Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

On Networked Evolutionary Games Part 1: Formulation ⋆

Hongsheng Qi * **Daizhan Cheng** * **Hairong Dong** **

* *Key Laboratory of Systems and Control, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, P.R.China (e-mail: qihongsh@amss.ac.cn, dcheng@iss.ac.cn)*
** *Beijing Jiaotong University, Beijing 100044, P.R.China (e-mail: hrdong@bju.edu.cn)*

Abstract: This paper presents a comprehensive modeling technique for networked evolutionary games (NEG). Three kinds of network graphs are considered, which are (i) undirected graph for symmetric games; (ii) directed graph for asymmetric games, and (iii) d-directed graph for symmetric games with partial neighborhood information. Three kinds of fundamental evolutionary games (FEGs) are discussed, which are (i) two strategies and symmetric (S-2); (ii) two strategies and asymmetric (A-2); and (iii) three strategies and symmetric (S-3). Three strategy updating rules (SUR) are introduced, which are (i) Unconditional Imitation (UI); (ii) Fermi Rule (FR); (iii) Myopic Best Response Adjustment Rule (MBRA). Then we review the fundamental evolutionary equation (FEE), and give the detailed formulation for different models. Finally, the network profile dynamics (NPD) of NEGs are investigated via their FEE.

Keywords: Networked evolutionary game, fundamental evolutionary equation, network profile dynamics, semi-tensor product of matrices

1. INTRODUCTION

In the last four decades or so, the investigation of evolutionary games (EG) has attracted a great attention from scientists in cross disciplines, because evolutionary game has wide background from biological systems (Taylor & Jonker, 1978; Charnov, 1982), economical systems (Sugden, 1986), social systems (Ohtsuki et al., 2006), physical systems (Nowak & May, 1992), *etc.*

In recent researches, the topological relationship among players of an EG is mostly ignored. That is, assume each player gambles with all others. In many practical cases the situation is not like this. Therefore, in recent years the networked EG (NEG) becomes a hot topic. Roughly speaking, an NEG adds a graph with players as its nodes and sides describing the neighborhoods of each players. Then each player only gambles with its neighbors (Nowak & May, 1992; Szabo & Toke, 1998; Santos et al., 2008). Since there are no many proper tools to deal with NEG, most of the researches are based on either simulations or statistics.

Recently, the semi-tensor product (STP) has been proposed for investigating (Boolean and k-valued logical) networks and network-based games (Cheng et al., 2011, 2012a). There are many other interesting developments such as (i) topological structure of notworks (Fornasini & Valcher, 2013b; Hochma et al., 2013); (ii) controllability and control design of various kinds of control networks (Laschov & Margaliot, 2012; Li & Sun, 2011a; Zhang & Zhang, 2013); (iii) optimal control and game related optimization (Laschov & Margaliot, 2012); (iv) network stability and stabilization (Li et al., 2013b); (v) technique for reducing complexity (Zhao et al., 2013); and (vi) various applications to control and signal processing *etc.* (Wang et al., 2012; Xu & Hong, 2013), just to quote a few.

In a very recent work, the STP has also been used to the modeling, analysis and control design of the NEGs (Cheng et al., Preprint2013). This paper is a development of Cheng et al. (Preprint2013). It provided a comprehensive discussion for various NEGs. The NEGs discussed could have three different graphs (i) undirected graph, which is used for the NEGs with symmetric fundamental network games (FNG); (ii) directed graph, which is used for the NEGs with asymmetric FNGs; and (iii) d-directed graph, which is used for symmetric games with partial neighborhood information. Three kinds of FNGs are discussed, which are (i) each player has two strategies and the game is symmetric (S-2); (ii) each player has two strategies and the game is asymmetric (A-2); and (iii) each player has three strategies and the game is symmetric (S-3). Three strategy updating rules (SUR) are introduced, which are (i) Unconditional Imitation (UI); (ii) Fermi Rule (FR); (iii) Myopic Best Response Adjustment Rule (MBRA). Though most of widely discussed kinds of NEGs will be discussed in detail, the technique developed is applicable for other cases.

⋆ This work was supported in part by National Natural Science Foundation (NNSF) of China under Grants 61074114, 61273013, 61104065, 61374168, and 61333001.

Then we review the fundamental evolutionary equation (FEE) introduced in Cheng et al. (Preprint2013) and construct the FEEs for various types of NEGs.

For statement ease, some notations and basic concepts are introduced first.

