3. Let r_o be the relative degree of an open loop of one node ($r_o = n - m$). Then the relative degree r of transfer function $T_{CO}(s)$ can be calculated as follows

$$r = (d_{C:O} + 1)r_o, \tag{38}$$

Proof. According to Lemma 2, the elements on positions $[O, C]$ in all powers of L lower than $d = d_{C:O}$ are zero. Therefore, all d leading terms in polynomial (35) are also zero. The numerator in (23) then has a form

$$\begin{aligned} N(s) = \phi^{1+d}(s)\Big(&p\psi^{N-1-d}(s) + h_{d+1}\psi^{N-2-d}(s)\phi(s) \\ &+ \ldots + h_{N-1}\phi^{N-1-d}(s)\Big) = \phi^{1+d}(s)P_\Delta(s) \end{aligned} \tag{39}$$

Since the polynomial $P_\Delta(s)$ has order $(N-1-d)n$, $\phi^{1+d}(s)$ has order $(1+d)m$ and the order of the denominator of the transfer function (23) is nN, the relative degree can be calculated as

$$r = Nn - (N-1-d)n - (1+d)m = (1+d)(n-m), \tag{40}$$

□

As the numerator of the open loop is present for $d+1$ times in (39), we have the following corollary.

Corollary 4. The transfer function $T_{CO}(s)$ has $d+1$ multiple zeros at the locations of the zeros of the open loop, i. e. roots of $b(s)q(s) = 0$.

4.3 Zeros which are not in the numerator of open loop

Except for the zeros of the open loop (6) there are also other zeros present in $T_{CO}(s)$. They are located in polynomial $P_\Delta(s)$, which is defined in (39) as

$$\begin{aligned} P_\Delta(s) = &p\psi^{N-1-d}(s) + h_{d+1}\psi^{N-2-d}(s)\phi(s) \\ &+ \ldots + h_{N-1}\phi^{N-1-d}(s). \end{aligned} \tag{41}$$

This can be factored into a product

$$P_\Delta(s) = p\prod_{i=1}^{N-1-d}\Big(a(s)p(s) + \gamma_i b(s)q(s)\Big), \tag{42}$$

where $-\gamma_i$ are the roots of the polynomial

$$Q(s) = ps^{N-1-d} + h_{d+1}s^{N-2-d} + \ldots + h_{N-1}. \tag{43}$$

The equality (42) can be viewed as a product of characteristic polynomials of closed loops with feedback gain $\gamma_i, \quad i = 1, \ldots, N-1-d$. It is analogous to how poles of the transfer function were obtained in (17). The development can be summarized in the main theorem of the paper.

Theorem 5. The transfer function $T_{CO}(s)$ can be written as

$$T_{CO}(s) = \frac{p[b(s)q(s)]^{1+d}\prod_{i=1}^{N-1-d}\Big(a(s)p(s) + \gamma_i b(s)q(s)\Big)}{\prod_{i=1}^{N}\Big(a(s)p(s) + \lambda_i b(s)q(s)\Big)}. \tag{44}$$

The locations of zeros were therefore related to the open-loop model and Laplacian matrix only. Our results are similar to the results for poles in Fax and Murray (2004). Clearly, if some mode is unobservable or uncontrollable, its characteristic polynomial must appear both in numerator and denominator.

4.4 Extensions from single integrator case

Based on the following proposition, we can easily extend many known results from single integrator dynamics to arbitrary agent dynamics.

Proposition 6. The polynomial $Q(s)$ defined in (43) is equal to the (O, C) cofactor of the matrix $(sI + L)$, i. e.

$$Q(s) = \mathrm{adj}(sI_N + L)_{(OC)} \tag{45}$$

The proof is given in Corollary 4 in the paper by Chebotarev and Agaev (2002). The coefficients there are identical to those in (31)-(34). In that paper it is shown that the coefficients h_i are equal to weight of the set of in-forests $\mathcal{F}_i^{C\to O}$ with i arcs, where the node C is the root of a tree and the node O lies in a tree diverging from the root. More on this can be found in Matrix-Forest Theorem in Agaev and Chebotarev (2000).

As a consequence, the roots of $Q(s)$ are identical to the zeros in the numerator of transfer functions from C to O in a model with single integrator dynamics. Therefore, all results regarding locations of zeros discussed in Briegel et al. (2011) are valid here. On the other hand, for higher order dynamics γ_i is not a location for zero, but the gain in the closed loop in (42).

Let us denote $\bar{L}^k_{(i:j)}$ a matrix, which is obtained from L by deleting rows and columns corresponding to the vertices on the kth path from vertex i to j.

Lemma 7. If the controlling and observing nodes are identical, $C = O$, the roots of polynomial Q are equal to the eigenvalues of $\bar{L}^1_{(C:C)}$.

Proof. Following the approach in Theorem 3.8 in Briegel et al. (2011), the numerator polynomial of the transfer function is given as $e_C^T\mathrm{adj}(sI_N+L)e_C$. This is equal to the (C, C) cofactor of L, that is characteristic equation with the form $(sI_{N-1} + \bar{L}^1_{(C:C)})$. Roots of this characteristic equation are roots of $Q(s)$. □

For undirected graphs, we have the following corollary.

Corollary 8. The poles and zeros of the transfer function $T_{C=O}(s)$ in undirected graph interlace each other on the root-locus-like curve, i. e. their order is (by increasing gain of the root-locus-like plot) $p_n, z_{n-1}, p_{n-1}, \ldots, z_1, p_1$.

