Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Output Regulation of Hybrid Linear Systems with Unpredictable Jumps

D. Carnevale * S. Galeani * L. Menini * M. Sassano *

* *Dipartimento di Ingegneria Civile e Ingegneria Informatica, Università di Roma "Tor Vergata", Via del Politecnico, 1 00133 Roma, Italy (Email: sergio.galeani, daniele.carnevale, mario.sassano@uniroma2.it, menini@disp.uniroma2.it).*

Abstract In this paper the output regulation problem for a class of hybrid linear systems in the presence of *uncertain time domain* is considered. Uncertainty is modeled by assuming that the time domain is not known to the controller and by allowing for arbitrarily close, even simultaneous, jumps. Considering the full information setting, the geometric characterization of the relevant regulation manifolds is given. Finally, the theory is illustrated and validated by means of numerical examples.

1. INTRODUCTION

Control of hybrid systems, characterized by the interaction between a continuous-time dynamics (here we will use the term "flow" dynamics) and a discrete-time dynamics (here named "jump" dynamics), is a topic widely studied in the last years, with several important results described, *e.g.*, in [Goebel et al., 2009, 2012, Johansson, 2004, Liberzon, 2003, Sun and Ge, 2005]. The problem of output regulation has also been recently addressed for different classes of hybrid systems, among others in the works [Cox et al., 2012, Galeani et al., 2008a,b, 2012, J.B. Biemond and Nijmeijer, 2011, Marconi and Teel, 2011, 2010, Morarescu and Brogliato, 2010] and references therein. When referring to hybrid systems, one very important issue is the difference between the case when the jump times are fixed and known and the case when the jump times depend on the state variables of the system itself. The second case is typically much more difficult to deal with, as a matter of fact, as discussed in [Galeani et al., 2012], the fact that the occurrence of a jump depends on the state, renders the overall system nonlinear, even when the flow dynamics and the jump dynamics are both linear. On the other hand, if the jump times are fixed and known to the controller "a priori", several simplifications occur: the overall system is essentially linear (and this has very nice consequences for stabilization issues) and, with an eye on regulation requirements, it is possible to use the redundant control inputs (if any) to steer the state of the plant to the proper values to be assumed right before the jump, in such a way that the regulated output and its derivatives are guaranteed to attain exactly the desired value right after the jump (see, e.g., [Carnevale et al., 2012a, 2013a,b]). Goal of this paper is to start filling the gap between the two cases just described: here the jump times are assumed to be not dependent from the state of the plant (thus linearity of the overall system is preserved), hence generated by some external source, but they are not known in advance to the controller. When the switching signal imposes a jump (one can imagine different sources, like an external operator deciding freely when to switch), this command is given at the same time to the exosystem, the plant and the compensator. The fact that the compensator has no information in advance about the jump times, renders necessary to drop some of the tools that have been used in the recent past, that are based on using explicit formulae for the solutions of LTI systems (see, e.g., [Carnevale et al., 2012a, 2013b]), and use a different approach, based on geometric control ideas.

⋆ This work was supported in part by ENEA-Euratom and MIUR.

Novel contribution

Goal of this paper is to solve the output regulation problem for a class of linear hybrid systems in which the time domain is not known in advance. The full information setting is assumed here, hence both the exogenous signal and the state of the plant are assumed to be measurable. The time domain does not have particular restrictions and the solution of the problem relies on two components: the characterization of the subspace of the extended state space (including both the state of the plant and of the exosystem) where output regulation is achieved, and the stabilization part. The novel contribution is concentrated in the first component, and relies heavily on the definition and geometric characterization of a proper subspace that is jointly controlled-invariant for the extended flow dynamics and jump dynamics: such a subspace is called here *hybridly controlled invariant*. Differently from what is standard for LTI systems, the solution of the problem will involve the characterization of the subspace of initial conditions of the exosystem for which the problem is solvable. As a matter of fact, requiring regulation for the whole set of initial conditions would result too restrictive in many cases. For greater generality, in this study also the presence of "impulsive" inputs (acting on the jump dynamics) is allowed, similarly to the works by [J. Bentsman and Rubinovich, 2011, R.G. Sanfelice and Heemels, 2013]. As for the system describing the flow dynamics, it is not assumed to be square, although, thanks to the presence of the impulsive inputs, the fact of having more control inputs than outputs is less crucial than in other output regulation problems for hybrid systems.

Notation A' denotes the transpose of matrix A. Given a subspace $\mathcal{V} \subset \mathbb{R}^n$, $A^{-1}(\mathcal{V})$ denotes the inverse image [1] of $\mathcal{V}$ through A, that is $A^{-1}(\mathcal{V})$ is the subspace of all vectors μ such that $A\mu \in \mathcal{V}$. For a square invertible matrix T, the usual inverse will be denoted as T^{-1} (without a subspace as argument).

2. PRELIMINARIES AND PROBLEM DEFINITION

2.1 Families of uncertain time domains

Following the notation of [Goebel et al., 2012], in the context of output regulation, a class of hybrid systems evolving on a single predetermined *hybrid time domain* of the form

$$\mathcal{T} := \{(t, k) : t \in [t_k, t_{k+1}], k \in \mathbb{N}, t_k := k\tau_M\}, \quad (1)$$

with $\tau_M > 0$ has been recently studied by [Marconi and Teel, 2010] and later considered in a series of papers [Cox et al., 2011a,b, 2012] as well as [Carnevale et al., 2012a,b, 2013b]. For such a class of systems, the dynamics can be unambiguously defined. Departing from the above mentioned papers, the hybrid systems considered in this paper will still be defined unambiguously with a fixed hybrid time domain $\mathcal{T}$, but our a priori knowledge of $\mathcal{T}$ will be limited to the fact that it has the form

$$\mathcal{T} := \{(t, k) : t \in [t_k, t_{k+1}], k \in \mathbb{N}\}, \quad (2)$$

for some sequence $\{t_k\}$ of jump times satisfying $t_k \leq t_{k+1}$, $k \in \mathbb{N}$, and it is unbounded (that is, for any $N \in \mathbb{R}$ it is possible to find $(t, k) \in \mathcal{T}$ such that $t + k > N$, so that it is possible to take limits for $t + k \to +\infty$ for $(t, k) \in \mathcal{T}$). An *unrestricted family* Θ of possible hybrid time domains is considered; hence, in particular it can happen that $t_k = t_{k+1}$ for some k (so that two or more consecutive jumps at the same flow time t are possible).