- Notations:
  (i) $\mathcal{M}_{m\times n}$: the set of $m \times n$ real matrices.
  (ii) $\mathrm{Col}(M)$ ($\mathrm{Row}(M)$) is the set of columns (rows) of M. $\mathrm{Col}_i(M)$ ($\mathrm{Row}_i(M)$) is the i-th column (row) of M.
  (ii) $\mathcal{D}_k := \{1, 2, \cdots, k\}, \quad k \geq 2$.
  (iii) δ_n^i: the i-th column of the identity matrix I_n.
  (iv) $\Delta_n := \{\delta_n^i | i = 1, \cdots, n\}$.
  (v) $\Upsilon_k = \left\{(r_1, \cdots, r_k) \middle| r_i \geq 0,\ i = 1, \cdots, k;\ \sum_{i=1}^{k} r_i = 1\right\}$ is called the set of k-th dimensional probabilistic vectors.
  (vi) A matrix $L \in \mathcal{M}_{m\times n}$ is called a logical matrix if the set of columns of L, denoted by $\mathrm{Col}(L)$, are of the form of δ_m^k. That is,
  $$\mathrm{Col}(L) \subset \Delta_m.$$
  Denote by $\mathcal{L}_{m\times n}$ the set of $m\times n$ logical matrices.
  (vii) If $L \in \mathcal{L}_{n\times r}$, by definition it can be expressed as $L = [\delta_n^{i_1}, \delta_n^{i_2}, \cdots, \delta_n^{i_r}]$. For the sake of brevity, it is briefly denoted as
  $$L = \delta_n[i_1, i_2, \cdots, i_r].$$
  (viii) A matrix $L \in \mathcal{M}_{m\times n}$ is called a probabilistic matrix if the columns of L are m-dimensional probabilistic vectors. That is,
  $$\mathrm{Col}(L) \subset \Upsilon_m.$$
  The set of $m\times n$ probabilistic matrices is denoted by $\Upsilon_{m\times n}$.
  (ix) If $L \in \Upsilon_{m\times n}$, if $\mathrm{Col}(L) = C_1 \cup C_2$, where $C_1 \subset \Delta_m$ and $C_2 \subset \Upsilon_m \backslash \Delta_m$, and $|C_2| \ll |C_1|$. Then for notational compactness, we still use the shorthand
  $$L = \delta_m[i_1, i_2, \cdots, i_n],$$
  where if $\mathrm{Col}_k(L) = \delta_m^s \in C_1$, $i_k = s$, else if $\mathrm{Col}_k(L) \in C_2$, that is, $\mathrm{Col}_k(L) = (r_1, \cdots, r_m)$, we express i_k as
  $$i_k = 1/(r_1) + 2/(r_2) + \cdots + m/(r_m).$$
- Operators:
  (i) Semi-tensor product of matrices (Cheng et al., 2011, 2012a):
  Definition 1. Let $M \in \mathcal{M}_{m\times n}$ and $N \in \mathcal{M}_{p\times q}$, and $t = \mathrm{lcm}\{n, p\}$ be the least common multiple of n and p. The semi-tensor product of M and N, denoted by $M \ltimes N$, is defined as
  $$\left(M \otimes I_{t/n}\right)\left(N \otimes I_{t/p}\right) \in \mathcal{M}_{mt/n\times qt/p}, \qquad (1)$$
  where $\otimes$ is the Kronecker product.
  (ii) Khatri-Rao Product of matrices (Ljung & Söderström, 1982)
  Definition 2. Let $M \in \mathcal{M}_{p\times m}$, $N \in \mathcal{M}_{q\times m}$. Then the Khatri-Rao Product is defined as
  $$M * N = [\,\mathrm{Col}_1(M) \ltimes \mathrm{Col}_1(N) \cdots \mathrm{Col}_m(M) \ltimes \mathrm{Col}_m(N)] \in \mathcal{M}_{pq\times m}.$$
  Proposition 3. Let $X \in \mathbb{R}^m$ be a column and M is a matrix. Then
  $$X \ltimes M = (I_m \otimes M)\, X. \qquad (2)$$
  (iii) Swap matrix (Cheng et al., 2011, 2012a):
  Definition 4. A matrix $W_{[m,n]} \in \mathcal{M}_{m\times n}$, defined by
  $$\begin{aligned} W_{[m,n]} = \delta_{mn}[&1, m+1, \cdots, (n-1)m+1;\\ &2, m+2, \cdots, (n-1)m+2;\\ &\cdots;\\ &n, m+n, \cdots, nm], \end{aligned} \qquad (3)$$
  is called the (m, n)-dimensional swap matrix.
  The basic function of the swap matrix is to "swap" two vectors. That is,
  Proposition 5. Let $X \in \mathbb{R}^m$ and $Y \in \mathbb{R}^n$ be two columns. Then
  $$W_{[m,n]} \ltimes X \ltimes Y = Y \ltimes X. \qquad (4)$$

The rest of this paper is organized as follows: Section 2 presents a mathematical framework for NEGs. Three basic components of an NEG, namely, network graph, FNG, and SUR, are discussed in detail. Section 3 is devoted to the FEE, which plays a key role in the investigation of NEGs. FEEs of all players are building block for constructing strategy profile dynamics of the overall networks. Section 4 is a brief conclusion.

2. MATHEMATICAL FRAMEWORK FOR NEG

This section is a comprehensive description of mathematical framework of NEGs. The main idea of which was firstly proposed in Cheng et al. (Preprint2013).

Definition 6. A networked evolutionary game, denoted by $((N, E), G, \Pi)$, consists of three ingredients as:

(i) a network (graph) (N, E);
(ii) a fundamental network game (FNG), G, such that if $(i, j) \in E$, then i and j play the FNG with strategies $x_i(t)$ and $x_j(t)$ respectively.
(iii) a local information based strategy updating rule (SUR).

