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LMI Conditions for Designing Rational Nonlinear Observers with Guaranteed Cost ⋆

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Abstract: This paper presents a technique for designing rational nonlinear observers for rational nonlinear systems with guaranteed cost performance. The approach is based on a Lyapunov function that is quadratic in the estimation error and rational in the system states. The design conditions are formulated as Linear Matrix Inequalities (LMIs). If the conditions are satisfied, then the estimation error is guaranteed to asymptotically converge to zero for initial conditions on an estimated region of attraction. An optimization procedure for enlarging the region of attraction is also provided. An example is used to illustrate the results.

1. INTRODUCTION

The problem of estimating the states of a nonlinear system is an important topic that has received a lot of attention leading to several interesting nonlinear observer design techniques. An interesting overview of the problem with the state-of-the-art of observers for nonlinear systems can be found in Kang et al. (2013). Despite some interesting results reported in the literature, many important aspects in the design of nonlinear observers need further research to be improved. For instance, some of the existing design techniques rely on transforming the nonlinear system into a linear one by using a nonlinear output injection as in Krener and Isidori (1983). However, the transformation requires solving partial differential equations that are difficult to be solved. Other techniques are based on the decomposition of the nonlinear system into a linear and a nonlinear part and high gain linear observers are used to attenuate the effects of the nonlinear part in the estimation error dynamics as in Khalil (1999) or even using approximations based on Lipschitz conditions as in Röbenack and Lynch (2007).

Techniques based on semidefinite programming were also reported, as in Arcak and Kokotović (2001), Ichihara (2007). Sector conditions are assumed in Arcak and Kokotović (2001) to cope with the nonlinear terms of the error dynamics. Using quadratic Lyapunov functions Ichihara (2007) proposes a nonlinear observer design method based on Sum of Squares (SOS) techniques. The method applies to the class of polynomial systems. In most engineering applications where the sensor nonlinearities cannot be neglected, polynomials are used to approximate the I/O characteristics of the sensors, although the advantage of using rational functions instead of polynomials has been discussed in Germani and Manes (2008). The case where the output is a rational function of the states is treated in Germani and Manes (2008), however the system is linear with respect to the state.

This paper presents a technique for designing rational nonlinear observers for rational nonlinear systems. The system output and the measurement vector can be represented as rational functions of the system states. The approach is based on the Lyapunov's stability theory and the design conditions are formulated as LMIs. The Lyapunov function considered is quadratic in the estimation error and rational in the system states. If the conditions are satisfied, then the estimation error is guaranteed to asymptotically converge to the origin, *i.e.* the observer states converge to the system states, for initial conditions in an estimated region of attraction. An optimization procedure for enlarging the region of attraction is also provided. The idea is to find the largest region of attraction satisfying a given guaranteed cost performance requirement. The results in this paper are extensions of the observer design technique in Dezuo and Trofino (2014) to include a guaranteed cost performance in the design.

The paper is organized as follows. The next section is devoted to some preliminaries and definitions. The main results on nonlinear observer design with guaranteed cost performance and maximization of its region of attraction are presented in the Section 3. In the Section 4 a numerical example illustrates the method. Finally, some concluding remarks end the paper.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidian space. $\mathbb{R}^{p\times q}$ is the set of $p \times q$ real matrices. $\mathbb{I}_q$ denotes the set of integers $\{1,\ldots,q\}$. M' denotes the transpose of M. $\|.\|$ represents the 2-norm of vectors. I_r denotes the $r \times r$ identity matrix. A $p\times q$ matrix of zeros is denoted by $0_{p\times q}$. The $i-th$ row of a matrix M is represented by $row_i(M)$. The notation $[\cdot]_{col}$, $[\cdot]_{row}$, $[\cdot]_{diag}$ denote matrices whose elements, indicated inside the brackets, are arranged as a column, row and diagonal arrays. $[M_i]_{col}^{i\in\mathbb{I}_n}$ is a compact notation for $[M_1,\ldots,M_n]_{col}$. $[M_i]_{row}^{i\in\mathbb{I}_n}$ is a compact nota-

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tion for $[M_1, \ldots, M_n]_{row}$. $[M]_{diag}^{\mathbb{I}_n}$ is a compact notation for $[M, \ldots, M]_{diag}$ where M is repeated n times. $M > 0$ means that M is a symmetric positive definite matrix. The symbol $\otimes$ denotes the Kronecker product. For two sets $\mathcal{U}, \mathcal{V}$ the notation $\mathcal{U} \subset \mathcal{V}$ denotes $\mathcal{U}$ is a subset of $\mathcal{V}$. For two polytopes $\Pi_1 \subset \mathbb{R}^{n_1}$ and $\Pi_2 \subset \mathbb{R}^{n_2}$ the notation $\Pi_1 \times \Pi_2$ represents a meta-polytope of dimension $n_1 + n_2$ obtained by the cartesian product of Π_1, Π_2. $\vartheta(\Pi)$ represents the set of all vertices of the polytope Π. $\mathbf{Co}\{v_i, i \in \mathbb{I}_q\}$ denotes the convex hull obtained from the set of q vectors $\{v_1, \ldots, v_q\}$. $\overline{\lambda}(M), \underline{\lambda}(M)$ denotes the maximum and minimum eigenvalues of a real symmetric matrix M.

2. PRELIMINARIES

Consider the nonlinear system

$$\begin{array}{l} \dot{x} = f(x) = A\,x + B\,\pi \\ 0 = G(x)\,x + F(x)\pi \end{array} \quad \begin{cases} y = C_y\,x + D_y\pi \\ w = C_w\pi \end{cases} \quad x(0) \in \mathcal{X} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state vector, with initial condition $x(0)$ and $\mathcal{X}$ is a given polytope. $\pi : \mathcal{X} \mapsto \mathbb{R}^p$ is a vector of nonlinear functions that can be viewed as a basis from which we can represent the set of nonlinear functions of interest. The dependence of π on (x) will be omitted to simplify the notation. A, B are coefficient matrices that are used to express $f(x)$ as a linear combination of (x, π). $G(x) : \mathcal{X} \mapsto \mathbb{R}^{p\times n}$ and $F(x) : \mathcal{X} \mapsto \mathbb{R}^{p\times p}$ are affine matrix functions of x. The vector y is the measurement vector and $w = w(y)$ is a vector that represents elements of the basis function π that can be expressed as a function of the measurements and thus w can be computed online. C_y, D_y, C_w are coefficient matrices that are used to express y, w in terms of x, π.

