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# A Preliminary Study of Barrier Stopping Points in Constrained Nonlinear Systems

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**Abstract:** We first recall results on the boundary of the so-called *admissible set* for state and input constrained nonlinear systems, namely that the boundary is made up of two parts: one included in the state constraints and its complement called the *barrier*, made of integral curves that satisfy a minimum-like principle. Then we define the notions of *barrier stopping points by intersection* and *by self-intersection*. We then prove that all regular intersection points of the integral curves running along the barrier are barrier stopping points. Then we present, on systems of two and three dimensions, examples where barriers with stopping points occur.

*Keywords:* barrier, admissible set, constrained nonlinear systems, stopping points

## 1. INTRODUCTION

The admissible set of a constrained nonlinear control system is the set of all points in the state space where there exists at least one control such that state constraints are satisfied for all time. Related ideas exist in other fields. These include *capturability* in differential games, see Isaacs (1965); *viability kernels* from viability theory, see Aubin (1991); and the *backwards reachable set*, see Mitchell et al. (2005) and the references therein. These fields find important application in verification of engineering designs as may be seen in the naïve academic example of section 4.2 where wall-avoidance of a nonholonomic car is studied.

Differential games is especially interesting to control engineers owing to its relation with H-∞ methods, see Başar and Bernhard (1995). See also Başar and Olsder (1999). An important construct in a differential game (more specifically, a two player pursuit-evasion differential game) is the *barrier*. Recently, De Doná and Lévine (2013) wrote a paper showing that the notion of *barrier*, as a *semi-permeable surface*, extends to purely qualitative state and input constrained nonlinear systems. As opposed to the theoretical development usually employed in the differential games literature, the approach taken by De Doná and Lévine (2013) does not initiate with the concept of a value function that satisfies an appropriate Hamilton-Jacobi partial differential equation. Moreover, in De Doná and Lévine (2013) it is shown that the barrier is obtained by solving minimum-like principle equations. This approach may provide useful insights into the value function-based theory.

It has been observed that optimal control problems with state constraints often have singularities of the value function, see chapter 1 of Vinter (2010). In our context without any cost to optimise, it appears that an analogous role to these singularities is played by intersection points of some integral curves satisfying the minimum-like principle, running along the barrier: such intersection points may be seen as particular cases of singularities of constrained Hamiltonian systems.

In this paper we prove a theorem that states that points where integral curves that run along the barrier intersect with one another or with themselves, are always *barrier stopping points by intersection* or *self-intersection* respectively.

The minimum-like principle result allows us to find a collection of $(n-1)$ dimensional oriented manifolds, including parts which may not lie in the barrier, especially if these manifolds intersect. The challenge in proving the above mentioned theorem consists of determining what parts of these intersecting manifolds need to be discarded, which is a matter of deducing the orientation of the barrier at points in a neighbourhood of the intersection. In other words, we want to know on which side of each of the above mentioned manifolds is the interior of the admissible set and on which side is its complement. This is especially challenging when dealing with problems of dimension higher than three.

We first cover the development as by De Doná and Lévine (2013), and state the important theorem that allows one to construct integral curves that run on the barrier via a minimum-like principle in section 2. We then introduce various notions of *barrier stopping point* in section 3 and state the theorem which is the main contribution of this paper. We then illustrate this result on an example of dimension 2 borrowed from De Doná and Lévine (2013) and then on the 3 dimensional example of the Dubins car, Dubins (1957), with state constraints. We then end with some concluding remarks.

## 2. PRELIMINARY THEORY

### 2.1 Constrained Dynamical Control Systems

We consider the constrained nonlinear system as specified by De Doná and Lévine (2013):

$$\dot{x} = f(x,u), \tag{1}$$
$$x(t_0) = x_0, \tag{2}$$
$$u \in \mathscr{U}, \tag{3}$$
$$g_i\big(x(t)\big) \leq 0 \quad \forall t \in [t_0,\infty), \quad \forall i \in \{1,\ldots,p\} \tag{4}$$

where $x(t) \in \mathbb{R}^n$. The input function $u$ is assumed to belong to the set $\mathscr{U}$: the set of Lebesgue measurable functions from $[t_0,\infty)$ to $U$, where $U$ is a compact convex subset of $\mathbb{R}^m$, and not a singleton.

