Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

The relaxed asymmetric Reeds–Shepp problem ⋆

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Abstract: The relaxed asymmetric Reeds–Shepp problem is the problem of finding a minimum-time trajectory for the Reeds–Shepp vehicle from a given initial state to a given final position, under the assumption that the backward motion of the vehicle is slower than the forward motion. This modification of the classic Reeds–Shepp problem aims at better capturing physical reality for vehicles that are less agile when moving backwards than when moving forwards—marine vehicles being a representative example—and leads to new insights into the motion planning of such vehicles. Specifically, it is shown that a family of optimal trajectories which is known to be sufficient for the relaxed Reeds–Shepp problem is also sufficient for the relaxed asymmetric Reeds–Shepp problem. It is also shown that the relaxed Reeds–Shepp problem and the relaxed asymmetric Reeds–Shepp problem cannot have the same minimum-time synthesis.

1. INTRODUCTION

The motivation behind the present work stems from motion planning problems for uninhabited surface vehicles (USVs) and, more specifically, marine vehicles. Autonomous marine missions are becoming increasingly important and challenging. Among other tasks, they include border patrolling, sea sampling, ocean debris tracking, and damage assessment after natural disasters such as hurricanes and tsunamis. The autonomous capabilities of uninhabited vehicles have matured over the past decade, thus the extensive use of USVs for marine missions is a natural path ahead. Especially in safety-critical rescue and relief operations.

Fast motion planning is an integral component of autonomous systems of vehicles and requires simple and effective kinematic models, along with robust and well understood algorithms. Many of these tools and algorithms, for some of the most common kinematic models, have been developed and thoroughly studied by mathematicians before their importance for path planning of uninhabited vehicles became ubiquitous. The well known Dubins paths [Dubins, 1957] are widely used as a basic tool for motion planning of aerial vehicles that can only move forwards, whereas the Reeds–Shepp paths [Reeds and Shepp, 1990] provide a convenient alternative for motion planning related to vehicles that can move both forwards and backwards.

Assuming that $\ell^{\pm}$, $r^{\pm}$, and $s^{\pm}$ denote left turns, right turns, and motion on a straight line, respectively, and the signs correspond to forward (plus sign) and backward (minus sign) of motion, then the main result in [Reeds and Shepp, 1990] states that a sufficient family of shortest paths between two configurations of a vehicle (a configuration being the position and orientation of a vehicle) consists of 48 words in $\ell^{\pm}$, $r^{\pm}$, and $s^{\pm}$, each word contains at most five letters, and there can be at most two points of direction reversal, that is, points where the vehicle changes from forward to backward motion or vice versa. In Sussmann and Tang [1991], the classification of the Reeds–Shepp paths is made systematic within the framework of geometric and optimal control theory, and further improved upon by eliminating two words ($\ell^-\ell^+\ell^-$ and $r^-r^+r^-$) from the sufficient family of optimal paths. The resulting 46 words represent the so-called Reeds–Shepp paths. These results reduce the basic motion planning problem of finding a shortest path between two given configurations of a vehicle to the computation of 46 paths and the selection of the shortest among them. This procedure is further refined in [Souères and Laumond, 1996] where a minimum-time "synthesis" is constructed in the sense that the state space $\mathbb{R}^2 \times \mathbb{S}^1$ of the Reeds–Shepp vehicle is partitioned into sets of initial conditions according to the type of Reeds–Shepp path that connects each initial condition to a given fixed configuration.

When the terminal constraint for the Reeds–Shepp vehicle is a position with unspecified orientation, as opposed to a configuration—in other words, if the terminal constraint in the optimal control formulation of the problem is a submanifold of $\mathbb{R}^2 \times \mathbb{S}^1$ diffeomorphic to $\mathbb{S}^1$—then a transversality condition at the final end-time complements the necessary conditions of the Maximum Principle and the sufficient family of optimal paths is significantly shortened and simplified [Souères et al., 1994]. The elements of the reduced sufficient family are called "relaxed Reeds–Shepp paths", in view of the relaxation of the terminal constraint.

Among other reasons, marine vehicles are inadequately described by the Reeds–Shepp kinematic model because, although these vehicles can move forwards and backwards, their maximum backward speed is smaller than their maximum forward speed. Therefore, motion planning based on the Reeds–Shepp paths will not be optimal in this case. The purpose of this work is to solve the relaxed

⋆ This research was partially supported by the Israeli government.