In the following we describe these three ingredients one by one.

2.1 Network Graph

We consider three kinds of network graphs.

(i) Undirected graph: $N = \{1, 2, \cdots, n\}$ $(n \leq \infty)$. It represents n players. If $(i, j) \in E$, then i is in the neighborhood of j, denoted by $i \in U(j)$. Simultaneously, $j \in U(i)$.
(ii) Directed graph: Note that the FNG is always played by two neighboring players. If the FNG is not symmetric, the directed edge is used to distinguish different roles of two players. Assume $(i, j) \in E$, *i.e.*, there is an edge from i to j, then in the game i is player 1 and j is player 2. Note that such directed graph does not affect the definition of neighborhoods.
(iii) D-directed graph: Assume the FNG is still symmetric, but the graph is not symmetric. That is, if $(i, j) \in E$, denoted by dot line arrow goes from i to j, it means

the information can go from i to j but not the other direction. In this case there exist $V(i) \subset U(i)$, such that player i can use only part of its neighborhood information, precisely, only the information from $V(i)$ can be used.

Definition 7. Consider an NEG with graph (N, E).

(i) If $(i, j) \in E$, then both $i \in U(j)$ and $j \in U(i)$ (no matter whether the graph is directed or not).
(ii) If there exist $\alpha_1, \cdots, \alpha_\lambda$, such that $i \in U(\alpha_1)$, $\alpha_1 \in U(\alpha_2)$, $\cdots$, $\alpha_s \in U(j)$, where $\lambda < s$, then i is said to be in the s-neighborhood of j, denoted by $i \in U_s(j)$.

Note that (i) if $i \in U_s(j)$, then $j \in U_s(i)$; (ii) if $i \in U_s(j)$, then $i \in U_h(j)$, $h > s$; (iii) it is assumed that $i \in U(i)$.

Definition 8. A graph is said to be homogeneous if the graph is undirected and each node has same degree, or the graph is directed and each node has same in-degree and same out-degree. If a graph is not homogeneous it is said to be heterogeneous.

The following example shows different kinds of network graphs.

Example 9. Assume there are 5 players. They form three kinds of graphs as shown in Fig. 1.

(i) A cycle of undirected network graph shown in Fig. 1 (a).
Consider the neighborhoods of 1:
$$U(1) = \{5, 1, 2\}; \quad U_2(1) = \{4, 5, 1, 2, 3\}.$$
(ii) A directed network graph shown in Fig. 1 (b).
Consider the neighborhoods of 1:
$$U(1) = \{1, 2\}; \quad U_2(1) = \{5, 1, 2, 3\};$$
$$U_3(1) = \{4, 5, 1, 2, 3\}.$$
(iii) D-directed network graph Fig. 1 (c).
Consider the neighborhoods of 1:
$$U(1) = \{1, 2, 3, 4, 5\}; \quad V(1) = \{1, 3, 4\}.$$

2.2 Fundamental Network Game

Definition 10. A fundamental network game (FNG) is a game with two players, *i.e.*, $N = (i, j)$, and each player has the same set of strategies as
$$S = S_i = S_j = (1, 2, \cdots, k).$$
It is symmetric if the payoff functions satisfy:
$$c_i(s_p, s_q) = c_j(s_q, s_p), \quad \forall s_p, s_q \in S.$$
Otherwise it is asymmetric.

In asymmetric case, only the directed graph can be used as the network graph.

The overall payoff of player i is assumed to be the average of its payoffs with all neighbors. Precisely,
$$c_i(t) = \frac{1}{|U(i)| - 1} \sum_{j \in U(i) \setminus i} c_{ij}(t), \quad i \in N. \tag{5}$$

An FNG is determined by two key factors: (i) k: the number of possible strategies; (ii) type: symmetric or asymmetric. So we classify the FNGs as:

- S-k: a symmetric game with k possible strategies;
- A-k: an asymmetric game with k possible strategies;