Now consider the nonlinear observer with the following structure.

$$\dot{z} = A\,z + B\,\phi + H_y(y - C_y z - D_y\phi) + H_w(w - C_w\phi) \tag{2}$$

$$0 = G(z)\,z + F(z)\phi \tag{3}$$

where $z \in \mathbb{R}^n$ is the vector of states of the observer, H_y, H_w are the observer gains to be designed and $\phi(.)$ has the same structure as $\pi(.)$. Therefore, $G(.), F(.)$ in (3) have the same structure as in (1). Define the linear and nonlinear estimation errors

$$e = x - z \quad , \quad \mu_e = \pi - \phi \tag{4}$$

and using (1), (2), (3) the error dynamics can be written as

$$\dot{e} = (A - H_y C_y)e + (B - H_w C_w - H_y D_y)\mu_e \tag{5}$$

$$h_J = C_e e + C_\mu \mu_e \quad , \quad x \in \mathcal{X}\ ,\ e \in \mathcal{E} \tag{6}$$

$$0 = G(x-e)\,x - G(x-e)\,e + F(x-e)\pi - F(x-e)\mu_e \tag{7}$$

where $\mathcal{E}$ is a given polytope defining the set of initial errors to be considered in the estimation problem, h_J is a performance output, in a sense to be specified later, and $C_e \in \mathbb{R}^{r_h\times n}, C_\mu \in \mathbb{R}^{r_h\times p}$ are given matrices.

Assumptions (8)

(a) $f(x)$ is a rational function well defined on $\mathcal{X}$ with $f(0) = 0$ and the origin of (1) is locally asymptotically stable. This assumption regards the class of systems for which the system decomposition (1) can be obtained, guarantee existence and uniqueness of the solutions of the differential equation in a neighborhood $\mathcal{X}$ of the equilibrium point $0 \in \mathcal{X}$ and the asymptotic stability of the system of (1) is a technical requirement (see the proof of Theorem 1 in Section 3 for details).

(b) The matrix $F(x)$ is invertible for all values of $x \in \mathcal{X}$. Under this regularity assumption the decomposition (1) of $f(x)$ in terms of the basis function π is well posed as $f(x) = (A - BF(x)^{-1}G(x))\,x$ is well defined $\forall(x) \in \mathcal{X}$.

(c) The matrix $F(z)$ is invertible for all values of $z = x - e$, $\forall(e, x) \in \mathcal{E} \times \mathcal{X}$. Under this condition the decomposition (2)-(3) in terms of ϕ is well posed.

Remark 1. As the initial condition $z(0) = 0$ is usually chosen for the observer, in this case we have $e(0) = x(0)$, and it seems natural to consider the polytope $\mathcal{E}$ equals to the polytope $\mathcal{X}$. □

We end this section with the following definition.

Definition 1. (Annihilator). Given a vector function $f(.) : \mathbb{R}^q \mapsto \mathbb{R}^s$ and a positive integer r, a matrix function $\aleph_f(.) : \mathbb{R}^q \mapsto \mathbb{R}^{r\times s}$ will be called an *annihilator of* $f(.)$ if $\aleph_f(z)\,f(z) = 0$, $\forall\, z \in \mathbb{R}^q$. If $\aleph_f(.)$ is a linear function it will be referred to as linear annihilator. □

Observe that the matrix representation of a linear annihilator is not unique. Suppose that $f(z) = z = [z_1\ \ldots\ z_q]' \in \mathbb{R}^q$. Taking into account all possible pairs z_i, z_j for $i \neq j$ without repetition, *i.e.* $\forall i, j \in \mathbb{I}_q$ with $j > i$, we get a linear annihilator given by the formula

$$\aleph_z(z) = \begin{bmatrix} \phi_1(z) & Y_1(z) \\ \vdots & \vdots \\ \phi_{(q-1)}(z) & Y_{(q-1)}(z) \end{bmatrix} \tag{9}$$

$$Y_i(z) = -z_i\, I_{(q-i)},\ i \geq 1\ \ ,\ \ \phi_1(z) = [\,z_2\ \ldots\ z_q\,]'$$

$$\phi_i(z) = \begin{bmatrix} & z_{(i+1)} \\ 0_{(q-i)\times(i-1)} & \vdots \\ & z_q \end{bmatrix},\ i \geq 2.$$

In this paper annihilators are used jointly with the Finsler's Lemma to reduce the conservativeness of state dependent LMIs. See for instance Trofino and Dezuo (2013), Oliveira and Skelton (2001) for details.

3. MAIN RESULTS

In this paper we are concerned with the problem of expressing the Lyapunov stability conditions of the origin of the error system (5) as an LMI problem. More precisely, we are interested in using LMIs to determine a suitable Lyapunov function $v(e, x)$ that satisfies the following conditions $\forall(e, x) \in \mathcal{E} \times \mathcal{X}$.

$$\begin{array}{c} \phi_3(e,x) \leq v(e,x) \leq \phi_1(e,x) \\ \dot{v}(e,x) \leq -\phi_2(e,x) \end{array} \tag{10}$$

where $\phi_1(.), \phi_2(.), \phi_3(.)$ are continuous positive definite functions on $\mathcal{E} \times \mathcal{X}$ and $\dot{v}(e, x)$ denotes the time derivative of $v(e, x)$. The above conditions imply from (Khalil, 1996, p. 152) the local uniform asymptotic stability of the equilibrium point $(0, 0) \in \mathcal{E} \times \mathcal{X}$.

