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Decomposition with Respect to Outputs for Boolean Control Networks ⋆

Yunlei Zou, Jiandong Zhu

Institute of Mathematics, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, PR China (e-mail: zouyl0903@163.com, zhujiandong@njnu.edu.cn)

Abstract: This paper investigates the decomposition with respect to outputs for Boolean control networks (BCNs). Firstly, based on the linear representation of BCNs, some algebraic equivalent conditions are obtained. Secondly, the concept of *perfect equal vertex partition* (PEVP) is proposed for BCNs. Thirdly, a necessary and sufficient graphical condition based on the PEVP for the decomposability with respect to outputs is obtained. Finally, an equivalent condition of PEVP is derived to help to calculate a PEVP for a BCN.

Keywords: Boolean control networks, Semi-tensor product, Decomposition with respect to outputs, Perfect equal vertex partition.

1. INTRODUCTION

Boolean networks (BNs) proposed by Kauffman (1969) are a kind of discrete logical dynamical systems, which are suitable for describing, analysing and simulating genetic regulatory networks. BNs with external inputs are usually called Boolean control networks (BCNs). Both BNs and BCNs have attracted great attention in the community of systems biology (Albert and Othmer (2003), Faure et al. (2006), Shmulevich and Kauffman (2004), Genoud and Metraux (1999), Datta et al. (2004), Pal et al. (2005), Snoussi (1989)). In recent years, a semi-tensor product method of BNs has been developed (Cheng and Qi (2010)) and a new theoretical framework of BCNs modeled by linear discrete systems has been established (Cheng et al. (2011a)). Based on the linear representation framework, many classical control problems are generalized to BCNs such as controllability, observability, stabilization, synchronization and optimal control (Cheng and Qi (2009), Laschov and Margaliot (2012), Fornasini and Valcher (2013b), Cheng et al. (2011b), Li and Chu (2012), Cheng et al. (2010), Zhao et al. (2011)). Furthermore, many of these results are extended to different kinds of BCNs (Chen and Sun (2012), Li and Sun (2012), Li and Wang (2012), Feng et al. (2012), Zhang et al. (2012)).

It is well-known that, in the traditional linear control system theory, system decompositions play an important role in system analysis. Many control problems are strongly related to system decompositions such as stabilization, designing observers, disturbance decoupling, minimum realization and identification. Indeed, these problems have been investigated for BCN systems (Cheng (2011), Cheng et al. (2010), Cheng and Zhao (2011), Fornasini and Valcher (2013a), Laschov et al. (2013), Zhao et al. (2013)). In Cheng et al. (2010), the decomposition forms called the controllable normal form, the observable normal form and the Kalman decomposition have been investigated. But some regularity conditions are imposed on the BCNs. In fact, not all the BCNs satisfy the regularity assumptions (Zou and Zhu (2014)). Moreover, with the method of Cheng et al. (2010), it is not easy to get the coordinate transformation constructively. In our recent paper Zou and Zhu (2014), the decomposition with respect to inputs is obtained without any regularity assumptions. Under the regularity assumption on the controllable sub-space, the maximum decomposition with respect to inputs is just the controllable normal form in Cheng et al. (2010). Similarly, how to remove the regularity condition for the observable normal form is an interesting issue.

In this paper, we consider the decomposition with respect to outputs in the framework of algebraic representation of BCNs without using any regularity assumptions. We focus on finding a method to obtain a coordinate transformation constructively to realize the maximum decomposition with respect to outputs. In section 2, we give some preliminaries and describe the decomposition with respect to outputs. In section 3, we obtain some equivalent algebraic conditions for the decomposability. In section 4, we obtain a necessary and sufficient graphical condition. In Section 5, an approach is proposed to get the maximum decomposition with respect to outputs. Finally, we give a summary of this paper.

2. PRELIMINARIES AND PROBLEM STATEMENT

Let $\mathcal{D} = \{\text{True} = 1, \text{False} = 0\}$. Consider a BCN described by the logical equations

$$\begin{aligned} x_1(t+1) &= f_1(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)), \\ &\vdots \\ x_n(t+1) &= f_n(x_1(t), \cdots, x_n(t), u_1(t), \cdots, u_m(t)), \\ y_1(t) &= h_1(x_1(t), \cdots, x_n(t)), \\ &\vdots \\ y_p(t) &= h_p(x_1(t), \cdots, x_n(t)), \end{aligned} \tag{1}$$

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where the state variables x_i, the output variables y_i and the controls u_i take values in $\mathcal{D}$, $f_i : \mathcal{D}^{n+m} \to \mathcal{D}$ and $h_i : \mathcal{D}^n \to \mathcal{D}$ are logical functions.