The *constraint set* is defined by:

$$G \triangleq \{x \in \mathbb{R}^n : g_i(x) \leq 0, i = 1,...,p\}$$

We introduce the notation $g(x) \doteq 0$ to indicate that $x$ satisfies $g_i(x) = 0$ for at least one $i \in \{1,...,p\}$ and $g_i(x) \leq 0$ for all $i \in \{1,...p\}$. $g(x) \prec 0$ (resp. $g(x) \preceq 0$) indicates that $g_i(x) < 0$ (resp. $g_i(x) \leq 0$) for all $i \in \{1,\ldots,p\}$. $\mathbb{I}(x)$ denotes the set of all indices $i \in \{1,\ldots,p\}$ such that $g_i(x) = 0$.

We also define the sets

$$G_0 \triangleq \{x \in \mathbb{R}^n : g(x) \doteq 0\}, \qquad G_- \triangleq \{x \in \mathbb{R}^n : g(x) \prec 0\}. \tag{5}$$

The constraint set is given by $G = G_0 \cup G_-$.

The assumptions made by De Doná and Lévine (2013) for the rigorous analysis of the barrier are:

**(A1)** $f$ is an at least $C^2$ vector field of $\mathbb{R}^n$ for every $u$ in an open subset of $\mathbb{R}^m$ containing $U$, whose dependence with respect to $u$ is also at least $C^2$.

**(A2)** There exists a constant $0 < C < +\infty$ such that the following inequality holds true:
$$\sup_{u\in U} |x^T f(x,u)| \leq C(1 + \|x\|^2), \quad \text{for all } x$$

**(A3)** The set $f(x,U)$, called the *vectogram* in Isaacs (1965), is convex for all $x \in \mathbb{R}^n$.

**(A4)** For each $i = 1,\ldots,p$, $g_i$ is an at least $C^2$ function from $\mathbb{R}^n$ to $\mathbb{R}$ and the set of points given by $g_i(x) = 0$ defines an $n-1$ dimensional manifold.

We denote by $x^{(u,x_0)}(t)$ the solution of the differential equation (1) with input $u \in \mathscr{U}$ and initial condition $x_0$.

### 2.2 The Admissible Set

The following definition and propositions in this section are from De Doná and Lévine (2013).

*Definition 1.* (Admissible States). We will say that a state-space point $\bar{x}$ is *admissible* if there exists, at least, one input function $v \in \mathscr{U}$, such that (1)–(4) are satisfied for $x_0 = \bar{x}$ and $u = v$:

$$\mathscr{A} \triangleq \{\bar{x} \in G : \exists u \in \mathscr{U},\ g\big(x^{(u,\bar{x})}(t)\big) \preceq 0, \forall t \in [t_0,\infty)\}. \tag{6}$$

Note that the Markovian property of the system implies that any point of the integral curve, $x^{(v,\bar{x})}(t_1)$, $t_1 \in [t_0,\infty)$, is also an admissible point.

We also recall that

$$\mathscr{A}^{\mathsf{C}} \triangleq \{\bar{x} \in G : \forall u \in \mathscr{U},\ \exists \bar{t} < +\infty, \exists i \in \{1,...,p\} s.t. \; g_i(x^{(u,\bar{x})}(\bar{t})) > 0\}. \tag{7}$$

*Proposition 1.* Assume that (A1)–(A4) are valid. The set $\mathscr{A}$ is closed.

We denote by $\partial\mathscr{A}$ the admissible set's boundary and define the two sets:

$$[\partial\mathscr{A}]_0 = \partial\mathscr{A} \cap G_0, \quad [\partial\mathscr{A}]_- = \partial\mathscr{A} \cap G_-. \tag{8}$$

It can be seen that $\partial\mathscr{A} = [\partial\mathscr{A}]_0 \cup [\partial\mathscr{A}]_-$.

### 2.3 The Barrier

We now look at the subset $[\partial\mathscr{A}]_-$ of the boundary of the admissible set. Again, the definition of the barrier, as well as the propositions and theorem 4 are from De Doná and Lévine (2013).