Reeds–Shepp problem (i.e., when the final orientation in unconstrained) for a kinematic model that accounts for differences in speed between forward and backward motion. The resulting algorithms may be applicable not only for the motivating example of the marine uninhabited vehicles, but also for some types of ground vehicles exhibiting the same kinematic constraint. For example, if the backwards-looking sensors of a ground vehicle have a shorter obstacle detection range than the forwards-looking sensors, the backward speed may be artificially limited to allow for sufficient reaction time if an unexpected obstacle is encountered. The main difficulty to be addressed is the lack of symmetry between forward and backward motion, a property extensively used in all previous work on the Reeds–Shepp problem. For example, if an optimal path contains a cusp, i.e., if there is a point on the path where the vehicle reverses its motion, then the circular arcs that form the cusp will be of different radii. This asymmetry complicates the analysis of the optimal paths.

The rationale behind the study of modifications of the classic Dubins kinematic model is to obtain more realistic representations of real-world scenarios and it has lead to several variations in the literature. It is worth noting that, from a mathematical point of view, many of these variations—for example, all the kinematic models mentioned thus far—amount to simply changing the set of control values or a boundary condition, however the practical implications are non-negligible. For example, the simpler form of the relaxed Reeds–Sheep paths makes path planning computationally simpler than with Reeds–Shepp paths. In this line of research, further generalisations have appeared in [Bakolas and Tsiotras, 2011, 2013; Dolinskaya and Maggiar, 2012]. In [Bakolas and Tsiotras, 2011] an optimal synthesis (in the sense of [Souères and Laumond, 1996] described above) is constructed for a Dubins vehicle that turns left or right with different minimum turning radii. An optimal synthesis that accounts for the presence of a constant drift field is presented in [Bakolas and Tsiotras, 2013], whereas time-optimal trajectories when the maximum speed and the minimum turning radius are functions of the orientation of the vehicle are analysed in [Dolinskaya and Maggiar, 2012].

2. PROBLEM STATEMENT

Consider the driftless control system

$$\gamma'(t) = u_1(t)X^1(\gamma(t)) + u_2(t)X^2(\gamma(t)) \tag{1}$$

on $M = \mathbb{R}^2 \times \mathbb{S}^1$, where $X^1, X^2 \in \Gamma^\omega TM$ are the real analytic vector fields with coordinate representations

$$X^1(x^1, x^2, x^3) = (\cos x^3, \sin x^3, 0)$$

and

$$X^2(x^1, x^2, x^3) = (0, 0, 1),$$

in the chart on TM induced by the chart $(V, \phi) = (\mathbb{R}^2 \times \mathbb{S}^1 \setminus \{(-1, 0)\}, \big((x^1, x^2), (y^1, y^2)\big) \mapsto (x^1, x^2, x^3 = \operatorname{atan}(y^2/y^1)))$ on M.[1] If I is an interval in $\mathbb{R}$, we denote by $L^1_{\mathrm{loc}}(I; \mathbb{R}^n)$ the locally integrable maps from I into $\mathbb{R}^n$, and by $W^{1,1}_{\mathrm{loc}}(I; M)$ the locally absolutely continuous curves from I into M. Then, the class $\mathcal{A}$ of **admissible controls** for (1) consists of maps $u(t) = (u_1(t), u_2(t))$ in $L^1_{\mathrm{loc}}(I; \mathbb{R}^n)$ that take values in the set $U \triangleq \{-c, 1\} \times [-1/\rho, 1/\rho] \subset \mathbb{R}^2$, where c and ρ are fixed, positive, real numbers such that $c \in]0, 1[$ and $\rho > 0$. Given a control u, the corresponding **trajectory** of (1) is an element of $W^{1,1}_{\mathrm{loc}}(I; M)$ and is denoted by γ. The projection on $\mathbb{R}^2$ of (the image of) a trajectory γ will be called the **path** that corresponds to γ. An **optimal path** is the projection of an optimal trajectory.

[1] To cover the entire state space M, i.e., the plane not covered by ϕ, a second chart can be chosen in an obvious manner.