We say that (1) is *decomposable with respect to outputs with order* $n-s$, if there exists a logical coordinate transformation $z_i = g_i(x_1, \cdots, x_n), i = 1, 2, \cdots, n$, such that (1) becomes

$$\begin{aligned} z_1(t+1) &= \hat{f}_1(z_1(t),\cdots,z_s(t),u_1(t),\cdots,u_m(t)),\\ &\vdots\\ z_s(t+1) &= \hat{f}_s(z_1(t),\cdots,z_s(t),u_1(t),\cdots,u_m(t)),\\ z_{s+1}(t+1) &= \hat{f}_{s+1}(z_1(t),\cdots,z_n(t),u_1(t),\cdots,u_m(t)),\\ &\vdots\\ z_n(t+1) &= \hat{f}_n(z_1(t),\cdots,z_n(t),u_1(t),\cdots,u_m(t)),\\ y_1(t) &= \hat{h}_1(z_1(t),\cdots,z_s(t)),\\ &\vdots\\ y_p(t) &= \hat{h}_p(z_1(t),\cdots,z_s(t)). \end{aligned} \tag{2}$$

The BCN (2) is called a *decomposition with respect to outputs with order* $n-s$. A decomposition with respect to outputs with the maximum order is called the *maximum decomposition with respect to outputs*. We say that BCN (1) is *undecomposable with respect to outputs* if the order of the maximum decomposition with respect to outputs is n.

Let $\mathrm{Col}(A)$ be the set of all the columns of matrix A, and denote the ith column of A by $\mathrm{Col}_i(A)$. Set $\triangle_k = \{\delta_k^i | i = 1, 2, \cdots, k\}$, where δ_k^i is the i-th column of $k \times k$ identity matrix I_k. A matrix $L \in \mathcal{R}_{m\times n}$ is called a logical matrix if $\mathrm{Col}(L) \subset \triangle_m$. The set of $m \times r$ logical matrices is denoted by $\mathcal{L}_{m\times r}$. For simplicity, we denote the logical matrix $L = [\delta_m^{i_1}, \delta_m^{i_2}, \cdots, \delta_m^{i_r}]$ by $\delta_m[i_1, i_2, \cdots, i_r]$.

Definition 1. (Cheng et al. (2011a)) Set $A \in \mathcal{R}_{m\times n}$, $B \in \mathcal{R}_{p\times q}$, and $\alpha = \mathrm{lcm}(n, p)$ be the least common multiple of n and p. The left semi-tensor product of A and B is defined as $A \ltimes B = (A \otimes I_{\frac{\alpha}{n}})(B \otimes I_{\frac{\alpha}{p}})$, where $\otimes$ is the Kronecker product.

Since the left semi-tensor product is a generalization of the traditional matrix product, we directly write $A \ltimes B$ as AB. In Cheng et al. (2011a), to express the logical variable with algebraic method, elements in $\mathcal{D}$ are identical with vectors True $\sim \delta_2^1$ and False $\sim \delta_2^2$. For simplicity, we denote $\Delta := \Delta_2 = \{\delta_2^1, \delta_2^2\}$.

Proposition 1. (Cheng and Qi (2010)) Let x_i and u_i take values in Δ and denote $x = \ltimes_{i=1}^n x_i$, $y = \ltimes_{i=1}^p y_i$, $u = \ltimes_{i=1}^m u_i$. Then the BCN (1) can be expressed in the algebraic form

$$\begin{aligned} x(t+1) &= Lu(t)x(t),\\ y(t) &= Hx(t), \end{aligned} \tag{3}$$

where $L \in \mathcal{L}_{2^n\times 2^{n+m}}$ and $H \in \mathcal{L}_{2^p\times 2^n}$.

Let $z = Tx$ be the algebraic form of the logical coordinate transformation $z_i = g_i(x_1, \cdots, x_n)$, where $z = \ltimes_{i=1}^n z_i$ and $T \in \mathcal{L}_{2^n\times 2^n}$ is a permutation matrix. Set $z^{[1]} = \ltimes_{i=1}^s z_i$ and $z^{[2]} = \ltimes_{i=s+1}^n z_i$. Then the decomposition form (2) can be rewritten in the algebraic form

$$\begin{aligned} z^{[1]}(t+1) &= G_1 u(t) z^{[1]}(t),\\ z^{[2]}(t+1) &= G_2 u(t) z(t),\\ y(t) &= M z^{[1]}(t), \end{aligned} \tag{4}$$

where $G_1 \in \mathcal{L}_{2^s\times 2^{s+m}}$, $G_2 \in \mathcal{L}_{2^{n-s}\times 2^{n+m}}$ and $M \in \mathcal{L}_{2^p\times 2^s}$.

For BCN (1) with the algebraic form (3), the *problem of decomposition with respect to outputs* is to find a permutation matrix T such that (3) has the form (4). The *problem of maximum decomposition with respect to outputs* is to find a decomposition with respect to outputs with the maximum order $n-s$.

In Cheng et al. (2010), the observable normal form of a BCN is proposed. Here, we rewrite the result in the algebraic form as follows.