*Definition 2.* The set $[\partial\mathscr{A}]_-$ is called the *barrier* of the set $\mathscr{A}$.

*Proposition 2.* Assume that (A1) to (A4) hold. The barrier $[\partial\mathscr{A}]_-$ is made of points $\bar{x} \in G_-$ for which there exists $\bar{u} \in \mathscr{U}$ and an arc of integral curve $x^{(\bar{u},\bar{x})}$ entirely contained in $[\partial\mathscr{A}]_-$ until it intersects $G_0$ at a point $x^{(\bar{u},\bar{x})}(\bar{t})$ for some $\bar{t} \in [t_0,+\infty)$.

In the following proposition $L_f h(x,u) \triangleq Dh(x) f(x,u)$ is the Lie derivative of a smooth function $h : \mathbb{R}^n \to \mathbb{R}$ along the vector field $f(\cdot,u)$ at the point $x$.

*Proposition 3.* Consider $\bar{x} \in [\partial\mathscr{A}]_-$ and $\bar{u} \in \mathscr{U}$ as in Proposition 2, i.e. such that the integral curve $x^{(\bar{u},\bar{x})}(t) \in [\partial\mathscr{A}]_-$ for all $t$ in some time interval until it reaches $G_0$. Then, there exists a point $z = x^{(\bar{u},\bar{x})}(\bar{t}) \in \mathsf{cl}([\partial\mathscr{A}]_-) \cap G_0$ for some finite time $\bar{t} \geq t_0$ such that

$$\min_{u\in U} \max_{i\in\mathbb{I}(z)} L_f g_i(z,u) = 0. \tag{9}$$

Then it is shown in De Doná and Lévine (2013) that the barrier can be obtained through the backward integration of the associated Hamiltonian system:

*Theorem 4.* Under the assumptions of Proposition 2, every integral curve $x^{\bar{u}}$ on $[\partial\mathscr{A}]_- \cap \mathsf{cl}(\mathsf{int}(\mathscr{A}))$ and the corresponding control function $\bar{u}$, as in Proposition 2, satisfies the following necessary condition.

There exists a (non zero) absolutely continuous maximal solution $\lambda^{\bar{u}}$ to the adjoint equation

$$\dot{\lambda}^{\bar{u}}(t) = -\left(\frac{\partial f}{\partial x}(x^{\bar{u}}(t),\bar{u}(t))\right)^T \lambda^{\bar{u}}(t), \quad \lambda^{\bar{u}}(\bar{t}) = (Dg_{i^*}(z))^T \tag{10}$$

such that

$$\min_{u\in U} \{(\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t),u)\} = (\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t),\bar{u}(t)) = 0 \tag{11}$$

at every Lebesgue point $t$ of $\bar{u}$ (i.e. for almost all $t \leq \bar{t}$).

In (10), $\bar{t}$ denotes the time at which $z$ is reached, i.e. $x^{\bar{u}}(\bar{t}) = z$, with $z \in G_0$ satisfying

$$g_i(z) = 0, \quad i \in \mathbb{I}(z), \quad \min_{u\in U} \max_{i\in\mathbb{I}(z)} L_f g_i(z,u) \triangleq L_f g_{i^*}(z,u^*) = 0.$$

Moreover, $\lambda^{\bar{u}}(t)$ is normal to $[\partial\mathscr{A}]_- \cap \mathsf{cl}(\mathsf{int}(\mathscr{A}))$ at $x^{\bar{u}}(t)$ for almost every $t \leq \bar{t}$.

*Remark 1.* By condition (11) we have $(\lambda^{\bar{u}}(t))^T f(x^{\bar{u}}(t),u) \geq 0$ for all $u \in U$, which means that $f(x,U)$ points in the direction of $\mathsf{cl}(\mathscr{A}^{\mathsf{C}})$ for all $x \in [\partial\mathscr{A}]_-$.