In addition to the stability requirements, we are also interested in designing observers with guaranteed cost with respect to the performance output h_J in (6). For a given constant γ , the problem of concern is to find the largest positively invariant set $\mathcal{R}$ such that

$$\max_{(e(0),x(0))\in\mathcal{R}} \int_0^\infty h_J(t)'h_J(t)dt < \gamma^{-1} \quad (11)$$

where $\mathcal{R}$ is defined as

$$\mathcal{R} = \{(e,x):\ v(e,x) \le 1\} \quad (12)$$

Remark 2. Under the hypothesis that the origin of the system is locally asymptotically stable, for any given γ there always exist a small enough neighborhood $\mathcal{R}$ of the origin leading the above criterion (11) to be satisfied. The problem of concern is then to find the largest possible neighborhood $\mathcal{R}$ for which (11) is satisfied for a given γ. □

Consider the Lyapunov function candidate

$$v(e,x) = v_q(e) + v_p(x) \quad (13)$$

where

$$v_q(e) := e'Qe \quad (14)$$

$$v_p(x) := x'\mathbf{P}(x)x = \pi_b' P\pi_b \ \ , \ \ P \in \mathcal{P} \ \ , \ \ \pi_b := \begin{bmatrix} x \\ \pi \end{bmatrix} \quad (15)$$

$$\mathbf{P}(x) = \begin{bmatrix} I_n \\ -F(x)^{-1}G(x) \end{bmatrix}' P \begin{bmatrix} I_n \\ -F(x)^{-1}G(x) \end{bmatrix} \quad (16)$$

$$\mathcal{P} := \{P\in \mathbb{R}^{(n+p)\times(n+p)}:\ P=P' \text{ and } \pi_b' P\pi_b=0 \text{ for } x=0\}$$

where $\mathcal{P}$ denotes a structure constraint on P such that $P \in \mathcal{P}$ implies $v_p(0) = 0$. The main result of the paper, summarized by the next theorem, proposes LMI conditions for the positiveness and decay of $v(e,x)$, $\forall(e,x) \in \mathcal{E} \times \mathcal{X}$, and for the inclusion $\mathcal{R} \subset \mathcal{E} \times \mathcal{X}$. Some auxiliary notation is presented in the sequel to simplify the presentation.

1) Consider the following LMIs for positivity of $v(e,x)$.

$$Q > 0 \quad (17)$$

$$P + L_bC_b(x) + C_b(x)'L_b' + \Gamma_b(x) > 0, \quad \forall x \in \vartheta(\mathcal{X}) \quad (18)$$

where $\Gamma_b(x) = M_b\aleph_{\pi_b}(x) + \aleph_{\pi_b}(x)'M_b'$ and

$$C_b(x) = [G(x)\ F(x)]\,, \quad C_b(x)\pi_b = 0, \quad \aleph_{\pi_b}(x)\pi_b = 0 \quad (19)$$

and $\aleph_{\pi_b}(x) \in \mathbb{R}^{s_b\times(n+p)}$ is an annihilator of the vector π_b defined as in Remark 3.1 of Trofino and Dezuo (2013). L_b, M_b are free scaling matrices to be determined with the dimensions of $C_b(x)', \aleph_{\pi_b}(x)'$, respectively. Due to space limitation, the procedure presented in Remark 3.1 of Trofino and Dezuo (2013) is omitted here. However, we exemplify the structure of the annihilators for the system given in the example in Section 4.

2) Consider the following LMI for decay of $v(e,x)$.

$$\Psi+\Psi'+\gamma H'H+L_dC_{\xi_a}(e,x)+C_{\xi_a}(e,x)'L_d'+\Gamma_d(e,x) < 0, \quad \forall(e,x) \in \vartheta(\mathcal{E} \times \mathcal{X}) \quad (20)$$

where $\Gamma_d(e,x) = M_d\aleph_{\xi_a}(e,x) + \aleph_{\xi_a}(e,x)'M_d'$, L_d, M_d are scaling matrices to be determined with dimensions of $C_{\xi_a}(e,x)', \aleph_{\xi_a}(e,x)'$, respectively, and

$$\Psi = \begin{bmatrix} QA-K_yC_y & QB-K_wC_w-K_yD_y & 0_{n\times n_a} \\ 0_{n_a\times n} & 0_{n_a\times p} & P_a \end{bmatrix} \quad (21)$$

$$P_a := \begin{bmatrix} PA_a \\ 0_{(p+n^2+np)\times n_a} \end{bmatrix}, \qquad \begin{matrix} H := [\, C_e\ C_\mu\ 0_{r_h\times n_a}\,] \\ n_a = n + 2p + n^2 + np \end{matrix}$$

$$A_a := \begin{bmatrix} A & B & 0_{n\times p} & 0_{n\times(n^2+np)} \\ 0_{p\times n} & 0_{p\times p} & I_p & 0_{p\times(n^2+np)} \end{bmatrix}$$

$$\begin{aligned} C_{\xi_a}(e,x) &= \begin{bmatrix} C_\xi(e,x) & 0_{p\times(p+n^2+np)} \\ 0_{(2p+n^2+np)\times(n+p)} & C_a(x) \end{bmatrix} \\ \aleph_{\xi_a}(e,x) &= \begin{bmatrix} \aleph_\xi(e,x) & 0_{(s_e+s_f+2n)\times(p+n^2+np)} \\ 0_{s_a\times(n+p)} & \aleph_{\pi_a}(x) \end{bmatrix} \\ C_\xi(e,x) &= [-G(x-e)\ -F(x-e)\ G(x-e)\ F(x-e)] \\ \aleph_\xi(e,x) &= \begin{bmatrix} \aleph_e(e) & 0_{s_e\times p} & 0_{s_e\times n} & 0_{s_e\times p} \\ 0_{s_f\times n} & -\aleph_\phi(x-e) & 0_{s_f\times n} & \aleph_\phi(x-e) \\ I_n\otimes x & 0_{2n\times p} & -e\otimes I_n & 0_{2n\times p} \end{bmatrix} \end{aligned} \quad (22)$$

where $\aleph_e \in \mathbb{R}^{s_e\times n}$ is a linear annihilator given by (9), $\aleph_\phi(.) \in \mathbb{R}^{s_f\times n}$ is an annihilator of ϕ with the same structure as $\aleph_\pi(.)$, obtained according to the Remark 3.1 of Trofino and Dezuo (2013).