Roughly speaking, it appears evident that the main difference between hybrid time domains of the form (1) and those of the form (2) belonging to Θ consists in the fact that in the former case jumps occur precisely every τ_M seconds while in the latter they may occur at any time, *i.e.* yielding non-uniformly spaced jumps. The main consequence is that, since the particular element $\mathcal{T}$ of the family Θ associated to the solutions of the hybrid linear system is not known beforehand, jumps occur *unexpectedly* as far as the controller is concerned. This immediately makes inapplicable the results developed along the lines traced in [Carnevale et al., 2012a, Marconi and Teel, 2010], which are based on exploiting the knowledge of τ_M.

2.2 Hybrid output regulation on uncertain time domains

Consider the hybrid linear plant $\mathcal{P}$ described by

$$\dot{x} = Ax + Bu + Pw, \quad (3a)$$
$$e = Cx + Qw, \quad (3b)$$
$$x^+ = Ex + Fu_d + Rw, \quad (3c)$$

with state $x(t, k) \in \mathbb{R}^n$, control inputs $u(t, k) \in \mathbb{R}^m$ and $u_d(t, k) \in \mathbb{R}^{m_d}$ acting on the continuous-time and discrete-time evolutions of the state x, respectively, and the performance output $e(t, k) \in \mathbb{R}^p$, $p \leq m$, where the disturbance/reference $w(t, k) \in \mathbb{R}^q$ is generated by the exosystem $\mathcal{E}$

$$\dot{w} = Sw, \quad (4a)$$
$$w^+ = Jw. \quad (4b)$$

As already specified, the above dynamics are associated to an unknown time domain $\mathcal{T}$ given by (2) about which the only available a priori knowledge consists in the fact that $\mathcal{T} \in \Theta$. Compared to [Carnevale et al., 2012a], the aim here is to achieve output regulation *robustly* with respect to the uncertain time domain in the considered family. Since, as it will become clear later, the problem might be solvable only for a subset of the solutions of (4), namely those ensuing from a suitable subspace of its state space (which turns out to be an invariant subspace), the determination of such a subset is also considered as part of the problem. It is stressed here that only *full information* output regulation problems will be considered in this paper.

Problem 1. (Hybrid Output Regulation with Uncertain Time Domain). Given plant $\mathcal{P}$ in (3) and the exosystem $\mathcal{E}$ in (4) find, if possible, a set $\overline{\mathcal{W}} \subseteq \mathbb{R}^q$, invariant with respect to (4), and a full information state feedback choice of the control inputs $\bar{u}(x, w)$, $\bar{u}_d(x, w)$ such that, for any $w(0, 0) \in \overline{\mathcal{W}}$, all the closed loop trajectories of (3)-(4) with the chosen $\bar{u}(x, w)$ and $\bar{u}_d(x, w)$ are such that $\lim_{t+k\to\infty} e(t, k) = 0$, with $(t, k) \in \mathcal{T}$, for any $\mathcal{T} \in \Theta$.

Obviously, Problem 1 is instrumental to solve the more meaningful following problem.

Problem 2. (Hybrid Output Regulation with Uncertain Time Domain and Stability). Given plant $\mathcal{P}$ in (3) and the exosystem $\mathcal{E}$ in (4) find, if possible, a set $\overline{\mathcal{W}} \subseteq \mathbb{R}^q$, invariant with respect to (4), and a full information state feedback choice of the control inputs $\bar{u}(x, w)$, $\bar{u}_d(x, w)$ which solve Problem 1 and ensure that the zero equilibrium of system (3) with $w = 0$ is globally asymptotically stable. ■

2.3 Some basic facts from geometric control

A short review of some tools and concepts from geometric control theory for linear time invariant systems is provided, in order to develop later the required analogous concepts for the considered class of hybrid systems. An in depth treatment of this topic can be found *e.g.* in [Basile and Marro, 1992, Trentelman et al., 2001]. Consider a continuous-time [respectively, discrete-time] linear time-invariant (LTI) system of the form

$$\dot{x} = Ax + Bu, \qquad [x^+ = Ax + Bu,] \quad (5)$$

with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. A subspace $\mathcal{V} \subset \mathbb{R}^n$ is an *invariant* for (5) if

$$A\mathcal{V} \subset \mathcal{V}, \quad (6)$$

and is a *controlled invariant* for (5) if

$$A\mathcal{V} \subset \mathcal{V} + \mathrm{Im}\,(B). \quad (7)$$

The practical relevance of controlled invariance consists in the fact that trajectories starting in the subspace $\mathcal{V}$ can be kept in $\mathcal{V}$ for all times by suitable control actions if and only if $\mathcal{V}$ is a controlled invariant. A feedback formulation of controlled invariance can be given, consisting in the fact that $\mathcal{V}$ is a controlled invariant if and only if there exists a feedback $u = Kx$ such that $\mathcal{V}$ is an invariant for $\dot{x} = (A+BK)x$ $[x^+ = (A+BK)x]$; any such a K is called

[1] Note: this definition *does not* require invertibility of matrix A.