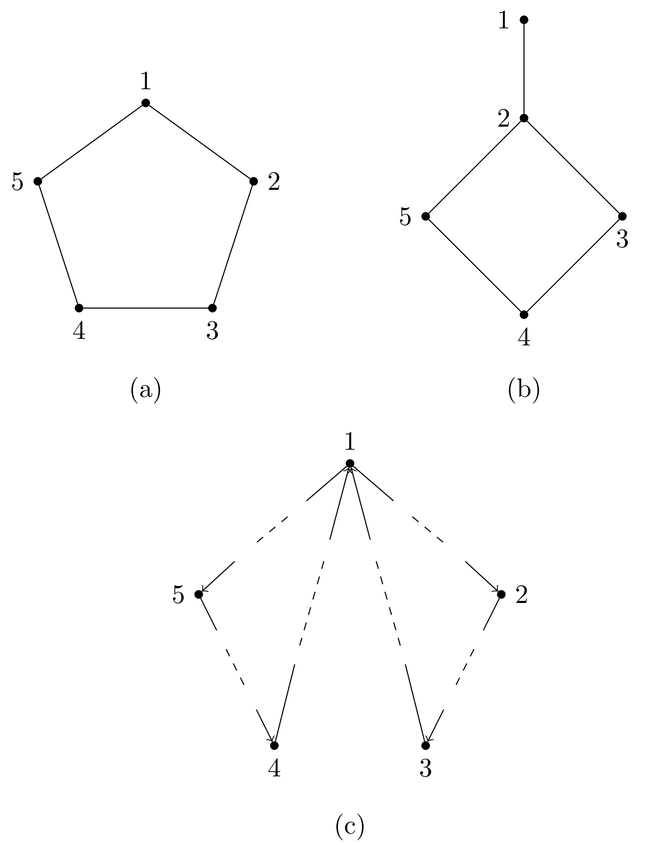

Fig. 1. Three kinds of network graphs

In the following example we collect some commonly used FNGs. More details about the practical meanings can be found in Rasmusen (2007); Smith (1982); Benoit & Krishna (1985).

Example 11. We consider three simplest kinds of FNGs as follows.

- S-2: The payoff bi-matrix of this kind of games is in Table 1.

Table 1. S-2 Games

$P_1 \backslash P_2$	1	2
1	$(R,\ R)$	$(S,\ T)$
2	$(T,\ S)$	$(P,\ P)$

It covers many well known games. For instance,
(1) if $2R > T + S > 2P$, it is the Game of Prisoner's Dilemma;
(ii) if $R = b - c$, $S = b - c$, $T = b$, $P = 0$, and $2b > c > b > 0$, it is the Snowdrift Game;
(iii) if $R = \frac{1}{2}(v - c)$, $S = v$, $T = 0$, $P = \frac{v}{2}$, and $v < c$, it is the Hawk-Dove Game.

- A-2: The payoff bi-matrix of this kind of games is in Table 2.

Table 2. A-2 Games

$P_1 \backslash P_2$	1	2
1	$(A,\ B)$	$(C,\ D)$
2	$(E,\ F)$	$(G,\ H)$

It also covers many well known games. For instance,

(i) if $A = H = a$, $B = G = b$, $C = D = E = F = 0$, and $a > b > 0$, it is the Battle of The Sexes;
(ii) if $E > A > C = D > B > 0 > F$, and $G = H = 0$, it is the Game of Boxed Pigs;
(iii) if $A = b$, $B = -b$, $C = b$, $D = -b$ $E = c$, $F = -c$, $G = a$, $H = -a$, and $a > b > c > 0$, it is the Game of Battle of the Bismark See,
(iv) if $A = D = F = G - a$, $B = C = E = H = a$, and $a \neq 0$, it is the Game of Matching the Pennies.

- S-3: The payoff bi-matrix of this kind of games is in Table 3.

Table 3. S-3 Games

$P_1\backslash P_2$	1	2	3
1	$(A,\ A)$	$(B,\ C)$	$(D,\ E)$
2	$(C,\ B)$	$(F,\ F)$	$(G,\ H)$
3	$(E,\ D)$	$(H,\ G)$	$(I,\ I)$

Some examples are
(1) if $A = F = I = 0$, $B = E = G = a$, $C = D = H = -a$, and $a \neq 0$, it is the Game of Rock-Scissor-Paper;
(ii) if $E = a$, $A = b$, $F = c$, $I = 0$, $B = G = H = D = d$, $C = e$, and $a > b > c > 0 > d > c$, it is the Benoit-Krishna Game.

2.3 Strategy Updating Rule

Denote by $x_i(t)$ the strategy of player i at time t. Then SUR is a rule which uses the local information to decide its next strategy. Precisely,

$$x_i(t+1) = f_i\left(\{x_j(t), c_j(t) \big| j \in U(i)\}\right),\ t \geq 0,\ i \in N. \tag{6}$$

There are some commonly used SURs.

- Unconditional Imitation (UI) (Nowak & May, 1992): The strategy of player i at time $t+1$, *i.e.*, $x_i(t+1)$, is selected as the best strategy from strategies of neighborhood players $j \in U(i)$ at time t. Precisely, if

$$j^* = \text{argmax}_{j \in U(i)}\, c_j(x(t)), \tag{7}$$

then

$$x_i(t+1) = x_{j^*}(t). \tag{8}$$

When the players with best payoff are not unique, say

$$\text{argmax}_{j \in U(i)}\, c_j(x(t)) := \{j_1^*, \cdots, j_r^*\}, \tag{9}$$

we may use the following 2 options:
(i) First Unconditional Imitation (UI-1): Choose one corresponding to a priority. For instance (as a default),

$$j^* = \min\{\mu \,|\, \mu \in \text{argmax}_{j \in U(i)}\, c_j(x(t))\}. \tag{10}$$

This method leads to a deterministic k-valued logical dynamics.
(ii) Second Unconditional Imitation (UI-2): Choose any one with equal probability. That is,

$$x_i(t+1) = x_{j_\mu^*}(t), \quad \text{with probability } p_\mu^i = \frac{1}{r}, \quad \mu = 1, \cdots, r. \tag{11}$$

This method leads to a probabilistic k-valued logical dynamics.