The control system (1) can be viewed as the kinematic model of a vehicle that moves forwards with constant unit speed and backwards with constant speed equal to c, along a planar curve whose curvature is bounded above by $|u_2/u_1|$. Therefore, when the vehicle moves forwards, the maximum curvature (resp. minimum turning radius) is $1/\rho$ (resp. ρ), whereas, when it moves backwards, the maximum curvature is $1/c\rho$ (resp. $c\rho$). In this paper, we are interested in the problem of determining a sufficient family of minimum-time trajectories from an initial state $p \in M$ to a terminal submanifold of the form $\{(x_f, y_f)\} \times \mathbb{S}^1 \subset M$. The physical interpretation is that the vehicle has an initial position and orientation and has to reach a final location at which the orientation is unconstrained. We call this problem the "relaxed asymmetric Reeds–Shepp problem" (RARSP); a precise formulation is as follows.

RARSP: *Given a point $p \in M$ and a submanifold $N = \{(x_f, y_f)\} \times \mathbb{S}^1 \subset M$, with $p \notin N$, minimise the time $T > 0$ over the set of trajectories $\gamma \in W^{1,1}_{\mathrm{loc}}([0, T]; M)$ of (1) that satisfy $\gamma(0) = p$ and $\gamma(T) \in N$.*

We show that, although forward and backward motions are not symmetric in the sense that the corresponding paths have different bounds on their curvature, there is a sufficient family of minimum-time paths which is identical to that of the classic Reeds–Shepp problem Souères et al. [1994]; Boissonnat et al. [1991]; Sussmann and Tang [1991]. However, for a given initial state and a given final position of the vehicle, a minimum-time path of (1) may differ substantially from a minimum-time path of the classic Reeds–Shepp vehicle.

It is standard practise in motion planning to describe a sufficient family of optimal paths as a collection of words from an alphabet that represents the motion primitives. The assignment of symbols to motion primitives for the relaxed asymmetric Reeds–Shepp problem is shown in Figure 1. The presence of a plus sign implies a circular arc of radius ρ, whereas the presence of a minus sign implies that the radius of a circular arc is $c\rho$. If a circular arc can correspond to either a left or right turn, it is denoted by C. Similarly, if a straight line segment corresponds to either forward or backward motion, it is denoted by S (e.g., $S^\pm$, $S^\mp$). A subscript to any one of the letters ℓ, r, and s corresponds to the length of the curve represented by the letter.

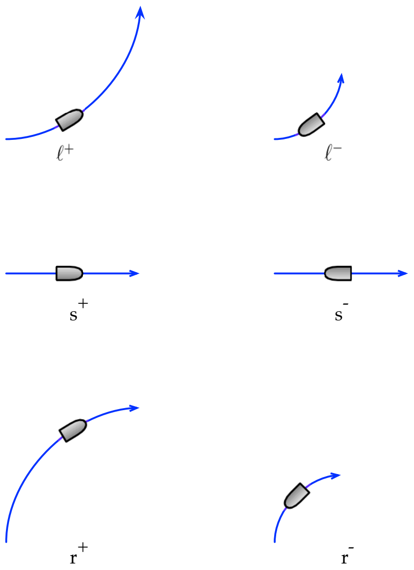

Fig. 1. Motion primitives for the relaxed asymmetric Reeds–Shepp vehicle.

3. EXISTENCE OF SOLUTIONS

Without loss of generality, we take $\rho = 1$ in the sequel. To guarantee the existence of solutions to the RARSP, we take the set of control values for (1) to be the convex hull $\mathrm{co}\,(U)$ of U, that is, $\mathrm{co}\,(U) = [-c, 1] \times [-1, 1]$. When the set of control values is $\mathrm{co}\,(U)$, the control system (1) is denoted by $(\mathrm{co}\Sigma)$ and the set of admissible controls by $\mathrm{co}\mathcal{A}$. The set $\mathrm{co}\,(U)$ is convex and compact and, therefore, classic results from Optimal Control theory guarantee the existence of optimal trajectories [Young, 1980]. However, it has to be shown that no additional extremals are introduced by doing so and this can be achieved by showing that (1) is small-time locally controllable. Small-time local controllability (STLC) of (1) implies that the minimum-time functions of (1) coincide when the set of control values is either U or $\mathrm{co}\,(U)$ [Sussmann and Tang, 1991]. In Sussmann and Tang [1991], c is equal to 1 and STLC follows from the fact that, in this case, (1) is symmetric and satisfies the Lie algebra rank condition [Nijmeijer and van der Schaft, 1990]. In our case, $c \in]0, 1[$ and symmetry of (1) does not follow directly from the form of the set of control values. We can show, however, that (1) is STLC by means of a different argument. As an example of a driftless system which is STLC, but whose set of control values is not symmetric, system (1) is, perhaps, of independent interest for theoretical investigations of such properties.