## 3. STOPPING POINTS

The result from the previous section allows the computation of integral curves that run along the barrier by construction. We assume in the remainder of the paper that $|\mathbb{I}(z)| = 1$ for all $z \in G_0$, where $|A|$ denotes the cardinality of a set $A$. Thus the mapping $z \mapsto \mathbb{I}(z)$ is piecewise constant on $G_0$ and it may be seen that the barrier $[\partial\mathscr{A}]_-$ is a piecewise $(n-1)$ dimensional manifold which is the envelope of backward integrated trajectories given by theorem 4. Several cases are possible, among which are:

- the barrier is made of maximal integral curves obtained from Theorem 4 by backward integration, that stop in finite time [1]. In this case we call the corresponding end-point a *barrier stopping point*.
- two or more distinct integral curves obtained as before intersect at a point, some arcs of these curves not forming part of the barrier. This phenomenon was shown in example 8.3 by De Doná and Lévine (2013), and was also observed by Isaacs (1965) in the context of differential games but without an in-depth explanation. Such a point corresponds to a *barrier stopping point by intersection*. See the next definition.
- an integral curve obtained as before intersects with itself at a later time and some arcs of this curve do not form part of the barrier. This corresponds to a *barrier stopping point by self-intersection*. If $\tilde{t}_1$ and $\tilde{t}_2$ are the distinct times at which this integral curve passes through the point $\xi$, then this case is possible if $f(\xi, u(\tilde{t}_1)) \neq f(\xi, u(\tilde{t}_2))$, where $u$ satisfies condition (11).

We now give precise definitions of these stopping point phenomena.

*Definition 3.*

(i) Consider two distinct integral curves $x^{(u_1,z_1)}$ and $x^{(u_2,z_2)}$ obtained from Theorem 4 by backward integration, running along the barrier $[\partial\mathscr{A}]_-$ from two distinct points $z_1, z_2 \in G_0$ at $\bar{t}_1$ and $\bar{t}_2$ respectively, i.e. $x^{(u_i,z_i)}(\bar{t}_i) = z_i$, $i = 1,2$, where $u_i$ is the corresponding control function that satisfies condition (11) for almost all $t \leq \bar{t}_i$, $i = 1,2$. Assume that there exists a point of intersection $\xi$ of these two curves at some time labeled $\tilde{t}$. $\xi$ is said to be *regular* if $\lambda^{u_i}(\tilde{t}) \neq 0$, $i = 1,2$. Moreover, $\xi$ is said to be a *barrier stopping point by intersection* if it is regular and either if the two maximal integral curves stop at $\xi$, or if $x^{(u_i,z_i)}(t) \in \mathsf{int}(\mathscr{A})$, $i = 1,2$, for all $t < \tilde{t}$, whereas $x^{(u_i,z_i)}(t) \in [\partial\mathscr{A}]_-$ for all $t \in [\tilde{t}, \bar{t}_i]$, $i = 1,2$.

(ii) Consider an integral curve $x^{(u,z)}$ obtained from Theorem 4 by backward integration, running along the barrier $[\partial\mathscr{A}]_-$ from a point $z \in G_0$ at $\bar{t}$, i.e. $x^{(u,z)}(\bar{t}) = z$, where $u$ is the corresponding control function that satisfies condition (11) for almost all $t \leq \bar{t}$. Assume that there exist times $\tilde{t}_1$ and $\tilde{t}_2$, with $\tilde{t}_1 < \tilde{t}_2$, such that $\xi = x^{(u,z)}(\tilde{t}_1) = x^{(u,z)}(\tilde{t}_2)$. $\xi$ is said to be *regular* if $\lambda^u(\tilde{t}_i) \neq 0$, $i = 1,2$. Moreover, $\xi$ is said to be a *barrier stopping point by self-intersection* if it is regular and if $x^{(u,z)}(t) \in \mathsf{int}(\mathscr{A})$, for all $t < \tilde{t}_2$, whereas $x^{(u,z)}(t) \in [\partial\mathscr{A}]_-$ for all $t \in [\tilde{t}_2, \bar{t}]$.

The next theorem states that all regular points where integral curves intersect with one another or with themselves are stopping points.

[1] we discard cases of blow-up in finite time

*Theorem 5.*

(i) Consider two distinct integral curves $x^{(u_1,z_1)}$ and $x^{(u_2,z_2)}$ as in Definition 3. If there exists a regular intersection point $\xi$ of these two curves at some time [2] $\tilde{t}$, i.e. $x^{(u_1,z_1)}(\tilde{t}) = x^{(u_2,z_2)}(\tilde{t}) = \xi$ and $\lambda^{u_i}(\tilde{t}) \neq 0$, $i = 1,2$, then $\xi$ is a *barrier stopping point by intersection*.