$$C_a(x) := \begin{bmatrix} G(x) & F(x) & 0_{p\times p} & 0_{p\times n^2} & 0_{p\times np} \\ W_1(x) & W_2(x) & F(x) & 0_{p\times n^2} & \bar{F}_a \\ W_3(x) & W_4(x) & 0_{n^2\times p} & I_{n^2} & 0_{n^2\times np} \\ 0_{np\times n} & 0_{np\times p} & 0_{np\times p} & -G_b(x) & F_b(x) \end{bmatrix}$$

$$\begin{matrix} W_1(x) = \bar{G}_a(x)A + G(x)A & W_3(x) = E_b(x)A \\ W_2(x) = \bar{G}_a(x)B + G(x)B & W_4(x) = E_b(x)B \end{matrix} \quad (23)$$

where the matrices $F(x), G(x)$, which are affine functions of x, are decomposed as

$$\begin{matrix} G(x) = G_0 + \bar{G}(x) & , & \bar{G}(x) := \sum_{i=1}^n \bar{G}_i x_i \\ F(x) = F_0 + \bar{F}(x) & , & \bar{F}(x) := \sum_{i=1}^n \bar{F}_i x_i \end{matrix} \quad (24)$$

where x_i are the entries of x and $G_0, \bar{G}_i, F_0, \bar{F}_i$ are constant matrices of structure that issue from the affine decompositions of $G(x), F(x)$.

$$E_b(x) := [xE_i]_{col}^{i\in\mathbb{I}_n},\ E_i := row_i(I_n),\ \bar{G}_a(x) := \sum_{i=1}^n \bar{G}_i x E_i$$

$$F_b(x) := [F(x)]_{diag}^{\mathbb{I}_n},\ G_b(x) := [G(x)]_{diag}^{\mathbb{I}_n},\ \bar{F}_a := [\bar{F}_i]_{row}^{i\in\mathbb{I}_n}$$

$\aleph_{\pi_a}(x) \in \mathbb{R}^{s_a\times(n+2p+n^2+np)}$ is an annihilator given by

$$\aleph_{\pi_a}(x) := \begin{bmatrix} \aleph_{\pi_b}(x) & 0_{s_b\times p} & 0_{s_b\times n^2} & 0_{s_b\times np} \\ W_a & \aleph_{\pi_b}(x)J_1 & \bar{H}_b & \bar{H}_a \\ 0_{s_en\times(n+p)} & 0_{s_en\times p} & \aleph_\mu(x) & 0_{s_en\times np} \\ 0_{s_fn\times(n+p)} & 0_{s_fn\times p} & 0_{s_fn\times n^2} & \aleph_\eta(x) \end{bmatrix}$$

$$W_a = [\aleph_{\pi_b}(x)J_0A \ \ \aleph_{\pi_b}(x)J_0B] \quad (25)$$

where with affine decomposition

$$\aleph_{\pi_b}(x) = H_0 + \bar{H}(x) \ \ , \ \ \bar{H}(x) := \sum_{i=1}^n \bar{H}_i x_i \quad (26)$$

where $H_0, \bar{H}_i \in \mathbb{R}^{s_b\times(n+p)}$ are fixed matrices of structure that issue from the affine decomposition of $\aleph_{\pi_b}(x)$ and

$$J_0 := \begin{bmatrix} I_n \\ 0_{p\times n} \end{bmatrix} \ ,\ J_1 := \begin{bmatrix} 0_{n\times p} \\ I_p \end{bmatrix} \ ,\ \begin{matrix} \bar{H}_a := [\bar{H}_iJ_1]_{row}^{i\in\mathbb{I}_n} \\ \bar{H}_b := -[\bar{H}_iJ_0]_{row}^{i\in\mathbb{I}_n} \end{matrix} \quad (27)$$

$$\aleph_\mu(x) := [\aleph_x(x)]_{diag}^{\mathbb{I}_n} \ \ , \ \ \aleph_\eta(x) := [\aleph_\pi(x)]_{diag}^{\mathbb{I}_n} \quad (28)$$

where $\aleph_x(x)$ is a linear annihilator of x given by (9) and $\aleph_\pi(x)$ is an annihilator of π obtained according to the Remark 3.1 of Trofino and Dezuo (2013).

3) Consider that $\mathcal{E}\times\mathcal{X}$ is a polytope that can be described as the convex hull of its vertices, or equivalently, as the intersection of half-spaces as indicated below.

$$\begin{aligned} \mathcal{E} \times \mathcal{X} &= \mathbf{Co}\{v_i,\ \forall i \in \mathbb{I}_h\} \\ &= \{(e,x):\ a_k'\, x_a \le 1\ ,\ \forall k \in \mathbb{I}_g\}\ \ ,\ x_a = \begin{bmatrix} e \\ x \end{bmatrix} \end{aligned} \quad (29)$$

where h is the number of vertices $v_i \in \mathbb{R}^{2n}$ of the polytope $\mathcal{E} \times \mathcal{X}$ and $a_k \in \mathbb{R}^{2n}$ are given vectors associated with the g facets $\mathcal{F}_k$ defined below.

$$\mathcal{F}_k = \{(e,x) \in \mathcal{E} \times \mathcal{X} :\ a_k' x_a = 1\} \ , \quad k \in \mathbb{I}_g \tag{30}$$

and then consider the following LMI for the estimation of the region of attraction.

$$Q_k'(P_{c_k}(x) + \Gamma_{c_k}(e,x))Q_k > 0 \quad \forall (e,x) \in \vartheta(\mathcal{F}_k) \ , \ \forall k \in \mathbb{I}_g \tag{31}$$

with the notation

$$\begin{aligned} C_k &= \left[\, a_k' C_0 \ \ -1 \,\right] \quad , \quad C_0 = [\, I_{2n} \ \ 0_{2n\times p}\,] \\ P_{c_k}(x) &= \left[Q, P + L_{b_k} C_b(x) + C_b(x)' L_{b_k}', -1\right]_{diag} \\ \Gamma_{c_k}(e,x) &= \left[0_{n\times n}, M_{b_k} \aleph_{\pi_b}(x) + \aleph_{\pi_b}(x)' M_{b_k}', 0\right]_{diag} \\ &\quad + N_{c_k} \left[\, C_0 \ \ -x_a\,\right] + \left[\, C_0 \ \ -x_a\,\right]' N_{c_k}' \end{aligned} \tag{32}$$

where Q_k is a basis for the null space of C_k and $M_{b_k} \in \mathbb{R}^{(n+p)\times s_b}$, $N_{c_k} \in \mathbb{R}^{(2n+p+1)\times 2n}$, $L_{b_k} \in \mathbb{R}^{(n+p)\times p}$, for $k \in \mathbb{I}_g$ are matrices to be determined.