Proposition 2. (Cheng et al. (2010)) Consider the BCN (1) with algebraic form (3). Assume that the largest unobservable subspace $\mathcal{O}_c$ is a regular subspace with $\{\tilde{z}_{\tilde{s}+1}, \tilde{z}_{\tilde{s}+2}, \cdots, \tilde{z}_n\}$ as its basis. Then, under the coordinate transformation $\tilde{z} = \tilde{T}x$, the BCN (3) becomes

$$\begin{aligned} \tilde{z}^{[1]}(t+1) &= \tilde{G}_1 u(t) \tilde{z}^{[1]}(t),\\ \tilde{z}^{[2]}(t+1) &= \tilde{G}_2 u(t) \tilde{z}(t),\\ y(t) &= \tilde{M} \tilde{z}^{[1]}(t), \end{aligned} \tag{5}$$

where $\tilde{z}^{[1]} = \ltimes_{i=1}^{\tilde{s}} \tilde{z}_i$, $\tilde{z}^{[2]} = \ltimes_{i=\tilde{s}+1}^n \tilde{z}_i$, $\tilde{G}_1 \in \mathcal{L}_{2^{\tilde{s}}\times 2^{\tilde{s}+m}}$, $\tilde{G}_2 \in \mathcal{L}_{2^{n-\tilde{s}}\times 2^{n+m}}$ and $\tilde{M} \in \mathcal{L}_{2^p\times 2^{\tilde{s}}}$. Eq. (5) is called an *observable normal form* of (3).

The basic concepts on regular subspace and the largest unobservable subspace $\mathcal{O}_c$ can be found in Cheng et al. (2011a). Comparing (4) with (5), we find that the maximum decomposition with respect to outputs and the observable normal form have the same structure. A natural question is whether they are the same notion. In fact, under the regularity assumption on the largest unobservable subspace $\mathcal{O}_c$, the two concepts are the same. We give the following proposition to illustrate this.

Proposition 3. Assume that the largest unobservable subspace $\mathcal{O}_c$ is regular, then a decomposition with respect to outputs is a maximum decomposition with respect to outputs if and only if it is an observable normal form.

Proof. Assume that (4) is a maximum decomposition with respect to outputs with order $n-s$. We first prove that $s = \tilde{s}$. By the definition of maximum decomposition with respect to outputs, we have $s \le \tilde{s}$. From the definition of largest unobservable subspace, it's easy to get that

$$z_{s+1}, \cdots, z_n \in \mathcal{O}_c = \mathcal{F}(\tilde{z}_{\tilde{s}+1}, \tilde{z}_{\tilde{s}+2}, \cdots, \tilde{z}_n). \tag{6}$$

Thus, (6) implies that $n-s \le n-\tilde{s}$, namely $s \ge \tilde{s}$. In the following, we say that (4) and (5) are equivalent. Since $\{\tilde{z}_{\tilde{s}+1}, \cdots, \tilde{z}_n\}$ is the sub-basis of state space $\mathcal{X} = \mathcal{F}(x_1, x_2, \cdots, x_n)$ and $\tilde{s} = s$, then (5) is a maximum decomposition with respect to outputs for the system (3). Conversely, by $\tilde{s} = s$, (6) and Theorem 13 of Cheng et al. (2010), we obtain that $\{z_{s+1}, \cdots, z_n\}$ is also a regular basis of $\mathcal{O}_c$, that is, (4) is an observable normal form. □

Proposition 3 implies that the decomposition with respect to outputs is a generalization of the observable normal form. In the remainder of this paper, we will give a method to realize the decomposition with respect to outputs for the system (3).

3. ALGEBRAIC EQUIVALENT CONDITIONS FOR THE DECOMPOSABILITY WITH RESPECT TO OUTPUTS

In this section, based on the definition of decomposability with respect to outputs, we derive some equivalent algebraic conditions. We denote the $n-$dimensional column vector whose entries are equal to 1 by $\mathbf{1}_n$.

Lemma 1. Assume that $M_1,\ M_2,\ \cdots,M_l \in \mathcal{R}_{m\times n}$ are non-negative matrices satisfying $\mathbf{1}_m^{\mathrm{T}} M_k = \mathbf{1}_n^{\mathrm{T}}$ for every $k = 1, 2, \cdots, l$. If $M_1 + M_2 + \cdots + M_l = lG$, with G being a logical matrix, then $M_1 = M_2 = \cdots = M_l = G$.

Swap matrix $W_{[m,n]}$ is an $mn \times mn$ logical matrix, defined as $W_{[m,n]} = [I_n \otimes \delta_m^1, I_n \otimes \delta_m^2, \cdots, I_n \otimes \delta_m^m]$.

Lemma 2. (Cheng et al. (2011a)) Let $W_{[m,n]} \in \mathcal{R}_{mn\times mn}$ be a swap matrix. Then $W_{[m,n]}^{\mathrm{T}} = W_{[m,n]}^{-1} = W_{[n,m]}$ and $W_{[m,1]} = W_{[1,m]} = I_m$, where I_m is an identity matrix.

Lemma 3. (Cheng et al. (2011a)) Let $A \in \mathcal{R}_{m\times n}$, $B \in \mathcal{R}_{p\times q}$. Then $W_{[m,p]}(A \otimes B)W_{[q,n]} = (B \otimes A)$.