Proof. The interlacing theorem 4.3.15 from Horn and Johnson (1990) holds here since the matrix for zeros of transfer functions was obtained by deleting principal rows and columns. Therefore the roots γ interlace with λ. Both must be real and as they act as gains in the closed-loop, we can use a root-locus-like explanation of their locations (see Herman et al. (2013)). The gains are interlacing, so also the locations of poles and zeros will be. □

Lemma 9. The multiplication factor p in the numerator is given as the sum of weights of shortest paths from the controlling node to the observing node.

Proof. This follows from theorem 3.4 in Briegel et al. (2011) and application of Proposition 6.

We remark that, unlike the case of undirected interaction discussed in the paper by Briegel et al. (2011), the transfer function can have a zero at the origin, provided the zero eigenvalue of the denominator is unobservable or uncontrollable from the other node.

4.5 Relations of minors and deleting rows

Now we state two theorems relating minors of Laplacian matrix for certain selection of the controlling and observing nodes to the deletion of principal rows and columns of $-L$.

Theorem 10. If there is only one path between controlling node and observing node, the roots of the polynomial $Q(s)$ are obtained as eigenvalues of $-\bar{L}^1_{(C:O)}$, which is obtained from $-L$ by deleting the columns and rows corresponding to the vertices on the path. That is

$$Q(s) = w(p_{C:O}) \det\left(sI_{N-d-1} + \bar{L}^1_{(C:O)}\right). \tag{46}$$

The second theorem is partly an extension of the previous one:

Theorem 11. If there are multiple paths from the controlling to observing node, then the numerator characteristic polynomial $Q(s)$ is a sum of characteristic polynomials of $-\bar{L}_i$ corresponding to all of the paths, i. e.

$$Q(s) = \sum_{i=1}^{P(\mathcal{G})_{C,O}} w_i(p_{C:O}) \det(sI + \bar{L}^i_{(C:O)}), \tag{47}$$

where $P(\mathcal{G})_{C,O}$ denotes the number of paths from C to O.

The importance of these two theorems can be seen especially when the communication graph is undirected. Their proofs are not shown due to the lack of space.

Corollary 12. If L is a symmetric matrix and conditions for Theorem 10 hold, the zeros interlace with poles on the root-locus-like plot.

The proof is similar to the proof of Corollary 8.

5. EXAMPLE

Consider a symmetric graph with five nodes shown in Fig. 2. The open-loop model is

$$M(s) = \frac{s+1}{s^2}. \tag{48}$$

Let us choose as the controlling node $C = 1$ and observing node $O = 3$. The transfer function is

$$T_{13}(s) = \frac{(s+1)^3 \prod_{i=1}^{2}(s^2 + \gamma_i s + \gamma_i)}{\prod_{i=1}^{5}(s^2 + \lambda_i s + \lambda_i)}. \tag{49}$$

Clearly, the terms in both numerator and denominator products have the structure of $a(s)p(s) + \lambda b(s)q(s)$, as indicated by (44). Moreover, since the distance between

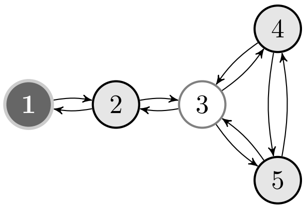

Fig. 2. Undirected graph used in the example

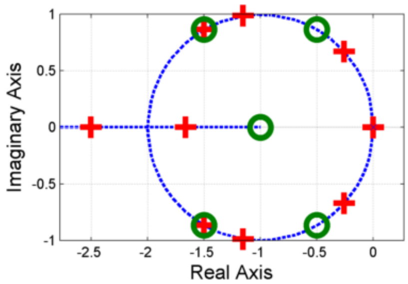

Fig. 3. Poles (crosses) and zeros (circles) of the graph in Fig. 2. The poles and zeros lie on the root-locus-like plot (dashed line).

nodes 1 and 3 is 2, there is also $(s+1)^{2+1}$ in the numerator, as follows from Lemma 4. The gains $\lambda_i = [4.17, 3.00, 2.31, 0.51, 0]$ can also be obtained as eigenvalues of Laplacian matrix of the graph

$$L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 3 & -1 & -1 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}. \quad (50)$$

The gains $\gamma = [3.00, 1]$ in the numerator can be obtained as negative of roots of polynomial $Q(s)$, given in (43), which in this case has a form

$$Q(s) = s^2 + 4s + 3. \quad (51)$$

Since there is only one path between C and O, we can use the Theorem 10 to calculate the characteristic polynomial in the numerator. It equals the determinant of a matrix $\bar{L}_{(1,2,3)}$, obtained from L by deleting rows and columns with indices $1, 2, 3$. The polynomial is given as

$$Q(s) = \det\left(sI_2 + \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\right) = s^2 + 4s + 3. \quad (52)$$

As the both L and $\bar{L}$ are symmetric matrices, their eigenvalues are real and interlace. The poles and zeros must therefore interlace on the root-locus like plot, as shown in Fig. 3. We remark that such transfer function is uncontrollable and unobservable, since there is a pole-zero cancellation of the poles $s^2 + 3s + 3$.

6. CONCLUSION

In this paper we considered transfer functions between two nodes in a formation or in general in network control systems. Both denominator and numerator polynomials are derived in a form of product of closed loops with non-unit feedback gain. These gains are in the denominator polynomial identical to the eigenvalues of the Laplacian matrix.

The gains in the numerator of the transfer function are related to the Laplacian matrix as well, but the relation is more complicated. Beside closed-loop-like polynomials, there is also open-loop numerator present in the transfer function between two nodes. Interlacing of poles and zeros was proved for undirected graphs with only one path between controlling and observing node.

As a future work, the main challenge remains to prove whether or not the zeros are minimal-phase if all the poles are stable. From simulations it seems they are and perhaps theorems 10 and 11 can be used in the proof.

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