- Fermi Rule (FM) (Szabo & Toke, 1998; Traulsen et al., 2006). That is, randomly choose a neighborhood $j \in U(i)$. Comparing $c_j(t)$ with $c_i(t)$ to determine $x_i(t+1)$ as

$$x_i(t+1) = \begin{cases} x_j(t), & \text{with probability } p_t, \\ x_i(t), & \text{with probability } 1 - p_t, \end{cases} \tag{12}$$

where p_t is decided by the Fermi function

$$p_t = \frac{1}{1 + \exp(-\zeta(c_j(t) - c_i(t)))}.$$

The parameter $\zeta > 0$ can be chosen arbitrarily. For simplicity, throughout this paper we set $\zeta = \infty$. Then

$$p_t = \begin{cases} 1, & c_j(t) > c_i(t), \\ 0, & c_j(t) \leq c_i(t). \end{cases}$$

This method leads to a probabilistic k-valued logical dynamics.

- Myopic Best Response Adjustment Rule (MBRA) (Young, 1993): Assume

$$\begin{aligned} &c_i\left(x_i = x^*, x_j = x_j(t),\ j \in U(i)\backslash\{i\}\right) \\ &= \max_{x \in S} c_i\left(x_i = x, x_j = x_j(t),\ j \in U(i)\backslash\{i\}\right), \end{aligned} \tag{13}$$

then we choose

$$x_i(t+1) = x^*. \tag{14}$$

When the strategies with best payoff are not unique, say, the set of best strategies is

$$S^* = \{x_1^*, \cdots, x_r^*\} \subset S, \tag{15}$$

we may use the following 2 options:
(i) First MBRA (MBRA-1): Choose one corresponding to a priority. For instance (as a default),

$$x_i(t+1) = \min\{x_j^* \in S^*\}. \tag{16}$$

This method leads to a deterministic k-valued logical dynamics.
(ii) Second MBRA (MBRA-2): Choose one with equal probability for best strategies. That is,

$$x_i(t+1) = x_j^*(t), \quad \text{with probability } p_\mu^i = \frac{1}{r}, \quad \mu = 1, \cdots, r. \tag{17}$$

This method leads to a probabilistic k-valued logical dynamics.

3. FUNDAMENTAL EVOLUTIONARY EQUATION

Observing equation (6), since $c_j(t)$ depends on $U(j)$ and $U(j) \subset U_2(i)$, (6) can be rewritten as

$$x_i(t+1) = f_i\left(\{x_j(t) \big| j \in U_2(i)\}\right),\ t \geq 0,\ i = 1, 2, \cdots, n. \tag{18}$$

We call (18) the fundamental evolutionary equation (FEE). One sees easily that the overall network evolutionary dynamics, called the network profile dynamics (NPD), is completely determined by the FEEs. It is also obvious that the FEEs are determined by network graph, FNG, and SUR. In the following we use some examples to depict the procedure for constructing FEEs.

Example 12. Consider an NEG. Assume the network graph is Fig.1 (a); FNG is S-2 with $R = S = -1$, $T = 2$, $P = 0$ (the game of Snowdrift). SUR is UI-1. Then the FEE can be determined via Table 4.

Table 4. Payoffs → Dynamics (Example 12)

Profile	11111	11112	11121	11122	11211	11212	11221	11222
c_{i-1}	-1	-1	-1	-1	-1	-1	-1	-1
c_i	-1	-1	-1	-1	2	2	1	1
c_{i+1}	-1	-1	2	1	-1	-1	1	0
f_i	1	1	2	2	2	2	2	2
$\vdots$								
Profile	22111	22112	22121	22122	22211	22212	22221	22222
c_{i-1}	1	1	1	1	0	0	0	0
c_i	-1	-1	-1	-1	1	1	0	0
c_{i+1}	-1	-1	2	1	-1	-1	1	0
f_i	2	2	2	2	2	2	2	2

Now identify $1 \sim \delta_2^1$ and $2 \sim \delta_2^2$, then we can express f into its algebraic form as

$$x_i(t+1) = M_f x_{i-2} x_{i-1} x_i x_{i+1} x_{i+2}, \quad i = 1,2,3,4,5, \tag{19}$$

where

$$\begin{aligned} M_f = \delta_2[&1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2 \\ &1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2]. \end{aligned} \tag{20}$$

Note that we use (mod 5) notation. That is, $x_0 = x_5$, $x_{-1} = x_4$ and so on.