Proposition 1. The control system (1) is small-time locally controllable.

Proof. To prove that (1) is STLC, we consider yet another control set, namely $\tilde{U} = \{-c, c\} \times [-c, c]$. The control system (1) is denoted by $(\tilde{\Sigma})$ when the set of control values is $\tilde{U}$, and it is STLC because it is symmetric and satisfies the Lie algebra rank condition. The goal is to show that every point q in M that can be reached from a given point $p \in M$ by a trajectory of $(\tilde{\Sigma})$ can also be reached by a trajectory of (1) in less time. First we recall the fact that $(\tilde{\Sigma})$ is STLC using measurable controls if and only if it is STLC using piecewise constant controls and, therefore, we restrict attention to piecewise constant controls without loss of generality. More specifically, let $T > 0$ be a positive real number and suppose that $p \in M$ and q is in the reachable set $R_{\tilde{\Sigma}}(p; [0, T[; \mathcal{A})$ from p using admissible controls defined on $[0, T]$. Then, q can be reached from p by means of a piecewise constant control [Grasse, 1992]

$$\tilde{u}^{p,q}_{\mathrm{pwc}} : [0, \tilde{T}] \to \{-c, c\} \times [-c, c].$$

Note that, by assumption, q can be reached from p in time less than T, therefore, $\tilde{T} < T$ (and, of course, the control can be extended to $[0, T]$ by setting $(\tilde{u}^{p,q}_{\mathrm{pwc},1}, \tilde{u}^{p,q}_{\mathrm{pwc},2}) = (0, 0)$ on $[\tilde{T}, T]$). From $\tilde{u}^{p,q}_{\mathrm{pwc}}$, we can construct a control

$$u^{p,q}_{\mathrm{pwc}} : [0, S] \to \{-c, 1\} \times [-1, 1],$$

where $0 < S < \tilde{T} < T$, that steers p to q in time less than T and is admissible for the control system of interest (1) since it takes values in $\{-c, 1\} \times [-1, 1]$. To this end, let $\tilde{t}_0 = 0 < \tilde{t}_1 < \cdots < \tilde{t}_{N-1} < \tilde{t}_N = \tilde{T}$ be the time instants where any one of the two components of $\tilde{u}^{p,q}_{\mathrm{pwc}}$ changes value.[2] If we set $I_i \triangleq]\tilde{t}_{i-1}, \tilde{t}_i]$, $i = 1, \ldots, N$, then the interval $[0, \tilde{T}]$ is the disjoint union

$$[0, \tilde{T}] = \{0\} \cup \left(\bigcup_{i=1}^{N} I_i \right).$$

First we describe the domain of $u^{p,q}_{\mathrm{pwc}}$. It is the disjoint union

$$[0, S] = \{0\} \cup \left(\bigcup_{i=1}^{N} J_i \right),$$

where J_i is of the form $]t_{i-1}, t_i]$, $i = 1, \ldots, N$, and the intervals J_i are contiguous in the sense that the right endpoint of J_i is equal to the left endpoint of J_{i+1}. Hence, the intervals J_i are uniquely defined as long as we specify their lengths. We do so as follows.

$$\mu(J_i) = \begin{cases} \mu(I_i), & \text{if } \tilde{u}^{p,q}_{\mathrm{pwc},1} = -c, \\ c\mu(I_i), & \text{if } \tilde{u}^{p,q}_{\mathrm{pwc},1} = c, \end{cases}$$

where μ denotes the Lebesgue measure on $\mathbb{R}$. We can now specify the values of the piecewise constant control $u^{p,q}_{\mathrm{pwc}}$ that steers p to q in time S and is admissible for (1)

$$u^{p,q}_{\mathrm{pwc}}(t) = \begin{cases} \tilde{u}^{p,q}_{\mathrm{pwc}}, & \text{if } \tilde{u}^{p,q}_{\mathrm{pwc},1} = -c, \\ \frac{1}{c}\tilde{u}^{p,q}_{\mathrm{pwc}}, & \text{if } \tilde{u}^{p,q}_{\mathrm{pwc},1} = c. \end{cases}$$

We conclude that, since the control system $(\tilde{\Sigma})$ is STLC, so is the control system (1).