a *friend* of the controlled invariant $\mathcal{V}$. If an output of the form $y = Cx$ is present, it is of interest to identify the *largest controlled invariant contained in* $\ker(C)$, namely the largest subspace $\mathcal{V}$ such that trajectories starting in $\mathcal{V}$ can be kept in $\mathcal{V}$ for all time by suitable control actions (since $\mathcal{V}$ is controlled invariant), meanwhile producing zero output (since $\mathcal{V} \subset \ker(C)$). Given a subspace $\mathcal{V}$, define the operator $\mathcal{A}_{(A,B)}(\cdot)$ mapping a subspace of $\mathbb{R}^n$ in another subspace of $\mathbb{R}^n$ according to the rule:

$$\mathcal{A}_{(A,B)}(\mathcal{V}) = \mathcal{V} \cap A^{-1}\left(\mathcal{V} + \mathrm{Im}\,(B)\right). \tag{8}$$

Since clearly $\mathcal{A}_{(A,B)}(\mathcal{V}) \subset \mathcal{V}$, either $\mathcal{A}_{(A,B)}(\mathcal{V}) = \mathcal{V}$, in which case $\mathcal{V}$ is a fixed point of $\mathcal{A}_{(A,B)}(\cdot)$, or $\mathcal{A}_{(A,B)}(\mathcal{V})$ has a dimension strictly smaller than the dimension of $\mathcal{V}$. Since the involved space $\mathbb{R}^n$ has finite dimension n, it follows that iterated application of $\mathcal{A}_{(A,B)}(\cdot)$ converges to a fixed point (which is a subspace) $\mathcal{V}^*$ in a finite number of iterations. Based on these facts, given a subspace $\mathcal{M} \subset \mathbb{R}^n$ and the dynamics (5), the *invariant subspace algorithm* determines the largest controlled invariant contained in $\mathcal{M}$ by defining

$$\mathcal{M}_0 = \mathcal{M}, \tag{9a}$$

$$\mathcal{M}_{k+1} = \mathcal{A}_{(A,B)}(\mathcal{M}_k) = \mathcal{A}^{k+1}_{(A,B)}(\mathcal{M}), \tag{9b}$$

for $k = 0, 1, 2, \ldots$, where $\mathcal{A}^{k+1}_{(A,B)}(\mathcal{M}) = \mathcal{A}_{(A,B)}(\mathcal{A}^{k}_{(A,B)}(\mathcal{M}))$. If $\mathcal{M}_{k+1} = \mathcal{M}_k$, then $\mathcal{M}^* = \mathcal{M}_k$ and $\mathcal{A}^{*}_{(A,B)}(\mathcal{M}) = \mathcal{A}^{k}_{(A,B)}(\mathcal{M})$. It can be seen that $k \leq n$.

3. HYBRIDLY CONTROLLED INVARIANT SUBSPACES

In this section, we first define an extended hybrid system including the plant and the exosystem, and then introduce the controlled-invariant subspaces of the extended state space which are crucial for solving the considered output regulation problems.

Define the *extended system* as

$$\begin{bmatrix} \dot{x} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} A & P \\ 0 & S \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u =: \hat{A} \begin{bmatrix} x \\ w \end{bmatrix} + \hat{B} u \,, \tag{10a}$$

$$\begin{bmatrix} x^+ \\ w^+ \end{bmatrix} = \begin{bmatrix} E & R \\ 0 & J \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} + \begin{bmatrix} F \\ 0 \end{bmatrix} u_d =: \hat{E} \begin{bmatrix} x \\ w \end{bmatrix} + \hat{F} u_d \,, \tag{10b}$$

$$e = [C \;\; Q] \begin{bmatrix} x \\ w \end{bmatrix} =: \hat{C} \begin{bmatrix} x \\ w \end{bmatrix}. \tag{10c}$$

The theory recalled in Section 2.3 applied to the flow dynamics (10a) with output (10c) allows to determine the largest *flow controlled invariant* contained in $\ker(\hat{C})$, henceforth denoted as $\mathcal{S}$, which is given by

$$\mathcal{S} = \mathcal{A}^*_{(\hat{A},\hat{B})}(\ker(\hat{C})) \subset \mathbb{R}^n \times \mathbb{R}^q. \tag{11}$$

Clearly, by construction, subspace $\mathcal{S}$ is such that $e = 0$ whenever $(x, w) \in \mathcal{S}$. In the classic (flow only) setting, the full information output regulation problem for (10a) with output (10c) is then solved by determining a friend $\hat{K} = [K \;\; \Gamma]$ of $\mathcal{S}$ such that the feedback control law $u = \Gamma w + Kx$ renders the subspace $\mathcal{S}$ invariant and attractive (attractivity can be achieved under suitable stabilizability hypotheses). In the hybrid case, matters are quite more involved since (x^+, w^+) does not necessarily belong to $\mathcal{S}$.

In the sequel, it will be useful to consider $\mathrm{proj}_{\mathcal{E}}(\mathcal{V})$, that is the natural projection (see [Basile and Marro, 1992]) of a subspace $\mathcal{V}$ of the extended state space on the state space of the exosystem. Such a projection is defined as

$$\mathrm{proj}_{\mathcal{E}}(\mathcal{V}) := \{w \in \mathbb{R}^q : [x' \; w'] \in \mathcal{V}, x \in \mathbb{R}^n\}.$$

If V is a basis matrix of $\mathcal{V}$, namely $\mathcal{V} = Im(V)$, a basis matrix of $\mathrm{proj}_{\mathcal{E}}(\mathcal{V})$ can be computed by retaining the linearly independent columns of the matrix $[0_{q\times n} \;\; I_q]\, V$.