Example 13. Assume network graph is Fig. 1 (b); FNG is A-2 with $A = H = 2$, $B = G = 1$, $E = F = C = D = 0$ (the game of Battle of The Sexes). SUR is Fermi Rule (FM). Since the game is not symmetric, the FEEs for individual nodes are different. We need to work out them one by one.

$$x_i(t+1) = M_i x_1 x_2 x_3 x_4 x_5 := M_i x, \quad i = 1,2,3,4,5, \tag{21}$$

where $x = \ltimes_{i=1}^5 x_i$ and M_i will be calculated in the following two steps:

(i) If the profile is known, then the payment for each player is known. For instance, if the profile

$$(x_1, x_2, x_3, x_4, x_5) = (1\ 1\ 2\ 2\ 2),$$

Then it is easy to calculate that

$$\begin{aligned} c_1 &= 2, \\ c_2 &= \frac{1}{3}(1+0+0) = \frac{1}{3}, \\ c_3 &= \frac{1}{2}(0+1) = \frac{1}{2}, \\ c_4 &= \frac{1}{2}(2+2) = 2, \\ c_5 &= \frac{1}{2}(0+1) = \frac{1}{2}. \end{aligned}$$

(ii) Comparing c_1 with c_2, we have $f_1 = x_1 = 1$. As for f_2 we have three choices:

$$\begin{aligned} j = 1 &\Rightarrow f_2 = x_1 = 1, \\ j = 3 &\Rightarrow f_2 = x_3 = 2, \\ j = 5 &\Rightarrow f_2 = x_5 = 2. \end{aligned}$$

Hence $f_2 = 1$ with probability $\frac{1}{3}$ and $f_2 = 2$ with probability $\frac{2}{3}$. We briefly express this by $f_2 = (\frac{1}{3}, \frac{2}{3})$.

Similarly, we can calculate f_i as in Table 5.

Finally, we have

$$\begin{aligned} M_1 = \delta_2[&1,1,1,1,1,1,1,1,1,2,1,2,2,2,2,2 \\ &1,1,1,1,1,2,1,2,2,2,2,2,2,2,2,2], \end{aligned} \tag{22}$$

$$\begin{aligned} M_2 = \delta_2[&1,1,1,1,1,1,1,(\tfrac{1}{3},\tfrac{2}{3}),(\tfrac{2}{3},\tfrac{1}{3}),(\tfrac{1}{3},\tfrac{2}{3}),2,2,(\tfrac{1}{3},\tfrac{2}{3}),2, \\ &2,2,1,1,1,1,1,1,1,(\tfrac{1}{3},\tfrac{2}{3}),(\tfrac{2}{3},\tfrac{1}{3}),2,2,2,2,2,2,2], \end{aligned} \tag{23}$$

Table 5. Payoffs → Dynamics (Example 13)

Profile	11111	11112	11121	11122	11211	11212	11221	11222
c_1	2	2	2	2	2	2	2	2
c_2	5/3	1	5/3	1	1	$\frac{1}{3}$	1	$\frac{1}{3}$
c_3	3/2	1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$	$\frac{1}{2}$
c_4	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	2
c_5	3/2	0	$\frac{1}{2}$	$\frac{1}{2}$	3/2	0	$\frac{1}{2}$	$\frac{1}{2}$
f_1	1	1	1	1	1	1	1	1
f_2	1	1	1	1	1	1	1	$(\frac{1}{3},\frac{2}{3})$
f_3	1	1	1	$(\frac{1}{2},\frac{1}{2})$	1	$(\frac{1}{2},\frac{1}{2})$	$(\frac{1}{2},\frac{1}{2})$	2
f_4	1	1	1	2	1	1	2	2
f_5	1	1	1	$(\frac{1}{2},\frac{1}{2})$	1	$(\frac{1}{2},\frac{1}{2})$	1	2
$\vdots$								
Profile	22111	22112	22121	22122	22211	22212	22221	22222
c_1	1	1	1	1	1	1	1	1
c_2	$\frac{2}{3}$	1	$\frac{2}{3}$	1	1	4/3	1	4/3
c_3	1	1	0	0	1	1	3/2	3/2
c_4	1	$\frac{1}{2}$	0	1	$\frac{1}{2}$	0	1	2
c_5	1	1	0	3/2	1	1	0	3/2
f_1	2	2	2	2	2	2	2	2
f_2	$(\frac{2}{3},\frac{1}{3})$	2	2	2	2	2	2	2
f_3	1	1	$(\frac{1}{2},\frac{1}{2})$	2	2	2	2	2
f_4	1	$(\frac{1}{2},\frac{1}{2})$	2	2	$(\frac{1}{2},\frac{1}{2})$	2	2	2
f_5	1	2	$(\frac{1}{2},\frac{1}{2})$	2	1	2	2	2

$$\begin{aligned} M_3 = \delta_2[&1,1,1,(\tfrac{1}{2},\tfrac{1}{2}),1,(\tfrac{1}{2},\tfrac{1}{2}),(\tfrac{1}{2},\tfrac{1}{2}),2,1,1,1,2,2,2,2,2 \\ &1,1,1,(\tfrac{1}{2},\tfrac{1}{2}),1,2,(\tfrac{1}{2},\tfrac{1}{2}),2,1,1,(\tfrac{1}{2},\tfrac{1}{2}),2,2,2,2,2], \end{aligned} \tag{24}$$

$$\begin{aligned} M_4 = \delta_2[&1,1,1,2,1,1,2,2,1,(\tfrac{1}{2},\tfrac{1}{2}),2,2,(\tfrac{1}{2},\tfrac{1}{2}),2,2,2 \\ &1,1,1,2,1,1,2,2,1,(\tfrac{1}{2},\tfrac{1}{2}),2,2,(\tfrac{1}{2},\tfrac{1}{2}),2,2,2], \end{aligned} \tag{25}$$

$$\begin{aligned} M_5 = \delta_2[&1,1,1,(\tfrac{1}{2},\tfrac{1}{2}),1,(\tfrac{1}{2},\tfrac{1}{2}),1,2,1,2,1,2,1,2,2,2 \\ &1,1,1,(\tfrac{1}{2},\tfrac{1}{2}),1,2,(\tfrac{1}{2},\tfrac{1}{2}),2,1,2,(\tfrac{1}{2},\tfrac{1}{2}),2,1,2,2,2]. \end{aligned} \tag{26}$$