(ii) Consider an integral curve $x^{(u,z)}$ as in Definition 3. If $x^{(u,z)}$ is self-intersecting at the regular point $\xi$, then $\xi$ is a *barrier stopping point by self-intersection*.

**Proof.** (i) We denote by $\lambda^{u_1}$ and $\lambda^{u_2}$ the two corresponding adjoint integral curves satisfying $\lambda^{u_j}(\bar{t}_j) = \left(Dg_{i_j^*}(z_j)\right)^T$, $j = 1,2$, with $i_j^* \in \mathbb{I}(z_j)$. For each $t \in [\tilde{t}, \bar{t}_j]$ the adjoint $\lambda^{u_j}(t)$ is the normal to the $(n-1)$ dimensional separating hyperplane $\Pi_j(t)$ tangent to the curve $x^{(u_j,z_j)}$ at the point $x^{(u_j,z_j)}(t)$, the vectogram given by $f(x^{(u_j,z_j)}(t), U)$ being included in the closed half space $\Pi_j^+(t)$ containing $\lambda^{u_j}(t)$, $j = 1,2$, since we have $(\lambda^{u_j}(t))^T f(x^{(u_j,z_j)}(t), v) \geq 0$ for all $v \in U$ by condition (11), and since $\lambda^{u_j}(t) \neq 0$ in a small interval around $\tilde{t}$ by assumption and by continuity of $t \mapsto \lambda^{u_j}(t)$, $j = 1,2$. Moreover, according to Remark 1, $f(x^{(u_j,z_j)}(t), v)$ points into $\mathsf{cl}(\mathscr{A}^\mathsf{C})$ for all $v \in U$ and all $t$ such that $x^{(u_j,z_j)}(t) \in [\partial\mathscr{A}]_-$, $j = 1,2$. Thus $f(\xi, U) \subset \Pi_1^+(\tilde{t}) \cap \Pi_2^+(\tilde{t})$ and $\mathsf{cl}\left(\mathscr{A}^\mathsf{C} \cap W(\xi)\right) \subset \Pi_1^+(\tilde{t}) \cap \Pi_2^+(\tilde{t})$, for all $W(\xi)$ sufficiently small neighbourhood of $\xi$.

If $\xi$ corresponds to a stopping point of both maximal integral curves, $x^{(u_1,z_1)}$ and $x^{(u_2,z_2)}$, then the theorem is proven. On the contrary, if $x^{(u_1,z_1)}$ and $x^{(u_2,z_2)}$ do not stop at $\xi$, by continuity there exists $\sigma$ and $\tau$ such that $\sigma < \tau < \tilde{t}$ and that $x^{(u_j,z_j)}(t) \notin W(\xi)$ for all $t \in ]\sigma, \tau[$, $W(\xi)$ being the arbitrary neighbourhood of $\xi$ previously introduced. Therefore, $x^{(u_j,z_j)}(t) \notin \mathsf{cl}(\mathscr{A}^c \cap W(\xi))$ for all $t \in ]\sigma, \tau[$, which readily implies that $x^{(u_j,z_j)}(t) \in \mathsf{int}(\mathscr{A})$. Since $W(\xi)$ can be made arbitrarily small, this proves that the two arcs of integral curves do not belong to the barrier $[\partial\mathscr{A}]_-$ for $t < \tilde{t}$, and thus that $\xi$ is a stopping point by intersection by Definition 3.

(ii) Let $\lambda^u$ denote the adjoint associated with the integral curve $x^{(u,z)}$ with $\lambda^u(\bar{t}) = Dg_{i^*}(z)^T$, and let $\Pi^+(t)$ denote the closed half space containing $\lambda^u(t)$ at time $t$. The proof of (i) may be adapted to a self-intersecting curve by replacing the two closed half spaces $\Pi_1^+(\tilde{t})$ and $\Pi_2^+(\tilde{t})$ by $\Pi^+(\tilde{t}_1)$ and $\Pi^+(\tilde{t}_2)$ respectively. The proof then follows the same lines.