Given a trajectory $\gamma : [a, b] \subset \mathbb{R} \to M$ of a control system, let $\mathrm{dom}\,(\gamma)$ denote the domain $[a, b]$ of γ, and $\mathrm{In}(\gamma)$ and $\mathrm{Term}(\gamma)$ denote $\gamma(a)$ and $\gamma(b)$, respectively. With this notation, the definitions of the minimum-time functions $V_\Sigma, V_{\mathrm{co}\Sigma} : M \times M \to [0, \infty[$ for (1) and $(\mathrm{co}\Sigma)$, respectively, are

$$V_\Sigma(p, q) \triangleq \inf_{u \in \mathcal{A}} \{\mu(\mathrm{dom}\,(\gamma)) \mid \mathrm{In}(\gamma) = p \text{ and } \mathrm{Term}(\gamma) = q\},$$

$$V_{\mathrm{co}\Sigma}(p, q) \triangleq \inf_{u \in \mathrm{co}\mathcal{A}} \{\mu(\mathrm{dom}\,(\gamma)) \mid \mathrm{In}(\gamma) = p \text{ and } \mathrm{Term}(\gamma) = q\}.$$

We have the following.

[2] It can be shown, and it is actually known from previous works, e.g., Sussmann and Tang [1991], that there can be only finitely many switchings.

Proposition 2. The minimum-time functions V_Σ and $V_{\text{co}\Sigma}$ coincide.

Proof. The result follows from Proposition 1 and Theorem 5 in Sussmann and Tang [1991].

4. NECESSARY CONDITIONS FOR OPTIMALITY

To analyse the optimal trajectories of (coΣ) we apply the Maximum Principle (MP) [Pontryagin et al., 1962; Agrachev and Sachkov, 2004] and we follow closely the approach of Sussmann and Tang [1991] with the necessary modifications to account for the asymmetric control set.

If $v \in T_xM$ and $\lambda \in T_x^*M$, then $\langle\lambda, v\rangle$ denotes the canonical pairing between λ and v. The Hamiltonian is the function

$$H(\lambda, x, u) = \langle\lambda, u_1X^1 + u_2X^2\rangle.$$

Theorem 3. (MP). If $\gamma \in W^{1,1}_{\text{loc}}([0,T];M)$ is a solution to the Relaxed Asymmetric Reeds–Shepp problem and $u \in \text{co}\mathcal{A}$ is the corresponding control,[3] then there exists a non-negative constant $\lambda^0 \geq 0$ and a non-trivial, absolutely continuous section $\lambda \in \Gamma T^*M$ of the cotangent bundle of M such that

$$\dot\lambda = -D_xH, \tag{2}$$

$$H(\lambda, x, u) = \min_{\omega\in\text{co}(U)} H(\lambda, x, \omega), \tag{3}$$

$$H(\lambda, x, u) = -\lambda^0, \tag{4}$$

$$\langle\lambda(T), v\rangle = 0, \text{ for every } v \in T_{\gamma(T)}N, \tag{5}$$

for almost every $t \in [0,T]$.

An important role in the analysis of the properties of extremals based on (2)-(5) is played by the switching functions

$$\phi_1 = \langle\lambda, X^1\rangle,$$
$$\phi_2 = \langle\lambda, X^2\rangle,$$

and the auxiliary function

$$\phi_3 = \langle\lambda, X^3\rangle,$$

where $X^3 = [X^2, X^1]$, that is, X^3 is the Lie bracket of X^2 and X^1. The functions ϕ_i, $i = 1,\ldots,3$, satisfy the following system of differential equations with discontinuous right-hand side [Sussmann and Tang, 1991]

$$\dot\phi_1 = u_2\phi_3, \tag{6}$$

$$\dot\phi_2 = -u_1\phi_3, \tag{7}$$

$$\dot\phi_3 = -u_2\phi_1. \tag{8}$$

In terms of the switching functions, the Hamiltonian can be written as

$$H(\lambda, x, u) = u_1\phi_1 + u_2\phi_2 \tag{9}$$

and the constancy of the Hamiltonian (4) is, then, equivalent to

$$u_1\phi_1 + u_2\phi_2 = -\lambda^0. \tag{10}$$

The minimisation condition (3) implies that $u_1 = 1$, when $\phi_1 < 0$ and $u_1 = -1/c$ when $\phi_1 > 0$, and $u_2 = -\text{sign}\,\phi_2$. Therefore, $u_1\phi_1 \leq 0$ and (10) can be written as

$$u_1\phi_1 - |\phi_2| = -\lambda^0. \tag{11}$$

[3] In the case of (coΣ), a trajectory defines uniquely a control.

