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Characterization of power systems near their stability boundary by Lyapunov direct method ⋆

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Abstract: We propose a new method for the small-signal stability analysis of power systems based on the spectral decomposition of a square H_2 norm of the transfer function. Compared with the dynamics of H_2 and H_∞ norms of the transfer functions, the analysis of the behavior of individual eigen-components allows the earlier identification of the pre-fault condition occurrence. Since each eigen-component is associated with a particular eigenvector, the potential sources of instability can easily be localized and tracked in real time. An important class of systems operating under the pre-fault conditions near the boundary of stability is considered. We demonstrate that in such cases several ill-stable modes can increase the system energy up to a critical level much earlier due to their synergetic effect. In particular, an ill-stable low-frequency mode can act as a catalyst increasing the energy in the system. An illustrative test for the stability analysis of a real small power grid at Russky Island is provided.

Keywords: power systems; Lyapunov direct method; small-signal stability analysis; sub-Gramian approach

1. INTRODUCTION

The small-signal stability analysis continues to be among major problems of the control theory (Lyapunov (1947); Andreyev (1976); Kwakernaak, Sivan (1991); Polyak, Shcherbakov (2002); Boyd *et al.* (1994)). It is of supreme theoretical and practical interest in electrical engineering, aerospace technology and power industry (Vostrikov *et al.* (2006)). For example, the simplified model of complex power grid can be composed of large number of the oscillatory systems, representing elastically connected generator groups (Kundur (1994)). The oscillatory systems have the resonance frequencies, corresponding to electromechanical oscillations of generator groups. The interaction between the ill-stable oscillatory modes in some circuits leads to the development of instabilities (Martins (1997)). The loss of a power grid stability leads to a voltage collapse and cascading failures (CIEE (2010)). In most cases of grid stability analysis, the linearized grid models for the normal and pre-fault conditions are used (Pavella *et al.* (2000)). In this paper we consider the ill-stable continuous linear dynamical systems, i.e. the stable systems with one or more eigenvalues having small negative real parts (CIEE (2010)). The classical approach to the stability analysis of power system is to investigate a system of equations, representing the dynamic behavior of this system as a whole. This general approach adequate and efficient for the stability analysis of small and medium-size power systems meets with difficulties being applied to the large-scale power systems. The modified Arnoldi method (Arnoldi (1951); Kundur *et al.* (1990)) and the matrix sign function technique (Misrikhanov, Ryabchenko (2008)) are the efficient algorithms for computing the ill-stable eigenvalues. Another approach employs the computation of the dominant pole spectrum of a power system (Martins (1997)).

In this paper we propose a new method for the small-signal stability analysis of the power systems based on the spectral decomposition of a square H_2 norm of the transfer function. We analyze the dynamic behavior of individual eigen-components. The proposed method can be considered as a special case of a more general approach of Gramians and sub-Gramians (Yadykin (2010); Yadykin, Galyaev (2013)) for solving the matrix differential and algebraic Lyapunov equations, based on the spectral decomposition of the Lyapunov integral, the Laplace transform and the expansion of the matrix resolvent of the dynamical system. However, the calculation of the Faddeev matrices required for obtaining sub-Gramians is a very ill-defined operation for the large matrices. Therefore, the scalar quadratic forms formed by the coefficients of a transfer function numerator are more convenient for calculation and analysis than the quadratic forms of the Faddeev matrices. Hence, the proposed method is more practical for the stability analysis of the large-scale dynamical systems. As a special case we analyze the behavior of power systems near their stability boundary and derive the asymptotic expressions for the eigen-components of a square H_2 norm of the transfer function. Finally in our paper we illustrate how the

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proposed method can be applied to the stability analysis of a real small power grid at Russky Island (Grobovoy *et al.* (2013)).