Theorem 1. Consider the nonlinear system (1) with Assumptions (8-a,b). Consider the nonlinear observer (2)-(3) with Assumption (8-c). Let γ be a given scalar specifying the desired level of performance in (11). Suppose that the LMIs (17),(18),(20),(31) are satisfied and define the observer gains as

$$H_y = Q^{-1} K_y \quad , \quad H_w = Q^{-1} K_w \tag{33}$$

Then the convergence properties

$$\lim_{t\to\infty} \begin{bmatrix} z(t) \\ \phi(z(t)) \end{bmatrix} = \begin{bmatrix} x(t) \\ \pi(x(t)) \end{bmatrix} \tag{34}$$

and the cost function (11) are satisfied. Moreover $v(e,x)$ in (13) is a Lyapunov function for the error system (5). □

Proof: The first part of the proof consists of showing that $v(e,x)$ with the structure (13)-(15) is a Lyapunov function that satisfies the stability conditions (10) $\forall (e,x) \in \mathcal{E} \times \mathcal{X}$. Thus the uniform asymptotic stability follows from (Khalil, 1996, p. 152). The second part of the proof consists of showing that $\mathcal{R}$ is an estimated region of attraction for (5) and that the performance criterion (11) is satisfied $\forall (e(0), x(0)) \in \mathcal{R}$.

Note that (17) is a sufficient condition for $v_q(e) > 0$, $\forall e \neq 0 \in \mathcal{E}$, and multiplying (18) by π_b to the right and its transpose to the left, and keeping in mind that $C_b \pi_b = 0$, $\pi_b' \Gamma_b \pi_b = 0$, we get $v_p(x) > 0$, $\forall x \neq 0 \in \mathcal{X}$. Therefore, considering the decomposition of the Lyapunov function in (13), we conclude that $v(e,x) > 0$ is satisfied $\forall (e,x) \neq (0,0) \in \mathcal{E} \times \mathcal{X}$.

The time derivative of $v(e,x)$ is given by

$$\dot{v}(e,x) = \dot{v}_q(e) + \dot{v}_p(x) \tag{35}$$

where, with (5), we have

$$\dot{v}_q(e) = 2e'Q\dot{e} = 2e'Q(A - H_y C_y)e + 2e'Q(B - H_w C_w)\mu_e \tag{36}$$

and the time derivative of $v_p(x)$ leads to

$$\dot{v}_p(x) = 2 \begin{bmatrix} x \\ \pi \end{bmatrix}' P \begin{bmatrix} Ax + B\pi \\ \dot{\pi} \end{bmatrix} \tag{37}$$

which can be rewritten as

$$\dot{v}_p(x) = \pi_a'(P_a + P_a')\pi_a \quad , \quad \pi_a = [\pi_b, \dot{\pi}, \mu, \eta]_{col} \tag{38}$$

$$\mu = [\mu_i]_{col}^{i\in\mathbb{I}_n} \ , \ \eta = [\eta_i]_{col}^{i\in\mathbb{I}_n} \ , \ \mu_i = -x\dot{x}_i \ , \ \eta_i := \pi\, \dot{x}_i \tag{39}$$

Observe that n, p are the dimensions of x, π respectively. Moreover, the time derivative of $v_p(x)$ has increased complexity and we need extra change of variables, namely $\dot{\pi}, \mu, \eta$, to render the expressions affine in x. By arranging in a single expression all the relations among the vectors $x, \pi, \dot{\pi}, \mu, \eta$ we get $C_a(x)\pi_a = 0$ with $C_a(x)$ from (23). Also, observe that $\aleph_{\pi_b}(x)\pi_b = 0$ and, according to (Trofino and Dezuo, 2013, p.14), $\aleph_{\pi_a}\pi_a = 0$. See (Trofino and Dezuo, 2013, p.13) for the detailed construction of the matrices $C_a(x)$ and $\aleph_{\pi_a}$, omitted here due to space limitation. Consider the vector $\xi := [e, \mu_e, x, \pi]_{col}$. Note that using the notation (22) we can rewrite (7) as $C_\xi(e,x)\xi = 0$. Also note that $\aleph_e(e)e = 0$, $\aleph_\phi(z)\phi = \aleph_\phi(x-e)(\pi - \mu_e) = 0$, and $(I_n \otimes x)e - (e \otimes I_n)x = 0$. Therefore $\aleph_\xi(e,x)\xi = 0$.

Now, with (36), (38), (21) and using the changes of variable

$$K_y = QH_y \quad , \quad K_w = QH_w \tag{40}$$

we can rewrite (35) as

$$\dot{v}(e,x) = \xi_a' \left(\Psi + \Psi'\right) \xi_a \quad , \quad \xi_a = [e, \mu_e, \pi_a]_{col} \tag{41}$$

To show that the performance criterion (11) is satisfied, consider the auxiliary condition

$$\dot{v}(e,x) + \gamma h_J' h_J < 0 \tag{42}$$

Noticing that $h_J = H\xi_a$, we can rewrite (42), with $\dot{v}(e,x)$ from (41), as

$$\dot{v}(e,x) = \xi_a' \left(\Psi + \Psi' + \gamma H'H\right) \xi_a < 0 \tag{43}$$

As $C_\xi(e,x)\xi = 0$, $\aleph_\xi(e,x)\xi = 0$, $C_a(x)\pi_a = 0$, $\aleph_{\pi_a}(x)\pi_a = 0$, we have that $C_{\xi_a}(e,x)\xi_a = 0$, $\xi_a'\Gamma_d(e,x)\xi_a = 0$. Therefore, from (43) and the Finsler's Lemma we get (20) as a sufficient LMI condition for the negativeness of $\dot{v}(e,x)$.