Because the vector fields X^1, X^2, and X^3 are everywhere linearly independent and the adjoint vector $t \mapsto \lambda(t)$ vanishes nowhere, we also have that

$$|\phi_1| + |\phi_2| + |\phi_3| \neq 0. \tag{12}$$

Property 1. Non-trivial abnormal extremals do not exist.

Proof. Suppose $\lambda^0 = 0$ along a non-trivial extremal. Then, $u_1\phi_1 = |\phi_2| \geq 0$ which implies $u_1\phi_1 = 0$ and this can only happen if $\phi_1 = 0$ (as a consequence of the minimisation condition (3) on the values of u_1). If $\phi_1 = \phi_2 = 0$, (12) implies that ϕ_3 does not vanish anywhere and it follows from (6)-(8) that $u_1\phi_3 = u_2\phi_3 = 0 \Rightarrow u_1 = u_2 = 0$ which is a contradiction since we assumed that the extremal is non-trivial.

Property 2. Along a non-trivial extremal, the functions ϕ_1 and ϕ_2 cannot have a common zero.

Proof. Suppose that there exists a time t such that $\phi_1(t) = \phi_2(t) = 0$. Then, (4) and (9) imply that $\lambda^0 = 0$ which contradicts Property 1.

Property 2 excludes all paths that involve simultaneous change in the values of u_1 and u_2. Using the, standard by now, notation to describe paths for the Dubins and the Reeds–Shepp vehicle, Property 2 implies that there cannot be paths or subpaths of the form $C^\pm S^\mp$ or of the form $C_1^\pm C_2^\mp$, if $C_1 \neq C_2$. That the latter form of paths cannot be optimal is intuitively obvious when $c = 1$, however it is less so in the present case because forward and backward motions have different bounds on their curvatures and, therefore, a simultaneous switching of u_1 and u_2, with $u_2 \neq 0$, does not mean that the vehicle traces the same circular arc in the opposite direction.

Property 3. Along an optimal trajectory, the control u_2 takes values in $\{-1, 0, 1\}$. In other words, optimal paths consist of circular arcs of maximum curvature and straight-line segments.

Proof. We already saw that $u_2 = -\text{sign}\,\phi_2$. If $\phi_2 = 0$ on an interval I of positive (Lebesgue) measure, then Property 2 implies that ϕ_1 cannot vanish anywhere in I and, therefore, $u_1 \in \{-c, 1\}$. Equation (7), then, implies that ϕ_3 is zero on I and, consequently, (8) implies that $u_2\phi_1 = 0 \Rightarrow u_2 = 0$ almost everywhere on I.

Property 4. Inflection points, straight line segments, and the final endpoint of an optimal path lie on the same straight line in the (x^1, x^2)-plane.

Proof. Let $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ be the coordinate representation of the adjoint vector. From the adjoint equation (2) it follows that λ_1 and λ_2 are constant along an optimal path, and

$$\lambda_3(t) = \lambda_1x^2(t) - \lambda_2x^1(t) + \lambda_3(0). \tag{13}$$

The transversality condition (5) which (in coordinates) says that $\lambda_3(T) = \phi_2(T) = 0$ (where T is the final time) together with (13) imply that

$$\begin{aligned} 0 = \lambda_3(T) &= \lambda_1x^2(T) - \lambda_2x^1(T) + \lambda_3(0) \\ &= \lambda_1x_f - \lambda_2y_f + \lambda_3(0). \end{aligned} \tag{14}$$

Hence, the points (x^1, x^2) where ϕ_2 vanishes (points where u_2 changes sign, points on straight line segments, and the final point of the path all lie on the same line in the

(x^1, x^2) plane, because at such point $\phi_2 = \lambda_3 = 0$. It follows directly that straight-line segments can only occur at the end of a path. It follows, also, that paths of the form $C^{\pm}C^{\pm}S$ cannot be optimal.