2. PROBLEM STATEMENT

Let us consider a mathematical model of the power system defined by a nonlinear algebro-differential system of equations

$$\begin{array}{c}\dot{x}\left(t\right)=f\left(x,u,t\right),\ x\left(t_0\right)=0\ ,\\ M\left(x,t\right)x\left(t\right)=N\left(x,t\right)u\left(t\right).\end{array} \tag{1}$$

The linearized model of this system about the equilibrium point can be represented as a linear algebro-differential system of equations

$$\begin{array}{c}\dot{x}\left(t\right)=A_1x\left(t\right)+Bu\left(t\right),\ x\left(t_0\right)=0\ ,\\ Mx\left(t\right)=Nu\left(t\right).\end{array} \tag{2}$$

For a nonsingular matrix N this system can be written as

$$\begin{array}{c}\dot{x}\left(t\right)=Ax\left(t\right),\ x\left(t_0\right)=0\ ,\\ A=A_1+BN^{-1}M.\end{array} \tag{3}$$

Let us consider a fully controllable and observable continuous linear time-invariant system with one input and one output defined by real matrices $A_{[n\times n]}, B_{[n\times 1]}, C_{[1\times n]}$

$$\begin{array}{c}\dot{x}=Ax+Bu,\ x\left(0\right)=0\ ,\\ y=Cx,\ x\in R^n,\ u\in R^1,\ y\in R^1.\end{array} \tag{4}$$

The finite and infinite controllability Gramians of this system in the time domain are defined (Talbot (1959); Hanzon, Peeters (1996)) as

$$\begin{array}{c}P^C\left(t\right)=\displaystyle\int_0^t e^{A\tau}BB^Te^{A^T\tau}d\tau\ ,\\ P^C\left(\infty\right)=\displaystyle\int_0^\infty e^{A\tau}BB^Te^{A^T\tau}d\tau\ .\end{array} \tag{5}$$

Direct substitution reveals that these Gramians are the solutions of the differential and algebraic Lyapunov equations (Antoulas (2005))

$$\frac{dP\left(t\right)}{dt}=AP\left(t\right)+P\left(t\right)A^T+BB^T,\ P\left(0\right)=0_{[n\times n]}\ , \tag{6}$$

$$AP\left(\infty\right)+P\left(\infty\right)A^T+BB^T=0\ . \tag{7}$$

A degree of system (4) stability can be defined as

$$d=\mid\max_i\{Re(\lambda_i)\}\mid,\ i=1,2,\ldots,n\ , \tag{8}$$

where λ_i are the system eigenvalues.

In this paper we consider two problems. The first problem is to find the degree of stability (8) of the system (4) with a Hurwitz matrix A. A straightforward calculation of eigenvalues for the matrix A could be a challenging problem for high-order systems. We examine another approach, that one of solving algebraic matrix Lyapunov equations

$$\left(A+dI\right)^TV+V\left(A+dI\right)=-BB^T \tag{9}$$

with a positive real parameter d (Ahmetzyanov *et al.* (2012)). It is well known that a system (4) has a degree of stability exceeding d if and only if a positive definite solution V of the system (9) exists for any positive definite matrix BB^T (Andreyev (1976)). Therefore the degree of stability can be found by solving the system of equations (9) at increasing d until its positive definite solution V ceases to exist.

The second problem is to obtain a stability index suitable for the power systems, which estimates the contribution of the individual ill-stable eigenmodes to the risk of stability loss. This contribution can be determined as a corresponding term in the spectral expansion of the Frobenius norm of the system transfer function. This representation allows the identification of the most dangerous modes in terms of the greatest contributions to the asymptotic variation of the system energy over an infinite time interval. Such modes will constitute the major part of the transfer function Frobenius norm. We are looking for the eigenmodes decomposition based on a solution of the corresponding differential or algebraic Lyapunov equation. Note that finding of the degree of stability d can be considered as a special case of this problem.

3. SPECTRAL DECOMPOSITION OF H_2 NORMS OF TRANSFER FUNCTIONS

In order to analyze the behavior of the ill-stable dynamical system we obtain a spectral decomposition of the H_2 norm of its transfer function by matrix A eigenmodes and analyze the properties of this decomposition.