Theorem 1. Consider the BCN (1) with algebraic form (3). Let $L = [L_1, L_2, \cdots, L_{2^m}], L_i \in \mathcal{L}_{2^n\times 2^n}$. Then the following statements are equivalent:
1) the system (1) is decomposable with respect to outputs with order $n - s$;
2) there exist a permutation matrix $T \in \mathcal{L}_{2^n\times 2^n}$ and logical matrices $G_1 \in \mathcal{L}_{2^s\times 2^{m+s}}$, $M \in \mathcal{L}_{2^p\times 2^s}$ such that

$$QL(I_{2^m} \otimes T^{\mathrm{T}}) = G_1(I_{2^{m+s}} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}}), \quad (7)$$
$$HT^{\mathrm{T}} = M(I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}}), \quad (8)$$

where $Q = (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})T$;
3) there exist a permutation matrix $T \in \mathcal{L}_{2^n\times 2^n}$ and logical matrices $G_1 = [A_1, A_2, \cdots, A_{2^m}] \in \mathcal{L}_{2^s\times 2^{m+s}}, A_i \in \mathcal{L}_{2^s\times 2^s}$, $M \in \mathcal{L}_{2^p\times 2^s}$ such that

$$QL_i = A_iQ, \quad (9)$$
$$H = MQ, \quad (10)$$

where $Q = (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})T$;
4) there exists a permutation matrix $T \in \mathcal{L}_{2^n\times 2^n}$ such that

$$\frac{1}{2^{n-s}} QL_iQ^{\mathrm{T}} \quad (11)$$

and

$$\frac{1}{2^{n-s}} HQ^{\mathrm{T}} \quad (12)$$

are logical matrices, where $Q = (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})T$.

Proof. 1) ⇔ 2) Assume that (3) has the decomposition (4) with respect to outputs with order $n - s$. By (3), we have

$$z^{[1]}(t + 1) = (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})z(t + 1)$$
$$= (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})TL(I_{2^m} \otimes T^{\mathrm{T}})u(t)z(t), \quad (13)$$
$$y(t) = HT^{\mathrm{T}} z^{[1]}(t)z^{[2]}(t). \quad (14)$$

By (4), we have

$$z^{[1]}(t + 1) = G_1(I_{2^{m+s}} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})u(t)z(t), \quad (15)$$
$$y(t) = M(I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})z^{[1]}(t)z^{[2]}(t). \quad (16)$$

From (13) and (15), we get (7). From (14) and (16), we get (8). Conversely, by (7), (8) and (13), (14), we get the decomposition (4) with respect to outputs.
2) ⇒ 3) Multiplying (7) on the right by $I_{2^m} \otimes T$ yields $QL = G_1[I_{2^m}\otimes((I_{2^s}\otimes\mathbf{1}_{2^{n-s}}^{\mathrm{T}})T)] = [A_1Q,\ A_2Q,\ \cdots,\ A_{2^m}Q]$, which implies $QL_i = A_iQ$, thus (9) is obtained. Multiplying (8) on the right by T gives (10).
3) ⇒ 4) Multiplying (9) on the right by $T^{\mathrm{T}}(I_{2^s} \otimes \mathbf{1}_{2^{n-s}})$ gives the logical matrix A_i shown by (11). Similarly, multiplying (10) on the right by $T^{\mathrm{T}}(I_{2^s} \otimes \mathbf{1}_{2^{n-s}})$ yields the logical matrix M shown by (12).
4) ⇒ 3) We Denote the logical matrix (11) and (12) by A_i and M respectively. Let

$$QL_iT^{\mathrm{T}}W_{[2^{n-s},2^s]} = [P_1, P_2, \cdots, P_{2^{n-s}}], \quad (17)$$

where $P_i \in \mathcal{L}_{2^s\times 2^s}$ are non-negative matrices. By (17) and Lemma 2, 3, we have

$$\begin{aligned} A_i &= \frac{1}{2^{n-s}} QL_iQ^{\mathrm{T}} \\ &= \frac{1}{2^{n-s}} [P_1, P_2, \cdots, P_{2^{n-s}}]W_{[2^s,2^{n-s}]}(I_{2^s} \otimes \mathbf{1}_{2^{n-s}}) \quad (18) \\ &= \frac{1}{2^{n-s}} [P_1, P_2, \cdots, P_{2^{n-s}}](\mathbf{1}_{2^{n-s}} \otimes I_{2^s}), \end{aligned}$$

that is,

$$\sum_{i=1}^{2^{n-s}} (\frac{1}{2^{n-s}} P_i) = A_i. \quad (19)$$

Multiplying (17) on the left by $\mathbf{1}_{2^s}^{\mathrm{T}}$ yields

$$\mathbf{1}_{2^s}^{\mathrm{T}}[P_1, P_2, \cdots, P_{2^{n-s}}] = \mathbf{1}_{2^n}^{\mathrm{T}} W_{[2^{n-s},2^s]} = \mathbf{1}_{2^n}^{\mathrm{T}}, \quad (20)$$

namely $\mathbf{1}_{2^s}^{\mathrm{T}} P_i = \mathbf{1}_{2^s}^{\mathrm{T}}$ for every $i = 1, 2, \cdots, 2^{n-s}$. Thus, by Lemma 1 and (19), it follows that $P_i = A_i$. Then

$$QL_iT^{\mathrm{T}}W_{[2^{n-s},2^s]} = A_i(\mathbf{1}_{2^{n-s}}^{\mathrm{T}} \otimes I_{2^s}). \quad (21)$$

By (21) and Lemma 2, 3, we have

$$QL_iT^{\mathrm{T}} = A_i(I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}}). \quad (22)$$