In summary, suppose the conditions of the Theorem 1 are satisfied. Then by convexity they are also satisfied $\forall (e,x) \in \mathcal{E} \times \mathcal{X}$. Define the positive constants

$$\begin{aligned} &\epsilon_1 = \max_{x\in\mathcal{X}} \overline{\lambda}(S) \ \ , \ \ \epsilon_3 = \min_{x\in\mathcal{X}} \underline{\lambda}(S) \ \ , \ \ \epsilon_2 = \max_{x\in\mathcal{X}} \overline{\lambda}(M'M) \\ &S := [Q, \ P + L_b C_b(x) + C_b(x)' L_b' + \Gamma_b(x)]_{diag} \\ &M := F(x)^{-1} G(x) \end{aligned} \tag{44}$$

Observe from Assumption (8-b) that $F(x)^{-1}$ is well defined $\forall x \in \mathcal{X}$ and thus ϵ_2 is a finite positive constant. As $Q > 0$ and $P + L_b C_b + C_b' L_b' + \Gamma_b > 0$, let us multiply S by $\xi_b = [e, x, \pi]_{col}$ to the right and by its transpose to the left. Keeping in mind that $C_b \pi_b = 0$, $\pi_b' \Gamma_b \pi_b = 0$ and $\pi_b' P \pi_b = x' \mathbf{P}(x) x$ as $\pi = -F(x)^{-1} G(x) x$, we get

$$\epsilon_3 \|\xi_b\|^2 \leq v(e,x) \leq \epsilon_1 \|\xi_b\|^2 \quad \forall (e,x) \in \mathcal{E} \times \mathcal{X} \tag{45}$$

On the other hand,
$\|e\|^2 + \|x\|^2 \leq \|\xi_b\|^2 \leq \|e\|^2 + (\epsilon_2 + 1)\|x\|^2$, $\forall (e,x) \in \mathcal{E} \times \mathcal{X}$
Thus $v(e,x)$ satisfies the bounds in (10) $\forall (e,x) \in \mathcal{E} \times \mathcal{X}$ with $\phi_3 = \epsilon_3 \left(\|e\|^2 + \|x\|^2\right)$, $\phi_1 = \epsilon_1 \left(\|e\|^2 + (\epsilon_2 + 1)\|x\|^2\right)$. Similar arguments are used to show the bounds on $\dot{v}(e,x)$. Define the positive constant ϵ_4 as

$$\epsilon_4 = \min_{e\in\mathcal{E}, x\in\mathcal{X}} \underline{\lambda}(-N(e,x)) \tag{46}$$

$$\begin{aligned} &N(e,x) := \\ &\Psi + \Psi' + \gamma H'H + L_d C_{\xi_a}(e,x) + C_{\xi_a}(e,x)' L_d' + \Gamma_d(e,x) \end{aligned}$$

Recall that $C_{\xi_a}\xi_a = 0$ and $\xi_a'\Gamma_d\xi_a = 0$. Thus from (43), (20), (46) we get

$$\dot{v}(e,x) = \xi_a' N \xi_a \leq -\epsilon_4 \|\xi_a\|^2 \tag{47}$$

As $\|\xi_a\|^2 = \|e\|^2 + \|\mu_e\|^2 + \|x\|^2 + \|\pi\|^2 + \|\dot{\pi}\|^2 + \|\mu\|^2 + \|\eta\|^2$ we conclude $\|\xi_a\|^2 > \|e\|^2 + \|x\|^2$ whenever $\|e\| \neq 0$ and $\|x\| \neq 0$, which in turn implies

$$\dot{v}(e,x) < -\epsilon_4 (\|e\|^2 + \|x\|^2) \tag{48}$$

and we conclude $\dot{v}(e,x)$ satisfies the bounds in (10) $\forall (e,x) \in \mathcal{E} \times \mathcal{X}$ with $\phi_2 = \epsilon_4 \left(\|e\|^2 + \|x\|^2\right)$, which completes the proof for local stability of the error dynamics.

Moreover, condition (20) requires the dynamics of the system (1) to be stable as well in order to be satisfied, hence the Assumption (8-a).

For the second part of the proof, notice in (12) that the surface of $\mathcal{R}$ is the unitary level set of $v(e,x)$. Thus, the condition that guarantee $\mathcal{R} \subset \mathcal{E} \times \mathcal{X}$ is

$$v(e,x) > 1 \quad , \quad \forall (e,x) \in \mathcal{F}_k \ , \ \forall k \in \mathbb{I}_g \tag{49}$$

Considering $\pi_c = [e, \pi_b, 1]_{col}$ and using the notation (32) we can rewrite (49) as

$$\pi_c'(P_{c_k}(x) + \Gamma_{c_k}(e,x))\pi_c > 0 \quad \forall \pi_c : \ C_k \pi_c = 0, \ \forall k \in \mathbb{I}_g \tag{50}$$

Observe $\aleph_{\pi_b}(x)\pi_b = 0$ and $C_0 \, [e' \ \pi_b']' = x_a$ which in turn implies $\pi_c' \Gamma_{c_k}(e,x)\pi_c = 0, \ \forall e, x, k$. Using the Finsler's Lemma we get the LMI condition (31). If (31) is satisfied, then $\mathcal{R} \subset \mathcal{E} \times \mathcal{X}$. Moreover, if the conditions (17), (18), (20) are satisfied, then $\mathcal{R}$ is positively invariant and (34) is satisfied $\forall (e(0), x(0)) \in \mathcal{R}$. In order to show that (42) implies the criterion (11) we use standard arguments. Integration of (42) from 0 to $T > 0$ leads to

$$v(e(T), x(T)) - v(e(0), x(0)) + \gamma \int_0^T h_J' h_J \ dt < 0 \tag{51}$$

As the system (5),(7) is stable in closed-loop $\forall (e(0), x(0)) \in \mathcal{R}$, we conclude that $\lim_{T\to\infty}(e(T), x(T)) = 0$ and thus $\lim_{T\to\infty} v(e(T), x(T)) = 0$. Therefore, (51) implies that

$$\int_0^\infty h_J' h_J \ dt < v(e(0), x(0))\gamma^{-1} \tag{52}$$

As $v(e(0), x(0)) < 1, \ \forall (e(0), x(0)) \in \mathcal{R}$, we have $v(e(0), x(0))\gamma^{-1} < \gamma^{-1}$. Therefore, we conclude from (52) that the criterion (11) is satisfied $\forall (e(0), x(0)) \in \mathcal{R}$. □