It is well known, that the matrix resolvent can be expanded (Faddeev, Faddeeva (1963); Hanzon, Peeters (1996)) as

$$\left(Is-A\right)^{-1}=\sum_{j=0}^{n-1}s^jA_j\times N^{-1}\left(s\right)\ , \tag{10}$$

where $N(s)=a_ns^n+\cdots+a_1s+a_0$ is the characteristic polynomial of the matrix A and the matrices $A_{j[n\times n]}$ are referred to as *Faddeev matrices* and can be found (Kwakernaak, Sivan (1991)) in the following form:

$$A_j=\sum_{i=j+1}^{n}a_iA^{i-j-1}\ . \tag{11}$$

Then the transfer function of the system can be written as

$$\begin{array}{c}W\left(s\right)=\dfrac{y\left(s\right)}{u\left(s\right)}=C(Is-A)^{-1}B=\\ =\dfrac{CA_{n-1}Bs^{n-1}+\ldots+CA_1Bs+CA_0B}{N\left(s\right)}=\\ =\dfrac{b_{n-1}s^{n-1}+\ldots+b_1s+b_0}{N\left(s\right)}\ .\end{array} \tag{12}$$

For convenience, we introduce the following notations:

$$W\left(s\right)\equiv\frac{b_{n-1}M\left(s\right)}{N\left(s\right)},\ \ \mathbf{b}^T\equiv\left[b_{n-1}\,b_{n-2}\,\ldots\,b_1\,b_0\right]\ . \tag{13}$$

The following theorem characterizes an eigenmode decomposition of the square H_2 norm of the system transfer function in the frequency domain.

Theorem 1. Let us consider a fully controllable and observable [stable] continuous linear dynamical system (4). Let A be a real square matrix with multiple eigenvalues s_δ with multiplicities $m_\delta, m_1+m_2+...+m_q=n$. Then, the square H_2 norm of the system transfer function (12) is given by

$$\|W(s)\|_2^2 = \sum_{\delta=1}^{q} G_\delta = \sum_{\delta=1}^{q} \sum_{k=1}^{m_\delta} \frac{(-1)^{m_\delta - k}}{(m_\delta - k)!(k-1)!}$$
$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \left[\frac{d^{k-1}}{ds^{k-1}} \left(\frac{(s-s_\delta)^{m_\delta} s^i}{N(s)} \right) \right]_{s=s_\delta} \times \tag{14}$$
$$\times \left[\frac{d^{m_\delta - k}}{ds^{m_\delta - k}} \left(\frac{s^j}{N(s)} \right) \right]_{s=-s_\delta} \times b_i b_j \ .$$

Proof. Follows from the Theorem 1 in (Yadykin (2010)).

Corollary. For a complex square matrix A the square H_2 norm of the system transfer function is given by a formula similar to (14) with the replacement of b_j by b_j^*.

If the matrix A spectrum contains only simple eigenvalues s_k, then an expression (14) for the square H_2 norm of transfer function takes the following form:

$$\|W\|_2^2 = \sum_{k=1}^{n} G_k,$$
$$G_k = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^i \, (-s_k)^j}{N'(s_k) N(-s_k)} b_i b_j \ , \tag{15}$$

where $N'(s)$ is the derivative of $N(s)$ and G_k represents an eigen-part of the square H_2 norm of the transfer function corresponding to the particular eigenvalue s_k.

Theorem 1 allows the square H_2 norm of the transfer function to be represented as a sum of the quadratic forms

$$\|W\|_2^2 = \sum_{k=1}^{n} G_k = \sum_{k=1}^{n} \frac{b(s_k) b(-s_k)}{N'(s_k) N(-s_k)}, \tag{16}$$
$$\text{where} \quad b(s) \equiv b_{n-1} s^{n-1} + \cdots + b_1 s + b_0 \ .$$

Each of this forms corresponds to a particular root of the system characteristic equation. The coefficients of the quadratic form corresponding to the eigenvalue s_k are given by a matrix

$$\tilde{G}_k = \left[\frac{s_k^i \, (-s_k)^j}{N'(s_k) N(-s_k)} \right]_{ij} , \quad G_k = \mathbf{b}^T \tilde{G}_k \mathbf{b} \ . \tag{17}$$

The basic properties of the eigen-parts G_k can be characterized by the following theorem.