Thus $QL_i = A_i(I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})T$, then (9) is proved. With a same procedure above, we can get (10) from (12).
3) ⇒ 2) Since $I_{2^m} \otimes T$ and T are nonsingular matrices, it's easy to get (7) and (8) from (9) and (10) respectively. □

Proposition 4. Let $R_k = \{q|\mathrm{Col}_q(H) = \delta_{2^p}^k\}, k = 1, 2, \cdots, 2^p$ and $|R_k| = h_k$. Assume that h is the greatest common divisor of $h_1, h_2, \cdots, h_{2^p}$. Let $r_0 = \max\{r\,|\,2^r \text{ is a factor of } h\}$. If (3) has the decomposition (4) with respect to outputs, then the order of (4) is at most r_0. Particularly, if $r_0 = 0$ we can directly say that the system is undecomposable with respect to outputs.
Proof. Multiplying (8) on the right by $\mathbf{1}_{2^n}$ gives $H\mathbf{1}_{2^n} = 2^{n-s}M\mathbf{1}_{2^s}$, i.e. $[h_1, h_2, \cdots, h_{2^p}]^{\mathrm{T}} = 2^{n-s}M\mathbf{1}_{2^s}$. Thus, by the definition of r_0, we have $n - s \le r_0$. □

Theorem 1 gives some equivalent algebraic conditions for the decomposability with respect to outputs. But it does not give a method to construct the transformation matrix T. In the following section, we try to give a procedure to compute T.

4. PERFECT EQUAL VERTEX PARTITION

We first introduce some concepts of graph theory.

Definition 2. Let A be the vertex set of a graph $\mathcal{G}$, and $\Phi_l, l = 1, 2, \cdots, \mu$ be subsets of A. $\{\Phi_l\}_{l=1}^{\mu}$ is called a *vertex partition* of A, if $\cup_{i=1}^{\mu}\Phi_l = A$ and $\Phi_i\cap\Phi_j = \emptyset$ for any $i \ne j$. A vertex partition $\{S_l\}_{l=1}^{\mu}$ of A is called an *equal vertex partition* if $|S_l| = |A|/\mu$ for every $l = 1, 2, \cdots, \mu$.

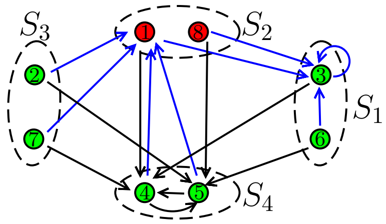

Fig.1. digraph corresponding to a BCN.

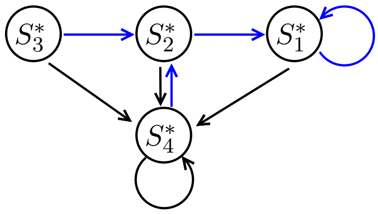

Fig.2. shrunken digraph of Fig.1.

For the definition of vertex partition, we admit that some Φ_l is empty set. This is just for the convenience of stating the following contents.

Proposition 5. Assume that A has two vertex partitions, denoted by $\{\tilde{\Phi}_l\}_{l=1}^{\tilde{\mu}}$ and $\{\Phi_l\}_{l=1}^{\mu}$ respectively. Suppose that, for each $l = 1, 2, \cdots, \tilde{\mu}$, there exists k_l such that $\tilde{\Phi}_l \subset \Phi_{k_l}$. Set $G_j = \{l | \ \tilde{\Phi}_l \subset \Phi_j\}$. Then we have $\Phi_j = \cup_{l \in G_j} \tilde{\Phi}_l$ for each $j = 1, 2, \cdots, \mu$.

Every logical matrix $L_i \in \mathcal{L}_{2^n \times 2^n}$ can be regarded as an adjacency matrix of a digraph $\mathcal{G}_i$. Here, the vertex set of $\mathcal{G}_i$ is $A = \{1, 2, 3, \cdots, 2^n\}$. $\mathcal{G}_i$ has a directed edge (q, k) if and only if $(L_i)_{kq} \neq 0$. We call $\mathcal{G}_i$ the *induced digraph* of matrix L_i. It is said that k is an out-neighbor of q if $(L_i)_{kq} \neq 0$. Denote the *out-neighborhood* of set S by $\mathcal{N}(S)$. By the definition of $\mathcal{N}(S)$, we have

$$q \in S, \quad p \notin \mathcal{N}(S) \Rightarrow (L_i)_{kq} = 0. \tag{23}$$

Definition 3. Consider a digraph $\mathcal{G}$ with the vertex set A. The equal vertex partition $\{S_l\}_{l=1}^{\mu}$ of A is called a *perfect equal vertex partition* (PEVP) if, for any $l = 1, 2, \cdots, \mu$, there exists an $\alpha_l \in \{1, 2, \cdots, \mu\}$ such that $\mathcal{N}(S_l) \subset S_{\alpha_l}$.