3.1 Enlarging the region of attraction

Once $\mathcal{R} \subset \mathcal{E} \times \mathcal{X}$ from (31), to enlarge the estimate we need to approach, as much as possible, the unitary level set of $v(e,x)$ from the facets of the polytope. As $v(e,x) > 1$ on the facets, the problem of concern is to minimize the largest level set of $v(e,x)$ on each facet, *i.e.* minimizing τ_k such that $v(e,x) < \tau_k, \ \forall (e,x) \in \mathcal{F}_k$, for all $k \in \mathbb{I}_g$. Proceeding as in (49), (50), (31) we can rewrite $v(e,x) < \tau_k, \ \forall (e,x) \in \mathcal{F}_k$ as

$$Q_k'(\tilde{P}_{c_k}(x) + \tilde{\Gamma}_{c_k}(e,x))Q_k > 0 \quad \forall (e,x) \in \vartheta(\mathcal{F}_k) \ , \ \forall k \in \mathbb{I}_g \tag{53}$$

where

$$\tilde{P}_{c_k}(x) = \left[-Q, -P + \tilde{L}_{b_k} C_b(x) + C_b(x)' \tilde{L}_{b_k}', \tau_k\right]_{diag}$$

$$\tilde{\Gamma}_{c_k}(e,x) = \left[0_{n\times n}, \tilde{M}_{b_k} \aleph_{\pi_b}(x) + \aleph_{\pi_b}(x)' \tilde{M}_{b_k}', 0\right]_{diag} + \tilde{N}_{c_k} \left[\, C_0 \ -x_a \,\right] + \left[\, C_0 \ -x_a \,\right]' \tilde{N}_{c_k}'$$

where $\tilde{M}_{b_k} \in \mathbb{R}^{(n+p)\times s_b}, \tilde{N}_{c_k} \in \mathbb{R}^{(2n+p+1)\times 2n}, \tilde{L}_{b_k} \in \mathbb{R}^{(n+p)\times p}$, for $k \in \mathbb{I}_g$ are matrices to be determined as in (31). Taking into account all facets of the polytope, the optimization problem can be formulated as minimizing the average value of τ_k as indicated below.

$$\underset{\tau_k, P_{c_k}, \Gamma_{c_k}, \tilde{P}_{c_k}, \tilde{\Gamma}_{c_k}}{minimize} \quad \frac{1}{g}\sum_{k=1}^{g} \tau_k \tag{54}$$
$$\text{subject to (17), (18), (20), (31), (53)}$$

Remark 3. Observe there is, in general, a trade off between the size of the estimated region of attraction $\mathcal{R}$ and the level of performance γ^{-1} in (11) we can achieve. In general, larger values of γ result in smaller sizes of the region of attraction $\mathcal{R}$. This natural trade off is illustrated in the numerical example in Section 4. □

4. NUMERICAL EXAMPLE

In the example that follows we have used SeDuMi with Yalmip interface from Löfberg (2004) to solve the LMIs and Simulink to obtain the state trajectories.

Example 1. Consider the rational system

$$\begin{aligned} \dot{x}_1 &= x_2 + 0.5\zeta(x) \\ \dot{x}_2 &= -x_1 - x_2 + 0.5x_1^2 \end{aligned} \ , \quad \zeta(x) = \frac{x_1}{x_2^2+1} \ , \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{55}$$

with the initial condition $x(0) = [0 \ 0.07]'$, $y = x_2$ is the measurement output and $h = x_1$ is the performance variable. Define the nonlinear function

$$\pi(x) = \left[x_1^2 \ x_1 x_2 \ x_2^2 \ \tfrac{x_1}{x_2^2+1} \ \tfrac{x_2}{x_2^2+1} \ \tfrac{x_1^2}{x_2^2+1} \ \tfrac{x_1 x_2}{x_2^2+1} \ \tfrac{x_2^2}{x_2^2+1}\right]' \tag{56}$$

and observe the following relations among the entries π_i of π:

$$\begin{aligned} &\pi_1 = x_1^2 \ , \quad \pi_2 = x_1 x_2 \ , \quad \pi_3 = x_2^2 \\ &\pi_4 + x_2 \pi_7 - x_1 = 0 \ , \quad \pi_5 + x_2 \pi_8 - x_2 = 0 \\ &\pi_6 = x_1 \pi_4 \ , \quad \pi_7 = x_2 \pi_4 \ , \quad \pi_8 = x_2 \pi_5 \end{aligned} \tag{57}$$

Note that $\pi_4 = \zeta(x)$. The relation $\pi_4 + x_2\pi_7 - x_1 = 0$ is obtained from the expression of the rational function π_4 rewritten as $\pi_4(x_2^2 + 1) = x_1$ and the change of variable $\pi_7 = x_2 \pi_4$. The relation $\pi_5 + x_2 \pi_8 - x_2 = 0$ is obtained in a similar way. Using the above relations and noticing that π_3, π_5, π_8 are available by measuring x_2, we get the system representation (1) with the following matrices:

$$\begin{aligned} &A = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & 0 & 0.5 & 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &C_y = [\, 0 \ 1 \,] \ , \qquad C_w = \begin{bmatrix} 0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \\ 0&0&0&0&0&0&0&1 \end{bmatrix} \\ &D_y = 0_{1\times p} \ , \\ &G(x) = \begin{bmatrix} \beta_1(x) \\ I_2 \\ 0_{3\times 2} \end{bmatrix}, \quad F(x) = \begin{bmatrix} -I_3 & 0_{3\times 2} & 0_{3\times 3} \\ 0_{2\times 3} & -I_2 & \beta_2(x) \\ 0_{3\times 3} & \beta_1(x) & -I_3 \end{bmatrix} \end{aligned} \tag{58}$$