3.1 *Maximal hybridly controlled invariant contained in* $\mathcal{S}$

As pointed out above, a solution moving in the flow controlled invariant $\mathcal{S}$ might leave $\mathcal{S}$ due to a jump, for any choice of u_d in (10b), since flow controlled invariance does not imply jump controlled invariance. It is then reasonable to look for a set $\mathcal{N}^*$ which is the largest *hybridly controlled invariant* contained in $\mathcal{S}$, so that suitable control actions exist preserving invariance of $\mathcal{N}^*$ during arbitrary intervals of flow and sequences of jumps.

The computation of $\mathcal{N}^*$ can be performed by repeated application of the invariant subspace algorithm, alternatively considering the flow dynamics (10a) and the jump dynamics (10b). Recalling the definition of the operator $\mathcal{A}^*_{(\cdot,\cdot)}(\cdot)$ in Section 2.3, the set $\mathcal{N}^*$, that is the largest hybridly controlled invariant contained in $\mathcal{S}$, can be computed as the fixed point of the iterations:

$$\mathcal{N}^{k+1} = \mathcal{A}^*_{(\hat{A},\hat{B})}\left(\mathcal{A}^*_{(\hat{E},\hat{F})}(\mathcal{N}^k)\right), \qquad \mathcal{N}^0 = \mathcal{S}\,, \tag{12}$$

for $k = 0, 1, ..., n-1$, which converge in a finite number of steps due to finite dimensionality of the extended state space $\mathbb{R}^n \times \mathbb{R}^q$; hence $\mathcal{N}^* := \mathcal{N}^i$ where i is the first index such that $\mathcal{N}^{i+1} = \mathcal{N}^i$.

The following proposition characterizes $\mathcal{N}^*$ as the key object for achieving output regulation as required in Problems 1 and 2.

Proposition 1. Consider the hybrid system (3) together with the exosystem (4). Let $\mathcal{N}^*$ be the limit of the sequence of subspaces $\mathcal{N}^k$ defined in (12). Then, there exist feedback control laws

$$u^i = K^i x + L^i w, \tag{13a}$$

$$u^i_d = K^i_d x + L^i_d w, \tag{13b}$$

with u^i making $\mathcal{N}^*$ flow controlled invariant and u^i_d making $\mathcal{N}^*$ jump controlled invariant, such that if the initial condition $[x'(0,0) \;\; w'(0,0)]'$ belongs to $\mathcal{N}^*$ then $e(t,k) = 0$ for all (t,k) in $\mathcal{T} \in \Theta$. In particular, this is possible for the set of initial conditions of the exosystem $\mathcal{E}$ given by $\overline{\mathcal{W}} = \mathrm{proj}_{\mathcal{E}}(\mathcal{N}^*)$. $\diamond$

3.2 *An example*

Consider the linear hybrid system described by the equations

$$\dot{x} = u, \qquad x^+ = x, \qquad e = x + Qw, \tag{14}$$

with $x \in \mathbb{R}$, $u \in \mathbb{R}$ and $e \in \mathbb{R}$, together with the exosystem

$$\dot{w} = (I_2 \otimes S_0)w\,, \qquad w^+ = \begin{bmatrix} I_2 & 0 \\ 0 & J_0 \end{bmatrix} w \tag{15}$$

$w \in \mathbb{R}^4$ with $Q = [1 \;\; 0 \;\; 1 \;\; 0]$ and

$$S_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad J_0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The exosystem injects on (14) a continuous-time sinusoidal signal, described by the first two components of w, and a discontinuous piecewise sinusoidal signal, described be the last two components of w. Note that for (14)-(15) the subspace $\mathcal{S}$ is defined by

$$\mathcal{S} = \{(x, w) \in \mathbb{R} \times \mathbb{R}^4 : x = -Qw\}$$

$$= \ker([1\ \ 1\ \ 0\ \ 1\ \ 0]) = \mathrm{Im}\left(\begin{bmatrix} -1 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\right). \tag{16}$$

Then, it can be easily checked that

$$\mathcal{A}^2_{(\hat{E},\hat{F})}(\mathcal{S}) = \mathcal{A}^1_{(\hat{E},\hat{F})}(\mathcal{S}) = \mathrm{Im}\left(\begin{bmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right),$$

hence $\mathcal{A}^*_{(\hat{E},\hat{F})}(\mathcal{S}) = \mathcal{A}^1_{(\hat{E},\hat{F})}(\mathcal{S})$. Moreover, $\mathcal{N}^0 = \mathcal{A}^*_{(\hat{E},\hat{F})}(\mathcal{S})$, which implies that $\mathcal{N}^* = \mathcal{A}^1_{(\hat{E},\hat{F})}(\mathcal{S})$. Interestingly, $\overline{\mathcal{W}} = \mathrm{proj}_{\mathcal{E}}(\mathcal{N}^*)$ selects the first two components of w, thus expressing the intuitive fact that (14), having a trivial jump map, may only *track* the continuous sinusoidal signal, by implementing a suitable feedback control law. If the jump map $x^+ = x$ of (14) is replaced by $x^+ = x + u_d$, so that $\hat{F} = [1\ 0\ 0\ 0\ 0]'$, repeating the computations it is found that $\mathcal{N}^* = \mathcal{A}^*_{(\hat{E},\hat{F})}(\mathcal{S}) = \mathcal{S}$ and $\overline{\mathcal{W}} = \mathrm{proj}_{\mathcal{E}}(\mathcal{N}^*) = \mathbb{R}^4$. Thus, in this case both the continuous and the discontinuous sinusoidal signals can be tracked by using suitable control laws u and u_d.

The conclusions of the previous example naturally lead to the comments discussed in the following remark.