*Remark 2.* Theorem 5 is applicable to points where more than two distinct integral curves obtained from Theorem 4 intersect. In this case, Theorem 5 can be applied to pairs of integral curves.

## 4. EXAMPLES

### *4.1 Two Dimensional Nonlinear Example*

Consider the problem from section 8.3 of De Doná and Lévine (2013). A system is specified with dynamics:

$$\begin{aligned} \dot{x}_1 &= 1 - x_2^2 \\ \dot{x}_2 &= u \end{aligned} \tag{12}$$

[2] in case of multiple intersection points, only the last one must be considered, i.e. for the largest time $\tilde{t} < \bar{t}_i$, $\imath = 1,2$.

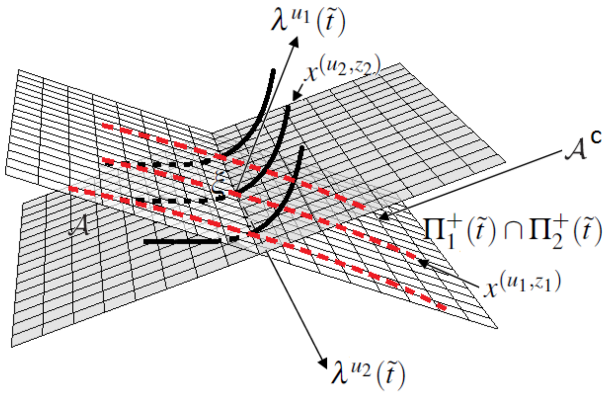


Fig. 1. Example of a stopping point by intersection occurring in a three dimensional system. For a small enough neighbourhood of the point $\xi$, denoted by $W(\xi)$, $\mathrm{cl}\left(\mathscr{A}^{\mathsf{C}}\cap W(\xi)\right)\subset \Pi_1^+(\tilde{t})\cap\Pi_2^+(\tilde{t})$

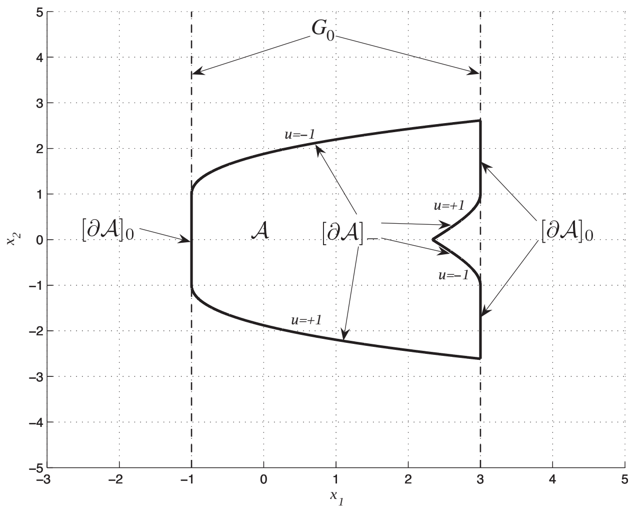


Fig. 2. Admissible set for (12), from De Doná and Lévine (2013)

$|u|\le 1$. The state is constrained to lie in the region $-1\le x_1\le 3$. It was shown that there are four points of tangential arrival, i.e. satisfying condition (9): $(-1,-1)$, $(-1,1)$, $(3,-1)$ and $(3,1)$. The costate dynamics are given by:

$$\begin{aligned}\dot{\lambda}_1 &= 0\\ \dot{\lambda}_2 &= 2x_2\lambda_1\end{aligned}\tag{13}$$

According to theorem 4 we must have $\min_u\{\lambda_1(1-x_2^2)+\lambda_2 u\}=0$, or $u(t)=-sign(\lambda_2)$.

It can be shown that for the curves initiating from $(3,1)$ and $(-1,-1)$ the corresponding control is given by $u(t)\equiv 1$, and for the curves initiating from $(-1,1)$ and $(3,-1)$ we get $u(t)\equiv -1$.

The barrier is shown in Fig. 2. Details of the barrier construction may be found in De Doná and Lévine (2013).