Theorem 2. Let us examine a fully controllable and observable stable system (4) with a diagonalizable matrix A. Then the eigen-parts (15) of the square H_2 norm of the transfer function have the following properties.

(i) If the matrix A is real, then both the eigen-parts G_k corresponding to the real eigenvalues and the sum of eigen-parts $G_{k1} + G_{k1^*}$ corresponding to pairs of complex conjugate eigenvalues are real. Therefore, the square H_2 norm of the transfer function can be represented as a sum of real numbers:

$$\|W\|_2^2 = \sum_{k=1}^{l} G_k + \sum_{k1=1}^{m} (G_{k1} + G_{k1^*}) \tag{18}$$

(ii) If the matrix A is Hurwitz, then the eigen-parts G_k can be found as

$$G_k = -\, C A_{(k)} B B^T \left[s_k I + A^T \right]^{-1} C^T$$
$$A_{(k)} \equiv \mathbf{Res}\ \left((Is - A)^{-1}, s_k \right) = \sum_{i=0}^{n-1} \frac{s_k^i A_i}{N'(s_k)} \tag{19}$$

Proof. Follows from the Theorem 3 in (Yadykin, Galyaev (2013)).

The square H_2 norm of the transfer function $\|W\|_2^2$ can be interpreted as a measure of the output energy produced by a unit energy input. Then the absolute values of eigen-parts $|G_k|$ characterize the contribution of the eigenmodes to the total variation of the system output energy. Therefore the eigenmode decomposition (15) allows an identification of the most dangerous modes in terms of the greatest contributions to the total output energy of the system. Such modes will constitute the major part of the sum. Conversely, those modes not making a significant contribution can be regarded as posing no threat to a system stability. Since each eigen-part is associated with a particular eigenvector, the potential sources of the instabilities can easily be localized and tracked over time. The representation (15) has a particular importance for analyzing the behavior of a power system operating near its stability boundary.

4. ASYMPTOTIC BEHAVIOR OF TRANSFER FUNCTIONS NEAR THE STABILITY BOUNDARY

Let us examine the asymptotic behavior of the eigen-parts (15) when one or more eigenvalues approach the imaginary axis from the left. We call these eigenvalues ill-stable. Formally, ill-stable eigenvalues can be defined as eigenvalues with small negative real parts. The denominator polynomial in (15) takes the following form:

$$N'(s_k) = \prod_{i \neq k} (s_k - s_i) \ ,$$
$$N(-s_k) = -2 s_k \prod_{i \neq k} (-s_k - s_i) \ , \tag{20}$$
$$N'(s_k) N(-s_k) = -2 s_k \prod_{i \neq k} (s_i^2 - s_k^2), k = 1, 2, \ldots, n.$$

For the nominator polynomial in (16) we obtain:

$$b(s_k) = \prod_{j=1}^{n-1} (s_k - \beta_j) \ ,$$
$$b(-s_k) = \prod_{j=1}^{n-1} (-s_k - \beta_j) \ , \tag{21}$$
$$b(s_k) b(-s_k) = \prod_{j=1}^{n-1} (\beta_j^2 - s_k^2), \ k = 1, 2, \ldots, n \ .$$

where β_j is a j-th root of the nominator polynomial $b(s)$ of the transfer function (12).

Let us consider some specific cases when ill-stable eigenvalues approach the imaginary axis from the left.