Fig.1 gives a digraph corresponding to the BCN $x(t+1) = [L_1, L_2]u(t)x(t)$, where $u \in \triangle, x \in \triangle_{2^3}$. Assume that $\mathcal{G}_i (i = 1, 2)$ is the induced digraph of matrix L_i. In Fig. 1, $\mathcal{G}_1$ is described by the blue edges and $\mathcal{G}_2$ by the black edges. Furthermore, the vertices with the same color have the same output. Fig.1 shows a PEVP $\{S_l\}_{l=1}^{4}$ for both $\mathcal{G}_1$ and $\mathcal{G}_2$. We shrink each S_l to a vertex S_l^* to construct another digraph $\mathcal{G}_i^*$, where S_l^* and S_j^* are adjacent in $\mathcal{G}_i^*$ if and only if there are some $v_l \in S_l$ and $v_j \in S_j$ such that v_l and v_j are adjacent in $\mathcal{G}_i$. Fig.2 shows the shrunken digraph $\mathcal{G}_i^*$ of $\mathcal{G}_i$.

Based on the above contents, we will propose an equivalent graphical condition for the decomposability with respect to outputs. Before the main result, we first give an intuitive explanation on the motivation. From the decomposition (2) with respect to outputs, we can see that, if $z_1, \cdots, z_s$ are fixed, the set

$$\{(z_1, \cdots, z_s, z_{s+1}, \cdots, z_n) \mid z_j \in \{0, 1\}, \ s+1 \leq j \leq n\} \tag{24}$$

has 2^{n-s} states. We denote (24) by $S_{z_1 \cdots z_s}$. Then the family

$$\{S_{z_1 \cdots z_s} \mid z_j \in \{0, \ 1\}, \ j = 1, 2, \cdots, s\}, \tag{25}$$

forms an equal partition of all the 2^n states. By the decomposition (2) with respect to outputs, we know that the state $z(t)$ transmits from one $S_{z_1 z_2 \cdots z_s}$ to another as the control $u(t) = \delta_{2^m}^i$ is fixed. Thus, the equal vertex partition (25) is perfect for any induced matrix $\mathcal{G}_i$ of L_i. Moreover, by the output expressions, the vertices in $S_{z_1 \cdots z_s}$ have the same output.

Assume $L = [L_1, L_2, \cdots, L_{2^m}], L_i \in L_{2^n \times 2^n}$ and $R_k = \{q | \mathrm{Col}_q(H) = \delta_{2^p}^k\}, k = 1, 2, \cdots, 2^p$. Let $\mathcal{G}_i$ be the induced digraph of L_i. We denote the out-neighborhood of vertex set S_l in digraph $\mathcal{G}_i$ by $N^i(S_l)$.

Theorem 2. Consider BCN (1) with the algebraic form (3). The system (1) is decomposable with respect to outputs with order $n - s$ if and only if there exists an equal vertex partition $\{S_l\}_{l=1}^{2^s} (|S_l| = 2^{n-s})$ such that
(i) $\{S_l\}_{l=1}^{2^s}$ is a PEVP for any digraph $\mathcal{G}_i$,
(ii) for any l, there exists an α_l such that $S_l \subset R_{\alpha_l}$.

Proof. (Necessity)By Theorem 1, there exist a permutation matrix $T \in \mathcal{L}_{2^n \times 2^n}$ and logical matrices $A_i \in \mathcal{L}_{2^s \times 2^s} (i = 1, 2, \cdots, 2^m)$, $M \in \mathcal{L}_{2^p \times 2^s}$ such that (9) and (10) hold. Set

$$Q = (I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})T = \delta_{2^s}[i_1, \cdots, i_{2^n}]. \tag{26}$$

Thus, (9) and (10) can be rewritten as

$$\delta_{2^s}[i_1, \cdots, i_{2^n}]L_i = A_i \delta_{2^s}[i_1, \cdots, i_{2^n}], \tag{27}$$

$$H = M\delta_{2^s}[i_1, \cdots, i_{2^n}]. \tag{28}$$

Let $S_l = \{q | i_q = l\}$. Then $\{S_l\}_{l=1}^{2^s}$ is an equal vertex partition of A with $|S_l| = 2^{n-s}$. For any l, we have

$$\exists \ \alpha_l^i, \ \alpha_l, \ \text{s.t.} \ A_i \delta_{2^s}^l = \delta_{2^s}^{\alpha_l^i}, \ M\delta_{2^s}^l = \delta_{2^p}^{\alpha_l}. \tag{29}$$

For any $k \in \mathcal{N}^i(S_l)$, there exists $q \in S_l$ $(i_q = l)$ such that $\mathrm{Col}_q(L_i) = \delta_{2^n}^k$. Then, we have

$$\delta_{2^s}[i_1, \cdots, i_{2^n}]\mathrm{Col}_q(L_i) = \delta_{2^s}^{i_k} = A_i \delta_{2^s}^{i_q} = A_i \delta_{2^s}^l = \delta_{2^s}^{\alpha_l^i}. \tag{30}$$

Thus $i_k = \alpha_l^i$, which implies $k \in S_{\alpha_l^i}$. Therefore, we have $\mathcal{N}^i(S_l) \subset S_{\alpha_l^i}$. By definition 3, (i) is proved.

For any $k \in S_l$, we have $i_k = l$ and $\mathrm{Col}_k(H) = M\delta_{2^s}^{i_k} = \delta_{2^p}^{\alpha_l}$, which implies $k \in R_{\alpha_l}$. Thus (ii) is proved due to $S_l \subset R_{\alpha_l}$.