An important remark is that the integral curves ending at $(3,1)$ and $(3,-1)$ intersect at the point $\xi \triangleq (2+\frac{1}{3},0)$, and it is thus a stopping point by intersection by Theorem 5.

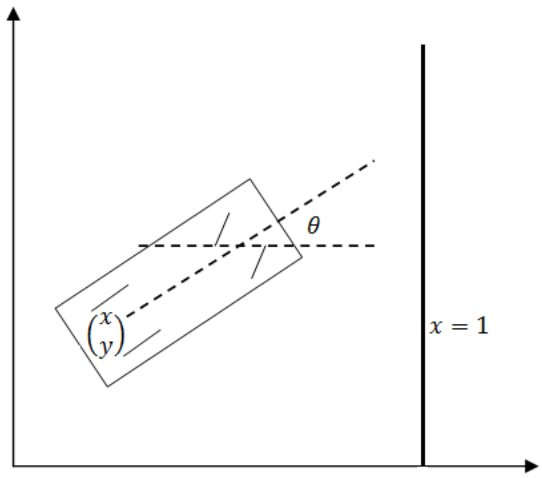


Fig. 3. Problem from section 4.2: wall avoidance for the Dubins car

### 4.2 *Nonholonomic Vehicle*

Consider the constrained system:

$$\begin{aligned}\dot{x} &= cos\theta\\ \dot{y} &= sin\theta\\ \dot{\theta} &= u\\ |u| &\le 1\end{aligned}$$

with constraint $g(x,y,\theta)=x-1$. This is generally refered to as the model of a nonholonomic vehicle of unit length moving at constant unit speed. The front wheels can instantaneously change their angle. The pair $(x,y)$ denotes the coordinates of the middle of the rear axle and $\theta$ is the orientation of the front wheels, Dubins (1957). See Figure 3. The constraint may be interpreted as a wall located at $x=1$ to be avoided.

The co-state at tangential arrival is given by (10): $\lambda(\bar{t}) = [g_x, g_y, g_\theta]^T = [1,0,0]^T$.

For almost all $t\le\bar{t}$:

$$\min_{|u|\le 1}\{\lambda_1 cos\theta+\lambda_2 sin\theta+\lambda_3 u\}=0\tag{14}$$

i.e.

$$\bar{u}(t)=\begin{cases}1 & \lambda_3(t)<0\\ -1 & \lambda_3(t)>0\\ anything & \lambda_3(t)=0\end{cases}$$

The co-state is given by:

$$\dot{\lambda}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \sin\theta & -\cos\theta & 0\end{bmatrix}\lambda$$

i.e. $\lambda_1(t)\equiv 1;\ \lambda_2(t)\equiv 0$

and therefore,

$$\dot{\lambda}_3 = \sin\theta$$

We can prove that $\lambda_3(t)$ vanishes only at isolated points. From equation (14) at the final time $\bar{t}$, $\theta(\bar{t}) = \pm\frac{\pi}{2}$

Therefore, we need to compute the integral curves of the system:

$$\begin{aligned}\dot{x} &= cos\theta \\ \dot{y} &= sin\theta \\ \dot{\theta} &= -sgn(\lambda_3) \\ \dot{\lambda}_3 &= \sin\theta\end{aligned}$$

with final conditions $\theta(\bar{t}) = \pm\frac{\pi}{2}$, $x(\bar{t}) = 1$, $y(\bar{t})$ arbitrary, $\lambda_3(\bar{t}) = 0$.

If $\theta(\bar{t}) = -\frac{\pi}{2}$, $\lambda_3(t) = -\sin(t-\bar{t})$ for $t \in [\bar{t}-\pi, \bar{t}]$. Similarly, if $\theta(\bar{t}) = \frac{\pi}{2}$ it can be shown that $\lambda_3(t) = \sin(t-\bar{t})$ for $t \in [\bar{t}-\pi, \bar{t}]$.