Case A. There is one real ill-stable eigenvalue $s_k = -\alpha$ with $\alpha \to +0$. All the other eigenvalues are fixed. From (16, 20-21) we obtain:

$$\|W\|_2^2 \sim G_k \sim \frac{b_0^2}{2\alpha \prod_{i\neq k} s_i^2} \to \infty \ (\alpha \to +0) \ . \tag{22}$$

The square H_2 norm of the transfer function is directly proportional to the constant term b_0^2 of the nominator polynomial and is inversely proportional to the absolute value α of the ill-stable real eigenvalue.

Case B. There are two real ill-stable eigenvalues $s_k = -\alpha_1$ and $s_l = -\alpha_2$ with $\alpha_1 \to +0$ and $\alpha_2 \to +0$. All the other eigenvalues are fixed. In this case we obtain:

$$\|W\|_2^2 \sim G_k + G_l \sim \frac{b_0^2}{2\alpha_1\alpha_2(\alpha_1+\alpha_2)\prod_{i\neq k,l} s_i^2} \to \infty \quad (\alpha_1, \alpha_2 \to +0) \ . \tag{23}$$

The square H_2 norm of the transfer function is inversely proportional to the product of absolute values of ill-stable eigenvalues and their absolute values sum. In contrast to the previous case $\|W\|_2^2$ grows much faster due to the synergetic effect of two real ill-stable modes.

Case C. There is a pair of ill-stable complex conjugate eigenvalues $s_k = -\alpha + j\omega$ and $s_{k^*} = -\alpha - j\omega$ with $\alpha \to +0$. All the other eigenvalues are fixed. In a similar way we obtain

$$\|W\|_2^2 \sim G_k + G_{k^*} \sim \frac{b(j\omega)\ b(-j\omega)}{4\alpha\omega^2 \prod_{i\neq k,k^*}(s_i^2+\omega^2)} \to \infty \quad (\alpha \to +0) \ . \tag{24}$$

If the frequency ω also approaches zero value, then we obtain the following asymptotic expression:

$$\|W\|_2^2 \sim G_k + G_{k^*} \sim \frac{b_0^2}{4\alpha(\omega^2+\alpha^2)\prod_{i\neq k,k^*} s_i^2} \to \infty \quad (\alpha,\ \omega \to +0) \ . \tag{25}$$

Therefore in the case of a pair of complex conjugate ill-stable low-frequency eigenvalues, $\|W\|_2^2$ is asymptotically inversely proportional to the product of these conjugate eigenvalues and the absolute value of their real part.

Case D. There is one real ill-stable eigenvalue $s_1 = -\alpha_0$ and one pair of ill-stable complex conjugate eigenvalues $s_2 = -\alpha + j\omega$ and $s_3 = -\alpha - j\omega$ with $\alpha_0, \alpha \to +0$. All the other eigenvalues are fixed. From (16) we obtain

$$\|W\|_2^2 \sim G_1 + G_2 + G_3 \sim \frac{1}{2\omega^4}\left(\frac{b_0^2}{\alpha_0 \prod_{i\neq 1,2,3} s_i^2} + \frac{b(j\omega)b(-j\omega)}{2\alpha\prod_{i\neq 1,2,3}(s_i^2+\omega^2)}\right) \to \infty \ (\alpha_0, \alpha \to +0). \tag{26}$$

If the frequency ω also approaches zero value, then we obtain the following asymptotic expression:

$$\|W\|_2^2 \sim G_1 + G_2 + G_3 \sim \frac{b_0^2}{\prod_{i\neq 1,2,3} s_i^2} \times \frac{\alpha_0^3 + \alpha_0(\omega^2 - 3\alpha^2) + 2\alpha(\alpha^2+\omega^2)}{4\alpha\alpha_0(\alpha^2+\omega^2)\left((\alpha_0^2+\omega^2-\alpha^2)^2 + 4\omega^2\alpha^2\right)} \quad (\alpha_0, \alpha, \omega \to +0) \ . \tag{27}$$

Comparing this with the case A one can see that an additional pair of complex conjugate ill-stable eigenvalues significantly increases the contribution of a single ill-stable real eigenvalue into $\|W\|_2^2$. In particular, low-frequency ill-stable modes can heavily increase the H_2 norm of the transfer function.