Example 14. Assume network graph is Fig. 1 (c); FNG is S-3 with $A = F = I = 0$, $B = E = G = 1$, $C = D = H = -1$ (the Game of Rock-Scissor-Paper), SUR is MBRA (It is easy to check that for this example MBRA-1 and MBRA-2 lead to the same FEE). We emphasize only one thing: based on the definition, the neighborhood for a player to play with maybe different from the neighborhood from which he can get of information. For instance, player 1 plays with players 2, 3, 4, and 5, but he can only get the information from 3 and 4. So

(i) if $(x_3(t) = 2) \cap (x_4(t) = 2)$ or $(x_3(t) = 2) \cap (x_4(t) = 1)$ or $(x_3(t) = 1) \cap (x_4(t) = 2)$, then

$$x_1(t+1) = 1;$$

(ii) if $(x_3(t) = 3) \cap (x_4(t) = 3)$ or $(x_3(t) = 3) \cap (x_4(t) = 2)$ or $(x_3(t) = 2) \cap (x_4(t) = 3)$, then

$$x_1(t+1) = 2;$$

(iii) if $(x_3(t) = 1) \cap (x_4(t) = 1)$ or $(x_3(t) = 1) \cap (x_4(t) = 3)$ or $(x_3(t) = 3) \cap (x_4(t) = 1)$, then

$$x_1(t+1) = 3.$$

We skip the detailed computation process and present the results as follows.

$$f_i(t+1) = M_i x(t), \quad i = 1,2,3,4,5, \tag{27}$$

where

$$
\begin{aligned}
M_1 = \delta_3[\ & 3,3,3,1,1,1,3,3,3,1,1,1,1,1,1,2,2,2,3,3,3,2,2,2,2\\
& 2,2,3,3,3,1,1,1,3,3,3,1,1,1,1,1,1,2,2,2,3,3,3,2,2\\
& 2,2,2,2,3,3,3,1,1,1,3,3,3,1,1,1,1,1,1,2,2,2,3,3,3\\
& 2,2,2,2,2,2,3,3,3,1,1,1,3,3,3,1,1,1,1,1,1,2,2,2,3\\
& 3,3,2,2,2,2,2,2,3,3,3,1,1,1,3,3,3,1,1,1,1,1,1,2,2\\
& 2,3,3,3,2,2,2,2,2,2,3,3,3,1,1,1,3,3,3,1,1,1,1,1,1\\
& 2,2,2,3,3,3,2,2,2,2,2,2,3,3,3,1,1,1,3,3,3,1,1,1,1\\
& 1,1,2,2,2,3,3,3,2,2,2,2,2,2,3,3,3,1,1,1,3,3,3,1,1\\
& 1,1,1,1,2,2,2,3,3,3,2,2,2,2,2,2,3,3,3,1,1,1,3,3,3\\
& 1,1,1,1,1,1,2,2,2,3,3,3,2,2,2,2,2,2];
\end{aligned}
$$

$$
\begin{aligned}
M_2 = \delta_3[\ & 3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3\\
& 3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3\\
& 3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3\\
& 3,3,3,3,3,3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\\
& 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\\
& 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\\
& 1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2\\
& 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\
& 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\
& 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2];
\end{aligned}
$$

$$
\begin{aligned}
M_3 = \delta_3[\ & 3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3\\
& 3,3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\\
& 1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\
& 2,2,2,2,2,2,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3,3\\
& 3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1\\
& 1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2\\
& 2,2,2,2,2,2,2,2,2,2,2,2,3,3,3,3,3,3,3,3,3,3,3,3,3\\
& 3,3,3,3,3,3,3,3,3,3,3,3,3,3,1,1,1,1,1,1,1,1,1,1,1\\
& 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,2,2,2,2,2,2,2\\
& 2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2,2];
\end{aligned}
$$

$$
\begin{aligned}
M_4 = \delta_3[\ & 3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3\\
& 1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1\\
& 2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2\\
& 3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3\\
& 1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1\\
& 2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2\\
& 3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3\\
& 1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1\\
& 2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2\\
& 3,2,3,1,1,1,1,2,2,3,2,3,1,1,3,2,2,2,3,2,3,1,1,3,2\\
& 2,2,3,2,3,1,1,3,2,2,2,3,2,3,1,1,3,2,2,2,3,2,3,1,1\\
& 3,2,2,2,3,2,3,1,1,3,2,2,2,3,2,3,1,1,3,2,2,2,3,2,3\\
& 3,1,2,3,1,2,3,1,2,3,1,2,3,1,2,3,1,2];
\end{aligned}
$$

$$M_5 = M_2.$$

4. CONCLUSION

This paper gives a comprehensive introduction for the modeling of networked evolutionary games. After an introduction to NEGs and to semi-tensor product, the NEG is formulated as a triplet: (i) network graph; (ii) fundamental network game; (iii) strategy updating rule. A detailed discussion with some illustrative examples are presented. Then the fundamental evolutionary equation (FEE) is proposed, which determines the overall network dynamics. The formulas to calculate FEE according to the triplet of NEG are obtained.