Remark 1. Recalling the construction of algorithm (12), we have that $\mathcal{N}^* = \mathcal{S}$ provided $\hat{E}\mathcal{S} \subseteq \mathcal{S} + Im(\hat{F})$. As a consequence, the subspace $\mathcal{S}$ is controlled invariant with respect to the jump dynamics (10b), in addition to the flow dynamics (10a). Hence, both $\hat{A}$ and $\hat{E}$ can be transformed to highlight the *decoupling* between the components of x that affect the error e and those that are completely *free* to evolve in $\mathcal{S}$, similarly to what is pursued in [Carnevale et al., 2012a] for the flow dynamics only. As a matter of fact, in this case the input redundancy is not crucial for generating a regulation error zeroing trajectory. ▲

4. HYBRID OUTPUT REGULATION WITH UNCERTAIN TIME DOMAIN

Having characterized the controlled invariant subspaces where zero output motion is possible, we now turn to the issue of rendering such subspaces externally stable (so that all solutions are attracted to them, and output regulation is achieved) and possibly internally stable (thus solving the output regulation problem with stability). Since an extensive literature is available about stabilization of the relevant class of hybrid linear systems, and the main focus of this paper (parallel to our previous work in [Carnevale et al., 2012a,b]) is on characterizing the relevant zero error motions which are crucial for regulation once coupled with suitable stabilizers, the discussion here will be necessarily much more sketchy.

Let $\mathcal{X} := \mathrm{Im}\,([I_n\ 0_{n\times q}]')$, $\mathcal{W} := \mathrm{Im}\,([0_{q\times n}\ I_q]')$ and define $\mathcal{N}^*_{\mathcal{X}} := \mathcal{N}^* \cap \mathcal{X}$, $\mathcal{N}^*_{\mathcal{W}}$ such that $\mathcal{N}^*_{\mathcal{W}} \oplus \mathcal{N}^*_{\mathcal{X}} = \mathcal{N}^*$, and finally $\mathcal{N}^c_{\mathcal{X}} \subset \mathcal{X}$ and $\mathcal{N}^c_{\mathcal{W}} \subset \mathcal{W}$ such that $\mathcal{N}^*_{\mathcal{X}} \oplus \mathcal{N}^c_{\mathcal{X}} = \mathcal{X}$ and $\mathcal{N}^*_{\mathcal{X}} \oplus \mathcal{N}^c_{\mathcal{X}} \oplus \mathcal{N}^*_{\mathcal{W}} \oplus \mathcal{N}^c_{\mathcal{W}} = \mathbb{R}^{n+q}$. Let $d^*_{\mathcal{X}}, d^c_{\mathcal{X}}, d^*_{\mathcal{W}}, d^c_{\mathcal{W}}$ denote the dimensions of $\mathcal{N}^*_{\mathcal{X}}, \mathcal{N}^c_{\mathcal{X}}, \mathcal{N}^*_{\mathcal{W}}$ and $\mathcal{N}^c_{\mathcal{W}}$, respectively. Let X_1, X_2, X_3 and X_4 be matrices whose columns span the subspaces $\mathcal{N}^*_{\mathcal{X}}, \mathcal{N}^*_{\mathcal{W}}, \mathcal{N}^c_{\mathcal{X}}$ and $\mathcal{N}^c_{\mathcal{W}}$, respectively; let U_1 and U_{1d} be matrices such that $\mathrm{Im}\,(U_1) = \mathrm{Im}\,(\hat{B}) \cap \mathcal{N}^*$ and $U_{1d} = \mathrm{Im}\,(\hat{F}) \cap \mathcal{N}^*$, and U_2, U_{2d} be matrices such that $[U_1\ U_2]$ and $[U_d\ U_{2d}]$ are invertible. Consider the change of coordinates

$$\begin{aligned} [x'\ w']' &= T^{-1}\chi = [X_1\ X_2\ X_3\ X_4]\,[\chi_1'\ \chi_2'\ \chi_3'\ \chi_4']', \\ u &= G^{-1}\nu = [U_1\ U_2]\,[\nu_1'\ \nu_2']', \\ u_d &= G_d^{-1}\nu_d = [U_{1d}\ U_{2d}]\,[\nu_{1d}'\ \nu_{2d}']'. \end{aligned}$$

In the new coordinates, system(10) becomes

$$\begin{aligned} \dot{\chi} &= \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix} + \begin{bmatrix} B_{11} & B_{12} \\ 0 & 0 \\ 0 & B_{32} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \nu_1 \\ \nu_2 \end{bmatrix} \\ &:= \bar{A}\chi + \bar{B}\nu, \end{aligned} \tag{17a}$$

$$\begin{aligned} \chi^+ &= \begin{bmatrix} E_{11} & E_{12} & E_{13} & E_{14} \\ 0 & E_{22} & 0 & E_{24} \\ E_{31} & E_{32} & E_{33} & E_{34} \\ 0 & 0 & 0 & E_{44} \end{bmatrix}\begin{bmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{bmatrix} + \begin{bmatrix} F_{11} & F_{12} \\ 0 & 0 \\ 0 & F_{32} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \nu_{1d} \\ \nu_{2d} \end{bmatrix} \\ &:= \bar{E}\chi + \bar{F}\nu_d, \end{aligned} \tag{17b}$$

$$e = [0\ 0\ C_3\ C_4]\,\chi := \bar{C}\chi, \tag{17c}$$

where the zero entries in the above block matrices are motivated by the fact that $\mathcal{N}^*$ is a hybridly controlled invariant contained in $\mathcal{S} \subset \ker(\hat{C})$, and $\mathrm{Im}\,(B) \subset \mathcal{X}$, $\mathrm{Im}\,(F) \subset \mathcal{X}$. Hybrid controlled invariance of $\mathcal{N}^*$ also implies the existence of matrices

$$\bar{K}^i = [\bar{K}^i_1\ \bar{K}^i_2], \quad \bar{K}^i_d = [\bar{K}^i_{d1}\ \bar{K}^i_{d2}], \tag{18}$$