The considered cases suggest several conclusions. The eigen-parts (15) of the square norm $\|W\|_2^2$ corresponding to the ill-stable modes have a similar asymptotic behavior. They infinitely grow near the stability boundary of the system. Our interpretation of this result is that the total energy of a dynamical system, operating under the pre-fault conditions, accumulates in the ill-stable modes. In this case the decomposition (15) of $\|W\|_2^2$ allows the identification of the most dangerous modes in terms of their greatest contributions to the total energy of the system. Such modes will constitute the major part of the sum. Conversely, those modes not making a significant contribution can be regarded as posing no threat to a system stability.

If there are several ill-stable modes they can increase the system energy up to a critical level much earlier due to their synergetic effect. In particular, according to (24, 26) a transfer function norm is inversely proportional to the frequency or even to the frequency squared. The lower the frequency of a given mode is, the stronger influence it has on the system transfer function norm. Therefore the ill-stable low-frequency modes can pose a special threat to the system stability because they can act as a catalyst increasing the energy in the system (Gaglioti *et al.* (2011)).

5. CASE STUDY

In order to illustrate how the proposed method can be applied to the stability analysis of power grids, we employ the Simulink model of a mini-grid at Russky Island (Grobovoy *et al.* (2013)). The one-line diagram of the power network model is presented in Fig.1.

In the existing power network at Russky Island, 35 kV rated voltage is used. The total length of the transmission lines is 19.65 km. In this investigation, the distributing network with the rated voltage of 10 kV and less are represented by the loads at the level of 35 kV, but some are combined with the nodes of the equivalent generators on the rated voltage of 10 kV or 6.3 kV. The power network model contains three two-winding power transformers, one three-winding transformer, and one autotransformer. Eight consumption loads represent the island electricity demands. Total electricity consumption in the model under investigation amounts to 45.65 MW. An electric power is generated in the system by four combined heat and power plants (CHPP). The CHPP-1 consists of five gas turbine with rated power 7.33 MWA, CHPP-2 is comprised

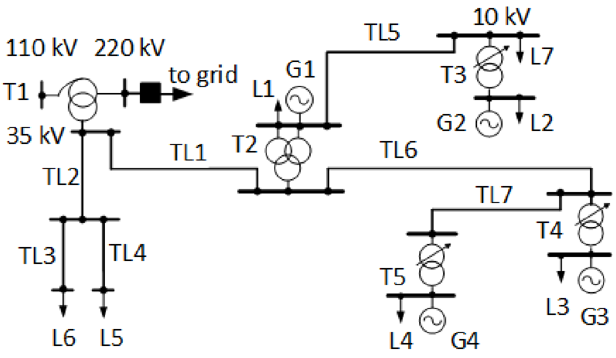

Fig. 1. One-line diagram power network: T1-T5, L1-L7, TL1-TL7, G1-G4 are equivalent transformers, loads, transmission lines and generators respectively

of two 2 MWA gas turbines, CHPP-3 consists of two gas turbines with rated power 7.33 MWA, and CHPP-4 has in its structure one 1.8 MWA gas turbine. For the purposes of the present examination, four equivalent generators represent these power generation units ($G1, G2, G3$ and $G4$ in Fig.1). Each of these is equipped by the excitation system and speed control which are available in the MATLAB software in the Simulink/SimPowerSystems package. The well-known Rowen's model (Yee *et al.* (2008)) for gas turbine has been used. In out simulation, however, we used only one control channel and neglected both channels of the temperature control of the exhaust gases and acceleration loop.

This grid is obviously too small and underpowered to have a lack of static stability in a real life, but it is possible to overload a computer model for the testing purposes. We studied the limit of the system stability by simultaneously increasing all of the loads and the active power of each generator while keeping the ratios between these fixed. We define the power increase coefficient as $\gamma = P/P_0$, where P and P_0 are the total active power and the initial total active power of the generators respectively. For each γ we linearize the system at $t = 100s$ and obtain a 64th order linear model. For the transfer function calculations we assume all elements of the matrices B and C equal ones.