The next step is to show that an optimal path cannot be a concatenation of three arcs. This fact is reported in [Souères et al., 1994] as Lemma 2, however we give here a simple and self-contained proof. First we make several observations that are used in the proof. Observe that

$$\phi_1^2 + \phi_3^2 = \lambda_1^2 + \lambda_2^2 > 0. \tag{15}$$

The sum cannot be zero, for otherwise the adjoint equation (2) and the transversality condition (5) would imply that $\lambda = 0$, however the MP precludes λ from being identically zero.

If $\phi_2(t) = 0$, for some t, then, from (11), $u_1(t)\phi_1(t) = -\lambda_0$. If we assume that $u_1 = 1$, (15) gives

$$\begin{aligned} \phi_3^2(t) &= \lambda_1^2 + \lambda_2^2 - \lambda_0^2 \Leftrightarrow \\ \left(\lambda_1 \sin x^3(t) - \lambda_2 \cos x^3(t)\right)^2 &= \lambda_1^2 + \lambda_2^2 - \lambda_0^2. \end{aligned} \tag{16}$$

From (16) one can compute the slope of the line described by (14). A similar equation is obtained if we assume $u_1 = -c$.

Property 5. Paths of the form CCC cannot be optimal.

Proof. Property 2 implies that paths of the form $C_1^{\pm}C_2^{\mp}C_3^{\pm}$ with $C_1 \neq C_2$ or $C_2 \neq C_3$ (for example, the paths $\ell^+r^-\ell^+$ and $r^-\ell^+\ell^-$) cannot be optimal. We show how to exclude paths of the form $C^{\pm}C^{\mp}C^{\pm}$; the other cases can be treated similarly. Consider first the case where $u_2 = 1$ along the whole path (in this case, the path is of the form $\ell^+\ell^-\ell^+$) and let $\sigma, \tau \in]0, T[$, $\sigma < \tau$, be the times when the control u_1 changes value from 1 to $-c$ and from $-c$ to 1, respectively. Along the whole path, (6) and (8) take the form

$$\dot{\phi}_1 = \phi_3, \tag{17}$$

$$\dot{\phi}_3 = -\phi_1, \tag{18}$$

and the solution in $[\sigma, \tau]$ to this system is given by

$$\begin{bmatrix} \phi_1(\tau) \\ \phi_3(\tau) \end{bmatrix} = \begin{bmatrix} \cos(\tau-\sigma) & \sin(\tau-\sigma) \\ -\sin(\tau-\sigma) & \cos(\tau-\sigma) \end{bmatrix} \begin{bmatrix} \phi_1(\sigma) \\ \phi_3(\sigma) \end{bmatrix}. \tag{19}$$

We have the boundary conditions $\phi_1(\sigma) = \phi_1(\tau) = 0$ (τ and σ are switching times for u_1) and $\phi_3(\sigma) = \phi_3(\tau) = \sqrt{\lambda_1^2 + \lambda_2^2}$ (from (15)). System (19) yields

$$\begin{aligned} 0 &= \sin(\tau-\sigma)\sqrt{\lambda_1^2 + \lambda_2^2}, \\ \sqrt{\lambda_1^2 + \lambda_2^2} &= \cos(\tau-\sigma)\sqrt{\lambda_1^2 + \lambda_2^2}. \end{aligned}$$

However, since $\sigma < \tau$, this is only possible if $\lambda_1 = \lambda_2 = 0$ and this was shown not to be possible because, together with the transversality condition, it implies the triviality of λ. So far we assumed that $u_2 \equiv 1$. If $u_2 \equiv -1$, then the only difference is that the equations in (17) change signs and the derivation remains the same. Also, it should be noted that some paths of the form CCC can be shown not to be optimal by simply using the fact that the inflection points and the final point lie on the same line in the (x^1, x^2) plane. One such case are the paths of the form $C^{\pm}C^{\pm}C^{\pm}$.

Observe also that, since ϕ_1 does not change sign along a sinlge arc, (19) implies that an arc cannot be longer than $\pi/2$. Recall that a cusp is a point where u_1 changes sign and an inflection point is a point where $\phi_2 = 0$. We have the following.