such that

$$[A_{31}\ A_{32}] = -B_{32}\bar{K}^i, \quad [E_{31}\ E_{32}] = -F_{32}\bar{K}^i_d, \tag{19}$$

which can be used to provide a state feedback action rendering $\mathcal{N}^*$ an hybrid invariant for the controlled system. In particular, recalling that any state feedback matrix transforming a controlled invariant $\mathcal{V}$ in a simple invariant for the controlled system is called a *friend* of $\mathcal{V}$, the set of *pairs of friends* of $\mathcal{N}^*$ in the new coordinates is given by:

$$\bar{K} = \begin{bmatrix} \bar{K}^s_1 & \bar{K}^s_2 & \bar{K}^s_3 & \bar{K}^s_4 \\ \bar{K}^i_1 & \bar{K}^i_2 & \bar{K}^s_5 & \bar{K}^s_6 \end{bmatrix}, \tag{20a}$$

$$\bar{K}_d = \begin{bmatrix} \bar{K}^s_{d1} & \bar{K}^s_{d2} & \bar{K}^s_{d3} & \bar{K}^s_{d4} \\ \bar{K}^i_{d1} & \bar{K}^i_{d2} & \bar{K}^s_{d5} & \bar{K}^s_{d6} \end{bmatrix}, \tag{20b}$$

so that applying the feedback $\nu = \bar{K}\chi$, $\nu_d = \bar{K}_d\chi$, equations (17a)-(17b) reduce to $\dot{\chi} = \bar{A}_{\bar{K}}\chi$, $\chi^+ = \bar{E}_{\bar{K}}\chi$ where $\bar{A}_{\bar{K}} := (\bar{A} + \bar{B}\bar{K})$, $\bar{E}_{\bar{K}} := (\bar{E} + \bar{E}\bar{K}_d)$, are given by

$$\bar{A}_{\bar{K}} = \begin{bmatrix} A_{11}+B_{11}\bar{K}^s_1 & A_{12}+B_{11}\bar{K}^s_2 & A_{13}+B_{11}\bar{K}^s_3 & A_{14}+B_{11}\bar{K}^s_4 \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33}+B_{32}\bar{K}^s_5 & A_{34}+B_{32}\bar{K}^s_6 \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \tag{21a}$$

$$\bar{E}_{\bar{K}} = \begin{bmatrix} E_{11}+F_{11}\bar{K}^s_{d1} & E_{12}+F_{11}\bar{K}^s_{d2} & E_{13}+F_{11}\bar{K}^s_{d3} & E_{14}+F_{11}\bar{K}^s_{d4} \\ 0 & E_{22} & 0 & E_{24} \\ 0 & 0 & E_{33}+F_{32}\bar{K}^s_{d5} & E_{34}+F_{32}\bar{K}^s_{d6} \\ 0 & 0 & 0 & E_{44} \end{bmatrix}. \tag{21b}$$

The feedback gains in the original coordinates are obviously given by $K = G^{-1}\bar{K}T$, $K_d = G_d^{-1}\bar{K}_dT$.

Remark 2. The state variables $z_x = [\chi_1'\ \chi_3']'$ and $z_w = [\chi_2'\ \chi_4']'$ are easily seen to have a direct correspondence with the state of the plant and of the exosystem, respectively. In particular, this is reflected by the fact that the dynamics of z_w is characterized by the matrices

$$\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, \quad \begin{bmatrix} E_{22} & E_{24} \\ 0 & E_{44} \end{bmatrix},$$

which are clearly unaffected by the input. The dynamics of z_x is characterized by the matrices

$$\begin{bmatrix} A_{11} + B_{11}\bar{K}_1^s & A_{13} + B_{11}\bar{K}_3^s \\ 0 & A_{33} + B_{32}\bar{K}_5^s \end{bmatrix},$$
$$\begin{bmatrix} E_{11} + F_{11}\bar{K}_{d1}^s & E_{13} + F_{11}\bar{K}_{d3}^s \\ 0 & E_{33} + F_{32}\bar{K}_{d5}^s \end{bmatrix},$$

which, under suitable hypothesis, can be stabilized by an appropriate choice of matrices $\bar{K}_1^s$, $\bar{K}_5^s$, $\bar{K}_{d1}^s$ and $\bar{K}_{d5}^s$ (whereas $\bar{K}_h^s$, $\bar{K}_{dh}^s$ for $h = 2, 3, 4, 6$ have no effect on stability). ▲

Remark 3. The state variables $z_1 = [\chi_1' \ \chi_2']'$ and $z_2 = [\chi_3' \ \chi_4']'$ can be given the following interesting interpretation. On one hand, the state z_1 represents the *internal* dynamics of $\mathcal{N}^*$, that is it represents those states that can *freely* evolve on the invariant subspace $\mathcal{N}^*$ after the output regulation task has been achieved (that is, when the regulated output e is zero). On the other hand, the state z_2 represents the *external* dynamics of $\mathcal{N}^*$, that is it describes the *off-the-subspace* motion, so that $\|z_2(t,k)\|$ is equal to the distance of the trajectory from the output-zeroing subspace $\mathcal{N}^*$ at any hybrid time instant $(t,k) \in \mathcal{T}$. Interestingly, the key component of any friend of $\mathcal{N}^*$, *i.e.* of any state feedback providing its invariance, intrinsically depends only on the internal variables z_1. ▲

Remark 4. By the construction of $\mathcal{N}^*$ and the previous remarks, it is apparent that χ_4 is associated to evolutions of the exosystem that cannot be actively compensated by the control input in such a way to ensure zero regulation output (since the largest hybridly controlled invariant contained in $\mathcal{S}$, hence in $\ker(C)$, is $\mathcal{N}^*$, but χ_4 is associated to off-the-subspace state variables) It follows that, provided that χ_4 is observable from e via C_4 with the dynamics given by A_{44} and E_{44}, and that for all $\chi_4(0,0) \neq 0$ there exists at least one $\mathcal{T} \in \Theta$ such that χ_4 does not converge to zero as $t + k \to +\infty$, $(t,k) \in \mathcal{T}$, the only way to achieve output regulation is by restricting the initial condition of the exosystem to a set $\overline{\mathcal{W}}$ such that $\chi_4(0,0) = 0$, as is done implicitly in Proposition 1. ▲