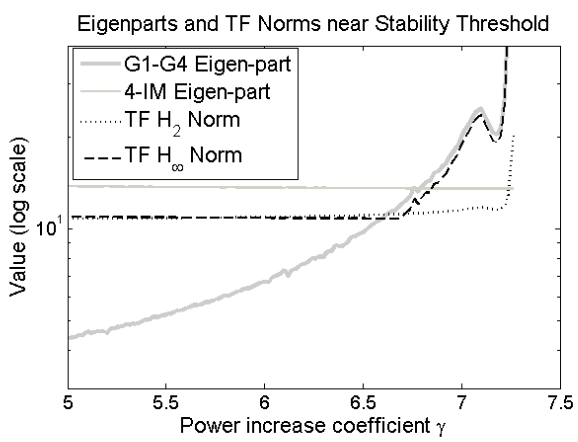

Fig. 2. H_2 and H_∞ norms of the transfer function and two largest eigen-parts as functions of power increase coefficient γ

The behavior of the H_2 and H_∞ norms of the transfer function as well as the behavior of the two largest eigen-parts (15) is shown in Fig.2 as functions of the power increase coefficient γ. The thick solid line represents the eigen-part corresponding to the inter-machine rotor angle oscillation between the generators G1 and G4 (or G1-G4 mode). The dotted line represents the eigen-part corresponding to the 4-inter-machine oscillatory mode (or 4-IM mode). For the most of loads the 4-IM eigenvalue is the closest to Im axis, but the G1-G4 mode becomes unstable for $\gamma \approx 7.25$ and the norm of the corresponding eigen-part approaches infinity. The H_2 and H_∞ norms of the transfer function (shown in Fig.2 by the solid and dashed lines respectively) approach infinity at $\gamma \approx 7.25$ as well (Ahmetzyanov *et al.* (2012)).

One can see that the analysis of the behavior of G1-G4 eigen-part allows the identification of the pre-fault conditions much earlier than the analysis of the H_2 and H_∞ norms of the transfer function. The proposed eigenmode decomposition also allows us to localize accurately the source of the potential instability (i.e., the rotor angle oscillation between G1 and G4 generators). Another interesting observation is that the eigen-part of the most dangerous mode seems to be a good approximation of the H_∞ norm of the transfer function. This observation (purely experimental by the moment) seems to be reasonable since both of them represent the worst-case gain of the system.

6. CONCLUSION

In this paper we propose a new method for the small-signal stability analysis of power systems based on the spectral decomposition of a square H_2 norm of the transfer function. We analyze the dynamic behavior of individual eigen-components. Compared with the dynamics of the full expression for the H_2 and H_∞ norms of the transfer functions, the analysis of the behavior of the individual eigen-parts allows the earlier identification of the pre-fault condition occurrence. In addition, since each eigen-part is associated with a certain characteristic state vector, the potential sources of instability can easily be located and tracked over time. The proposed method can be considered as a special case of a more general approach of Gramians and sub-Gramians proposed in (Yadykin (2010); Yadykin *et al.* (2013)). However, it is more convenient for the practical calculations and can be used to study the stability of large-scale dynamical systems.

The proposed decomposition has a particular importance for analyzing behavior of a power system operating near its stability boundary. We explore the asymptotic behavior of the ill-stable systems, when one or more eigenvalues approach the imaginary axis from the left. In this case the total energy of a dynamical system accumulates in the ill-stable modes, and it is sufficient to know only the corresponding eigenvalues, as they provide the principal contributions to the total energy of the system. Analyzing the asymptotic expressions, we observed that several ill-stable modes can increase the system energy up to a critical level much earlier due to their synergetic effect. In particular, the ill-stable low-frequency modes can pose a special threat to the system stability because they can act as a catalyst increasing the energy in the system. Finally in our paper we illustrate how the proposed method can be applied to the stability analysis of a real small power grid at Russky Island.

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