where

$$\beta_1(x) = \begin{bmatrix} x_1 I_2 \\ row_2(I_2) x_2 \end{bmatrix}, \qquad \beta_2(x) = [\, 0_{2\times 1} \ -x_2 I_2 \,]$$

and the following annihilators

$$\aleph_{\pi_b} = \begin{bmatrix} \aleph_{\pi_d} & 0 \\ 0 & \aleph_\zeta \\ \Xi & 0 \\ 0 & \Xi \end{bmatrix} \qquad \begin{aligned} &\aleph_{\pi_d} = \aleph_\zeta = [\aleph_x, \aleph_{\pi_2}]_{diag} \\ &\aleph_\pi = [\aleph_{\pi_2}, \aleph_\zeta]_{diag} \\ &\aleph_{\pi_2} = \begin{bmatrix} x_2 & -x_1 & 0 \\ 0 & x_2 & -x_1 \end{bmatrix} \end{aligned}$$
$$\Xi = [x_2 \ 0 \ 0 \ -1 \ 0]$$

Through rows and columns manipulations it is possible to check that $det(F(x)) = x_2^2 + 1$ and thus the Assumption (8-b) on $F(x)$ invertibility holds globally. The same is true for $F(x - e)$ and Assumption (8-c). Observe $x_2^2 + 1$ is the denominator of the rational function in (55), (56). Consider the observer structure given in (2)-(3), with $z(0) = 0$ and the nonlinear function

$$\phi(z) = \left[z_1^2 \ z_1 z_2 \ z_2^2 \ \tfrac{z_1}{z_2^2+1} \ \tfrac{z_2}{z_2^2+1} \ \tfrac{z_1^2}{z_2^2+1} \ \tfrac{z_1 z_2}{z_2^2+1} \ \tfrac{z_2^2}{z_2^2+1}\right]'$$

and noticing that $\phi(.)$ has the same structure as $\pi(.)$ in (56), the matrices $G(z), F(z)$ have the same structure given in (58) with x replaced by z.

The polytope considered for x is a hypercube of the form $\mathcal{X} = \{x : \|x_i\| \le \alpha, \ i \in \mathbb{I}_n\}$ with $\alpha = 0.5$, and $\mathcal{E} = \mathcal{X}$ according to Remark 1. See Remark 5.2 of Trofino and Dezuo (2013) for an algorithm to improve the choice of the vertices of $\mathcal{X}$. Solving the optimization problem (54) for the above polytope and fixing the level of performance in (11) with the choice $\gamma = 10$, a feasible solution is found, leading to the observer gains

$$H_y = \begin{bmatrix} -43.656 \\ 27.312 \end{bmatrix}, \ H_w = \begin{bmatrix} -0.430 & -5.581 & 8.442 \\ -1.396 & 3.174 & -3.833 \end{bmatrix}$$

The trajectories of the states of the system and of the observer are shown in Fig. 1(a) and 1(b) for x_1, z_1 and x_2, z_2, respectively, for the given initial condition $x(0)$ and $z(0) = 0$. Note that z converges to x, as expected. Fig. 1(e) shows the estimated region of attraction $\mathcal{R}_0$ for the case of $z(0) = 0$, therefore $e(0) = x(0)$. Note that $\mathcal{R}_0 \subset \mathbb{R}^n$, $\mathcal{R} \subset \mathbb{R}^{2n}$ and that $\mathcal{R}_0 \subset \mathcal{R}$ because $\mathcal{R}_0$ represents the set of initial conditions for a particular case of $z(0)$. Fig. 1 also shows the trajectories of z and the estimated region of attraction $\mathcal{R}_0$ for $\gamma = 0$ (case without guaranteed cost performance). Comparing the results, note the trade off between the level of guaranteed cost and size of the region of attraction. The larger the estimate of the region of attraction the slower may become the observer response for initial conditions near the boundary of the estimate. $\square$

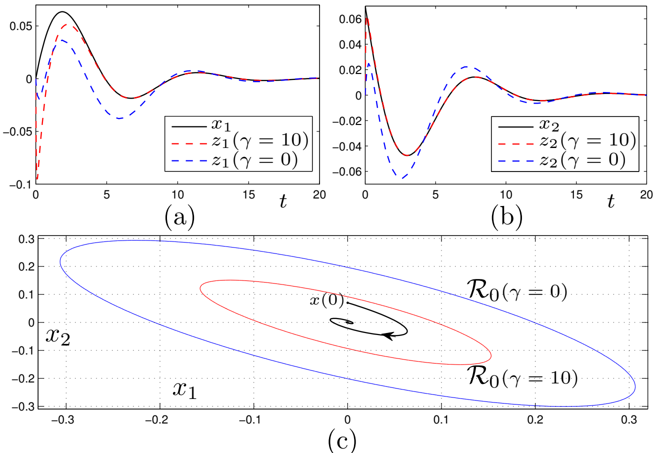

Fig. 1. Trajectories: (a) x_1, z_1, (b) x_2, z_2. (c) Estimated regions of attraction $\mathcal{R}_0$ for initial conditions considering $z(0) = 0$.

5. CONCLUDING REMARKS

This paper proposes LMI conditions for designing rational nonlinear observers for rational nonlinear systems with guaranteed cost performance. The measurement output can be expressed as a rational function of the states. The results guarantee convergence of the estimation error to zero for initial conditions inside of an estimated region of attraction. An example is used to illustrate the approach. The estimate of the region of attraction is based on a quadratic function of the error and a rational function of the system states. For this reason the proposed method requires the local asymptotic stability of the system whose states are to be estimated. The use of a quadratic Lyapunov function for the error dynamics introduces a certain conservatism in estimating the region of attraction. Although, the use of a rational Lyapunov function of the error seems to be possible, some technical difficulties arises and we are currently investigating this point. It is worth to emphasize that the optimization procedure presented in Section 3.1, is based on Trofino and Dezuo (2013) and is very effective for the characterization of regions of attraction (see Trofino and Dezuo (2013) for details). Other points of current research are the use of $\mathcal{H}_\infty$ performance requirement to the observer design and the inclusion of uncertain parameters in the system.

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