Property 6. Every arc between a cusp and an inflection point is $\pi/2$ radians

Proof. Equations (15) and (6)-(8) show that the vector $\boldsymbol{v} = (\phi_1, \phi_3)$ has constant length and rotates with angular velocity u_2. For concreteness, consider an arc between a cusp and an inflection point such that $u_1 = 1$ and $u_2 = 1$; other cases are proven in a completely analogous manner. Let $t = 0$, $t = \tau$ be the times when the cusp and the inflection point occur, respectively. At the cusp, we have $\boldsymbol{v}(0) = (0, \sqrt{\lambda_1^2 + \lambda_2^2})$ from the fact that $t = 0$ is assumed to be a switching time for u_1 (hence $\phi_1(0) = 0$) and from equation (15). At the inflection point we have $\boldsymbol{v}(\tau) = (\lambda_0, 0)$. That $\phi_3(\tau) = 0$ follows from the fact that $t = \tau$ is a critical point of ϕ_2 ($\dot{\phi}_2(\tau) = 0$) and from (7) together with the fact that $u_1(\tau) \neq 0$. Hence, $\boldsymbol{v}(0) \cdot \boldsymbol{v}(\tau) = 0$ and since $\lambda_0 > 0$, $\boldsymbol{v}$ must rotate by $\pi/2$.

The previous analysis shows that the relaxed asymmetric Reeds–Shepp problem and the relaxed Reeds–Shepp problem admit the same sufficient family of optimal paths as given in Souères et al. [1994].

Theorem 4. A sufficient family of optimal paths for the relaxed asymmetric Reeds–Sheep problem is given by: $C_\alpha^{\pm}C_{\pi/2}^{\mp}S$ with $0 \leq \alpha < 2\pi$ and $d \geq 0$; $C_\alpha^{\pm}C_\beta^{\mp}$ with $\alpha, \beta \leq \pi$; $C_\alpha^{\pm}S_d^{\pm}$ with $\alpha < \pi/2$ and $d \geq 0$.

5. A COMMENT ON OPTIMAL SYNTHESIS

Theorem 4 should not be interpreted as saying that a given instance of the relaxed asymmetric Reeds–Shepp problem admits the same solutions as the relaxed Reeds–Shepp problem when the boundary conditions are the same for the two problems. This is a point where the difference in speed during forward and backward motion has a significant effect on the form of optimal trajectories, as we now explain by means of an example.

Consider the case where the initial state of the vehicle is $(0,0,0)$ and the final position is $(x_f, 0) = (-d, 0)$, $d > 0$. Then, it can be shown that there exists a positive real number number d^* such that

(1) if $d < d^*$, the shortest path is of the form s_d^-, that is, the vehicle moves backwards on a straight line segment the for time $T = \frac{d}{c}$, and
(2) if $d > d^*$, a shortest path is of the form $r_\alpha^- r_{\pi/2}^+ s_\delta^+$, $\alpha > 0$, $\delta > 0$. An illustration of this path is given in Figure 2.

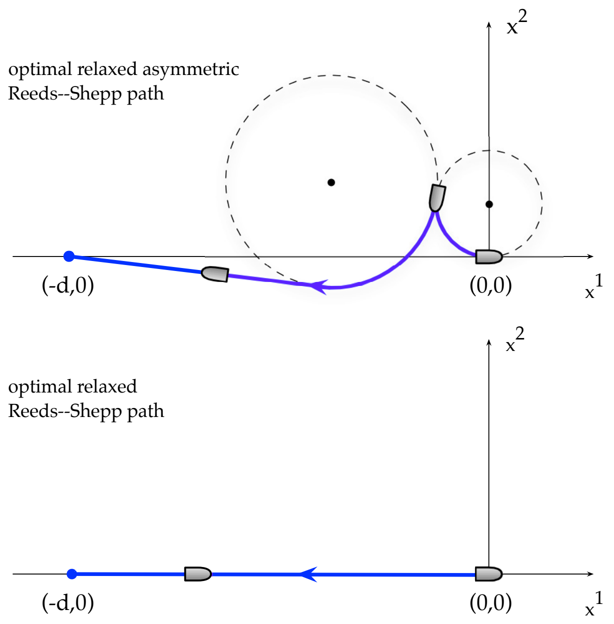

Fig. 2. An optimal path from $(0, 0, 0)$ to $(-d, 0)$ for $d > d^*$.

It follows that an optimal path starting with a straight-line segment backwards will continue until the final point, or it cannot be an optimal path.

The above example shows that a minimum-time synthesis for the relaxed asymmetric Reeds–Shepp problem cannot be the same as that for the relaxed Reeds–Shepp problem [Souères et al., 1994].

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