In light of the previous remark, it should be evident that hybrid output regulation in the presence of uncertain time domains $\mathcal{T} \in \Theta$ can be achieved by enforcing *attractivity* of $\mathcal{N}^*$, that is, using the framework employed in [Basile and Marro, 1992], achieving *external stabilization* of the controlled-invariant subspace $\mathcal{N}^*$. Moreover, the additional requirement of closed-loop stability can be achieved if, in addition to external stabilization of $\mathcal{N}^*$, it is also possible to achieve *internal stabilization* of the controlled invariant subspace $\mathcal{N}^* \cap \mathcal{X}$ (which is a controlled invariant being the intersection of a controlled invariant and an invariant, see again [Basile and Marro, 1992] for the analogous property in the non hybrid setting). As illustrated above, such external [internal] stabilization problem can be reformulated as the problem of finding matrices $\bar{K}_5^s$, $\bar{K}_{d5}^s$, $[\bar{K}_1^s$, $\bar{K}_{d1}^s]$ such that the dynamics described by the matrices $A_{33} + B_{32}\bar{K}_5^s$, $E_{33} + F_{32}\bar{K}_{d5}^s$ $[A_{11} + B_{11}\bar{K}_1^s$, $E_{11} + F_{11}\bar{K}_{d1}^s]$ is asymptotically stable for any $\mathcal{T} \in \Theta$. Such matrices $\bar{K}_5^s$, $\bar{K}_{d5}^s$, $[\bar{K}_1^s$, $\bar{K}_{d1}^s]$ can be used, together with $\bar{K}^i$ $[\bar{K}_d^i]$ from (18), (19), to define a friend of $\mathcal{N}^*$, as in (20), yielding the required invariance and stability properties; note that the choice of other submatrices in (20) is arbitrary.

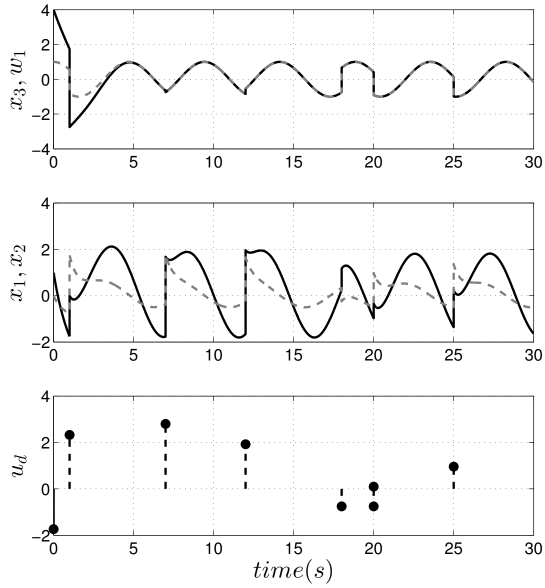

Figure 1. Top graph: time histories of $x_3(t,j)$ and $w_1(t,j)$, solid and dashed line, respectively. Middle graph: time histories of $x_1(t,j)$ and $x_2(t,j)$, solid and dashed line, respectively. Bottom graph: sequence of impulsive control inputs u_d corresponding to jumps of hybrid system (22)-(23).

Theorem 1. Consider hybrid system (3) together with the exosystem (4). If K, K_d, are a pair of friends of $\mathcal{N}^*$ achieving external stability of $\mathcal{N}^*$ in $\mathcal{N}^* + \mathcal{X}$ (that is, for $\chi_4(0,0) = 0$) for all $\mathcal{T} \in \Theta$, then $u = K\,[x' \ w']'$, $u_d = K_d\,[x' \ w']'$, solve Problem 1 with $\overline{\mathcal{W}} = \mathrm{proj}_{\mathcal{E}}(\mathcal{N}^*)$. If K, K_d, also achieve internal stability of $\mathcal{N}^* \cap \mathcal{X}$ for all $\mathcal{T} \in \Theta$, then $u = K\,[x' \ w']'$, $u_d = K_d\,[x' \ w']'$, solve Problem 2 with $\overline{\mathcal{W}} = \mathrm{proj}_{\mathcal{E}}(\mathcal{N}^*)$. ◇

5. NUMERICAL SIMULATIONS

Consider a linear hybrid system described by the equations (3) with the matrices

$$[A \mid B \mid P] = \left[\begin{array}{ccc|cc|cc} -1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -0.5 & 0 & 1 & 1 & 0 \end{array}\right], \tag{22a}$$

$$[C \mid Q] = [0 \ 0 \ 1 \mid -1 \ 0], \tag{22b}$$

$$[E \mid F \mid R] = \left[\begin{array}{ccc|c|cc} 0 & 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 0 & 0 & 0 \end{array}\right]. \tag{22c}$$

The linear hybrid system is driven by an exosystem described by equations of the form (4) with

$$S = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{23}$$

To begin with, note the maximal controlled-invariant subspace with respect to (3a) contained in $\ker([C\ Q])$ is $\mathcal{S} = \ker([0\ \ 0\ \ 1\ \ -1\ \ 0])$. Suppose that the hybrid time domain may be any domain belonging to the family Θ. Hence simultaneous jumps (*i.e.*, more than one jump having the same value of t) can occur. Towards the construction of a feedback control law solving Problem 2, consider the computational algorithm (12) as described in Proposition 1 and notice that

$$\mathcal{N}^* = \mathrm{Im}\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right). \tag{24}$$