(Sufficiency) For any $q \in S_l, l = 1, 2, \cdots, 2^s$, let $i_q = l$. We denote $Q = \delta_{2^s}[i_1, \cdots, i_{2^n}]$. Since $|S_l| = 2^{n-s}$, there exists a permutation matrix T such that $(I_{2^s} \otimes \mathbf{1}_{2^{n-s}}^{\mathrm{T}})T = Q$. For any $l \in \{1, 2, \cdots, 2^s\}$, it follows that

$$\begin{aligned}
\mathrm{Col}_l(QL_iQ^{\mathrm{T}}) &= QL_i\mathrm{Col}_l(Q^{\mathrm{T}}) = \sum_{q \in S_l} QL_i\delta_{2^n}^q \\
&= \sum_{q \in S_l} Q\mathrm{Col}_q(L_i) \\
&= \sum_{q \in S_l} Q\delta_{2^n}^k \quad (k \in N^i(q) \subset N^i(S_l) \subset S_{\alpha_l^i}) \\
&= \sum_{q \in S_l} \delta_{2^s}^{i_k} = \sum_{q \in S_l} \delta_{2^s}^{\alpha_l^i} = 2^{n-s}\delta_{2^s}^{\alpha_l^i},
\end{aligned}$$

$$\begin{aligned}
\mathrm{Col}_l(HQ^{\mathrm{T}}) &= H\mathrm{Col}_l(Q^{\mathrm{T}}) = \sum_{q \in S_l} H\delta_{2^n}^q \\
&= \sum_{q \in S_l} \mathrm{Col}_q(H) \quad (q \in S_l \subset R_{\alpha_l}) \\
&= \sum_{q \in S_l} \delta_{2^s}^{\alpha_l} = 2^{n-s}\delta_{2^s}^{\alpha_l},
\end{aligned}$$

which implies $\frac{1}{2^{n-s}}QL_iQ^{\mathrm{T}}$ and $\frac{1}{2^{n-s}}HQ^{\mathrm{T}}$ are logical matrices. Therefore, by 4) of Theorem 1, the sufficiency is proved. □

Compared with the state-space method proposed by Cheng et al. (2010), this Theorem gives a constructive procedure to calculate the transformation matrix. If the graphical condition of Theorem 1 is satisfied, it's easy to construct Q. Thus, we can get T by $(I_{2^s}\otimes \mathbf{1}^{\mathrm{T}}_{2^{n-s}})T=Q$.

To display the effectiveness of Theorem 2, we reconsider Example 10.3 of Cheng et al. (2011a) and construct the coordinate transformation using the graphical method.

Example 1. Consider the following BCN

$$\begin{cases} x_1(t+1)=x_3(t)\vee u(t),\\ x_2(t+1)=(x_1(t)\wedge\neg x_3(t))\vee(\neg x_1(t)\wedge(x_3(t)\leftrightarrow u(t))),\\ x_3(t+1)=x_3(t)\rightarrow u(t),\\ y(t)=(x_1(t)\leftrightarrow x_3(t))\rightarrow(x_2(t)\bar{\vee}x_3(t)).\end{cases} \quad (31)$$

Let $x(t)=x_1(t)x_2(t)x_3(t)$. Then we have $x(t+1)=Lu(t)x(t)$ and $y(t)=Hx(t)$, where $L=[L_1,L_2]\in\mathcal{L}_{8\times 16}$, with $L_1=\delta_8[3\ 1\ 3\ 1\ 1\ 3\ 1\ 3]$, $L_2=\delta_8[4\ 5\ 4\ 5\ 4\ 5\ 4\ 5]$ and $H=\delta_2[2\ 1\ 1\ 1\ 1\ 1\ 1\ 2]$. The digraph corresponding to the BCN is just shown in Fig.1. The partition $S_i(i=1,2,3,4)$ given in Fig.1 is a PEVP for both $\mathcal{G}_1$ and $\mathcal{G}_2$. Furthermore, by the color of vertices, the vertices in S_l have the same output. Thus, the system is decomposable with respect to outputs with order 1. Let

$$(I_4\otimes\mathbf{1}_2^{\mathrm{T}})T=Q=\delta_4[2\ 3\ 1\ 4\ 4\ 1\ 3\ 2],$$

it follows that $T=\delta_8[3\ 6\ 1\ 8\ 7\ 2\ 5\ 4]$. The coordinate transformation matrix T is the same as that given in Example 10.3 of Cheng et al. (2011a). Therefore, the decomposition with respect to outputs is obtained as

$$\begin{cases} z_1(t+1)=u(t),\\ z_2(t+1)=z_1(t)\wedge u(t),\\ z_3(t+1)=z_3(t)\rightarrow u(t).\\ y(t)=z_1(t)\rightarrow z_2(t).\end{cases}$$

5. FINDING A PEVP

In this section, we first give an equivalent condition of the concept PEVP and then provide an effective method to calculate a PEVP.