REFERENCES

J.P. Benoit, V. Krishna (1985). Finitely repeated games, *Econometrica*, vol. 17, no. 4, pp. 317-320.

E.L. Charnov (1982). *The Theory of Sex Allocation*, Princiton Univ. Press, New Jersey.

D. Cheng, H. Qi, Z. Li (2011). *Analysis and Control of Boolean Networks: A Semi-tensor Product Approach*, Springer, London, 2011.

D. Cheng, H. Qi, Y. Zhao (2012a). *An Introduction to Semi-tensor Product of Matrices and Its Applications*, World Scientific, Singapore.

D. Cheng, F. He, H. Qi, T. Xu (Preprint2013). Modeling, analysis and control of networked evolutionary games, `http://lsc.amss.ac.cn/~dcheng/preprint/NTGAME02.pdf` (IEEE TAC, provisionally accepted).

E. Fornasini, M.E. Valcher (2013b). Observability, reconstructibility and state observers of Boolean control networks, *IEEE Trans. Aut. Contr.*, vol. 58, no. 6, pp. 1390-1401.

G. Hochma, M. Margaliot, E. Fornasini, M.E. Valcher (2013). Symbolic dynamics of Boolean control networks, *Automatica*, vol. 49, no. 8, pp. 2525-2530. .

D. Laschov, M. Margaliot (2012). Controllability of Boolean control networks via the Perron-Frobenius theory, *Automatica*, vol. 48, no. 6, pp. 1218-1223.

F. Li, J. Sun (2011a). Controllability of Boolean control networks with time delays in states, *Automatica*, vol. 47, no. 3, pp. 603-607.

R. Li, M. Yang, T. Chu (2013b). State feedback stabilization for Boolean control networks, *IEEE Trans. Aut. Contr.*, vol. 58, no. 7, pp. 1853-1857.

Z. Liu, Y. Wang (2012). Disturbance decoupling of mix-valued logical networks via the semi-tensor product method, *Automatica*, vol. 48, no. 8, pp. 1839-1844.

L. Ljung, T. Söderström (1982). Theory and Practice of Recursive Identification, MIT Press.

M.A. Nowak, R.M. May (1992). Evolutionary games and spatial chaos, *Nature*, vol. 359, pp. 826-829..

H. Ohtsuki, C. Hauert, E. Lieberman, M.A. Nowak (2006). A simple rule for the evolution of cooperation on graphs and social networks, *Nature*, vol. 441, pp. 502-505.

E. Rasmusen (2007). *Games and Information: An Introduction to Game Theory*, 4th Ed., Basil Blackwell, Oxford.

F.C. Santos, M.D. Santos, J.M. Pacheco (2008). Social diversity promotes the emergence of cooperation in public goods games, *Nature*, vol. 454, pp. 213-216.

J.M. Smith (1982). *Evolution and the Theorem of Games*, Cambridge Univ. Press, Cambridge.

R. Sugden (1986). *The Economics of Rights, Cooperation and Welfre*, Blackwwell, Oxford.

G. Szabo, C. Toke (1998). Evolutionary prisoner's dilemma game on a square lattice, *Phys. Rev. E*, vol. 58, pp. 69.

Pl Taylor, L. Jonker (1978). Evolutionary stable strategies and game dynamics, *Math. Biosci.*, vol. 16, pp. 76-83.

A. Traulsen, M.A. Nowak, J.M. Pacheco (2006). Stochastic dynamics of invasion and fixation, *Phys. Rev., E*, vol. 74, pp. 011909.

Y. Wang, C. Zhang, Z. Liu (2012). A matrix approach to graph maximum stable set and coloring problems with application to multi-agent systems, vol. 48, no. 7, pp. 1227-1236.

X. Xu, Y. Hong (2013). Matrix approach to model matching of asynchrollous sequential machines, *IEEE Trans. Aut. Contr.*, vol. 58, no. 11, pp. 2974-2979.

H.P. Young (1993). The evolution of conventions, *Econometrica*, vol. 61, pp. 57-84.

L. Zhang, K. Zhang (2013). L_2 stability, H_∞ control of switched homogeneous nonlinear systems and their semi-tensor product of matrices representation, *Int. J. Robus. Nonli. Contr.*, vol. 23, pp. 638-652.

Y. Zhao, J. Kim, M. Filippone (2013). Aggregation algorithm towards large-scale Boolean network analysis, *IEEE Trans. Aut. Contr.*, vol. 58, no. 8, pp. 1976-1985.