Interestingly, $\overline{\mathcal{W}} = \mathbb{R}^2$ which implies that all the exogenous signals generated by the exosystem (23) may be tracked/rejected by system (22) for a suitable selection of feedback control laws u and u_d, which are obtained as described in Section 4. In the first simulation we let $x(0,0) = [1\ \ 0\ \ 4]$ and $w(0,0) = [1\ \ 0]$. The hybrid time domain is set *a priori* but it is unknown to the controller, as discussed in the previous sections. We summarize the time domain by providing the sequence of time instants at which jumps in the plant and in the exosystem occur, namely $T_J = \{0,\ 1,\ 7,\ 12,\ 18,\ 20,\ 20,\ 25\}$. Note that there are two simultaneous jumps at $t = 20s$. The top graph of Figure 1 shows the time histories of the third component of the state of system (22), *i.e.* $x_3(t,j)$, in closed-loop with the control inputs u and u_d as above and the first component of the state of the exosystem (23), *i.e.* $w_1(t,j)$, solid and dashed line, respectively. In particular, according to the structure of C and Q the former should be regulated to the value of the latter. The middle graph of Figure 1 displays the time histories of the first and second components of the state of system (22), *i.e.* $x_1(t,j)$ and $x_2(t,j)$ in closed-loop with the control inputs u and u_d, solid and dashed line, respectively. Finally, the sequence of impulsive control inputs u_d corresponding to jumps of hybrid system (22) -(23) is shown in Figure 1.

6. CONCLUSIONS

A novel problem of hybrid output regulation with unpredictable jumps has been introduced for arbitrary hybrid time domains, and the key geometric objects for its solution have been described. Future work will be focused on extending the proposed result to different classes of time domains.

REFERENCES

G. Basile and G. Marro. *Controlled and Conditioned Invariants in Linear System Theory.* Prentice-Hall, 1992.

D. Carnevale, S. Galeani, and L. Menini. Output regulation for a class of linear hybrid systems. Part 1: trajectory generation. In *Conf. on Decision and Control*, 2012a.

D. Carnevale, S. Galeani, and L. Menini. Output regulation for a class of linear hybrid systems. Part 2: stabilization. In *Conf. on Decision and Control*, 2012b.

D. Carnevale, S. Galeani, and L. Menini. A case study for hybrid regulation: Output tracking for a spinning and bouncing disk. In *Mediterranean Conf. on Control and Automation*, 2013a.

D. Carnevale, S. Galeani, and M. Sassano. On semiclassical solutions of hybrid regulation equations. In *Mediterranean Conf. on Control and Automation*, pages 868–876, 2013b.

N. Cox, L. Marconi, and A. Teel. Hybrid output regulation with unmeasured clock. In *Conf. on Decision and Control*, pages 7410 –7415, 2011a.

N. Cox, A. R. Teel, and L. Marconi. Hybrid output regulation for minimum phase linear systems. In *American Control Conf.*, pages 863–868, San Francisco (CA), USA, June, 29-July, 01 2011b.

N. Cox, L. Marconi, and A. R. Teel. Hybrid internal models for robust spline tracking. In *Conf. on Decision and Control*, pages 4877 –4882, 2012.

S. Galeani, L. Menini, and A. Potini. Trajectory tracking in linear hybrid systems: an internal model principle approach. In *American Control Conf.*, 2008a.

S. Galeani, L. Menini, A. Potini, and A. Tornambe. Trajectory tracking for a particle in elliptical billiards. *International Journal of Control*, 81(2):189–213, 2008b.

S. Galeani, L. Menini, and A. Potini. Robust trajectory tracking for a class of hybrid systems: an internal model principle approach. *IEEE Trans. Aut. Cont.*, 57(2):344–359, 2012.

R. Goebel, R. Sanfelice, and A. R. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, 29(2):28–93, 2009.

R. Goebel, R. Sanfelice, and A. Teel. *Hybrid Dynamical Systems: Modeling, Stability, and Robustness.* Princeton University Press, 2012. ISBN 9781400842636. URL `http://books.google.it/books?id=JwrOYO3fuGQC`.

B. M. M. J. Bentsman and E. Rubinovich. Dynamical systems with active singularities: Input/ state/output modeling and control. *Automatica*, 44(8):1741–1752, 2011.

W. H. J.B. Biemond, N. Van de Wouw and H. Nijmeijer. Tracking control for hybrid systems with state-triggered jumps. *IEEE Trans. Aut. Cont.*, 58(4):876–890, 2011.

K. H. Johansson. *Encyclopedia of Life Support Systems (EOLSS)*, chapter Hybrid Control Systems. H. Unbehauen, Ed., 2004.

D. Liberzon. *Switching in Systems and Control.* Birkhäuser, 2003.

L. Marconi and A. Teel. Internal model principle for linear systems with periodic state jumps. *IEEE Trans. Aut. Cont.*, 58(11):2788–2802, 2011.

L. Marconi and A. R. Teel. A note about hybrid linear regulation. In *Conf. on Decision and Control*, pages 1540–1545, Atlanta (GA), USA, December, 15-17 2010.

I. Morarescu and B. Brogliato. Trajectory tracking control of multiconstraint complementarity lagrangian systems. *IEEE Trans. Aut. Cont.*, 55(6):1300–1313, 2010.

N. V. d. W. R.G. Sanfelice, J.B. Biemond and W. Heemels. An embedding approach for the design of state-feedback tracking controllers for references with jumps. *Int. J. Robust and Nonlinear Control*, 2013.

Z. Sun and S. S. Ge. *Switched Linear Systems: Control and Design.* Springer, 2005.

H. L. Trentelman, A. A. Stoorvogel, and M. L. J. Hautus. *Control Theory for Linear Systems.* Springer, 2001.