We deduce that trajectories on the barrier are helices in the $(x, y, \theta)$ space:

Notice that for a curve ending at $[x(\bar{t}), y(\bar{t}), \theta(\bar{t})]^T = [1, y_1, -\frac{\pi}{2}]^T$, we get $x(t) = \cos(t-\bar{t})$; $y(t) = -\sin(t-\bar{t}) + y_1$; $\theta(t) = -(t-\bar{t}) - \frac{\pi}{2}$ and for a curve ending at $[x(\bar{t}), y(\bar{t}), \theta(\bar{t})]^T = [1, y_1, \frac{\pi}{2}]^T$, we get $x(t) = \cos(t-\bar{t})$; $y(t) = \sin(t-\bar{t}) + y_2$; $\theta(t) = t - \bar{t} + \frac{\pi}{2}$ for $t \in [\bar{t}-\pi, \bar{t}]$. These two curves intersect when $t = \bar{t} - \frac{\pi}{2}$, and $y_2 - y_1 = 2$.

The envelopes of the backwards integrated trajectories, which are indexed by their initial $y$ coordinates, form two manifolds that intersect in a line denoted by $\mathscr{S} = \{(x, y, \theta) : (x, y, \theta)^T = (0, 1, 0)^T s,\ s \in \mathbb{R}\}$. We can conclude that all points $\xi \in \mathscr{S}$ are stopping points by intersection.

We can interpret the result as follows: the car is allowed to do what it pleases, unless it comes too close to the wall ($x > 0$). If $x > 0$ and the front wheels are not oriented appropriately, then the car is guaranteed to hit the wall regardless of control chosen. This corresponds to being in the set $\mathscr{A}^c$.

For a certain distance close to the wall ($0 < x < 1$) there are two orientations for the front wheels along with appropriate controls ($u = \pm 1$) that guarantee that the car will arrive tangentially to the wall, and any other control will result in collision. This corresponds to being on the barrier, $[\partial\mathscr{A}]_-$.

The line $\mathscr{S}$ of stopping points are special points on the barrier. From here, the car can choose between two different controls that will guarantee tangential arrival to the wall. However, backward prolongation of the barrier curves that intersect on $\mathscr{S}$ are clearly contained in the interior of $\mathscr{A}$, and therefore cannot be considered as part of the barrier. Note that the barrier $[\partial\mathscr{A}]_-$ is not $C^1$ in a neighbourhood of $\mathscr{S}$, hence the analogy with singularities of the value function in optimal control, where the iso-value surfaces, constructed by the maximum principle or by dynamic programming, and which play an analogous role to the barrier, display discontinuities of the gradient of the value function.

## 5. CONCLUSION

In this paper we have conducted a preliminary study of singularities that occur in the construction of barriers for state and input constrained nonlinear systems, namely stopping points which are created by the intersection or self-intersection of some integral curves running along the barrier, producing a continuous but only piecewise differentiable manifold. We have derived a theorem showing that such points of intersection are always barrier stopping points. Future work will focus on applications, as well as comparisons with singularities occurring in optimal control with state constraints and differential games.

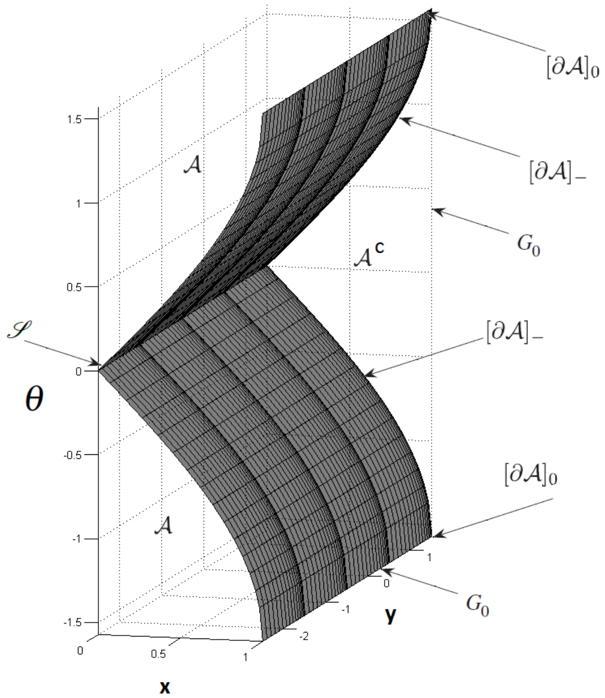


Fig. 4. The barrier for the nonholonomic vehicle, showing the intersection of the two surfaces in a line

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