Theorem 3. A given equal vertex partition $\{S_l\}_{l=1}^{2^s}$ is perfect for the digraph $\mathcal{G}_i$ if and only if there exists a vertex partition $\{\Phi_l^i\}_{l=1}^{2^s}$ of $\mathcal{G}_i$ satisfying

$$\forall\ 1\le l\le 2^s,\exists\ \alpha_l^i,\ \text{s.t.}\ \mathcal{N}^i(\Phi_l^i)\subset S_l\subset\Phi^i_{\alpha_l^i}. \quad (32)$$

Proof. (Sufficiency) Assume that (32) holds. It is easy to get that $\mathcal{N}^i(S_l)\subset\mathcal{N}^i(\Phi^i_{\alpha_l^i})\subset S_{\alpha_l^i}$. Thus $\{S_l\}_{l=1}^{2^s}$ is a PEVP for the digraph $\mathcal{G}_i$.

(Necessity) Assume that $\{S_l\}_{l=1}^{2^s}$ is a PEVP for the digraph $\mathcal{G}_i$, it follows that

$$\forall\ 1\le l\le 2^s,\exists\ \alpha_l^i,\ \text{s.t.}\ \mathcal{N}^i(S_l)\subset S_{\alpha_l^i}. \quad (33)$$

Set

$$P_l^i=\{k\mid\mathcal{N}^i(S_k)\subset S_l\},\ \Phi_l^i=\bigcup_{k\in P_l^i}S_k. \quad (34)$$

Then, $\{\Phi_l^i\}_{l=1}^{2^s}$ is a partition of $\mathcal{G}_i$. It follows that

$$\mathcal{N}^i(\Phi_l^i)=\bigcup_{k\in P_l^i}\mathcal{N}^i(S_k)\subset S_l\subset\bigcup_{\mu\in P^i_{\alpha_l^i}}S_\mu=\Phi^i_{\alpha_l^i}. \quad (35)$$

Thus, (32) is proved. □

In Theorem 3, every PEVP $\{S_l\}_{l=1}^{2^s}$ of $\mathcal{G}_i$ corresponds to a vertex partition $\{\Phi_l^i\}_{l=1}^{2^s}$ satisfying (32). In the following, we will give some properties of $\{\Phi_l^i\}_{l=1}^{2^s}$, which is very useful for finding a PEVP.

Assume that the BCN (1) with algebraic form (3) is decomposable with respect to outputs with order s. Then by Theorem 1, there exist a permutation matrix $T\in\mathcal{L}_{2^n\times 2^n}$ and logical matrices $A_\mu\in\mathcal{L}_{2^s\times 2^s}(\mu=1,2,\cdots,2^m)$ such that (9) holds. We denote

$$Q=\delta_{2^s}[i_1,\cdots,i_{2^n}], \quad (36)$$

$$A_\mu Q=\delta_{2^s}[j_{\mu 1},\cdots,j_{\mu 2^n}]. \quad (37)$$

Then, from (9), it follows that

$$\delta_{2^s}[i_1,\cdots,i_{2^n}]L_\mu=\delta_{2^s}[j_{\mu 1},\cdots,j_{\mu 2^n}]. \quad (38)$$

By the necessity of Theorem 2, set $S_l=\{q|i_q=l\}$, then $\{S_l\}_{l=1}^{2^s}$ is a PEVP for each $\mathcal{G}_\mu$.

Proposition 6. Consider the PEVP $\{S_l\}_{l=1}^{2^{n-s}}$ with each $S_l=\{p|i_p=l\}$ constructed as in the necessity proof of Theorem 2. Let $\Phi_l^\mu=\{p|j_{\mu p}=l\}$. Then $\{\Phi_l^\mu\}_{l=1}^{2^{n-s}}$ is a vertex partition satisfying (32).

Proof. From (32), one can easily see that

$$\forall\ 1\le l\le 2^{n-s},\ \exists\ \alpha_l^\mu,\ \text{s.t.}\ S_l\subset\Phi^\mu_{\alpha_l^\mu}.$$

Moreover, for any $p\in\mathcal{N}(\Phi_l^\mu)$, there exists $q\in\Phi_l^\mu$, i.e. $j_{\mu q}=l$, such that $(L_\mu)_{pq}\neq 0$. Thus it follows from (38) that

$$2^m\delta^l_{2^{n-s}}=2^m\delta^{j_{\mu q}}_{2^{n-s}}=\sum_{k=1}^{2^n}\delta^{i_k}_{2^{n-s}}(L_\mu)_{kq},$$

which implies $i_p=l$, i.e. $p\in S_l$ due to $(L_\mu)_{pq}\neq 0$. Therefore we have $\mathcal{N}(\Phi_l^\mu)\subset S_l$. □

Proposition 7. Denote $R_k^i=\{q|\mathrm{Col}_q(L_i)=\delta^k_{2^n}\},k=1,2,\cdots,2^n$. Then the following statements hold:

(i) for any R_k^i, there exists a ξ_k such that $R_k^i\subset\Phi^i_{\xi_k}$ and $\mathcal{N}^i(R_k^i)\subset\mathcal{N}^i(\Phi^i_{\xi_p})$;

(ii)

$$\Phi_l^i=\bigcup_{k\in G_l^i}R_k^i,\quad \mathcal{N}^i(\Phi_l^i)=\bigcup_{k\in G_l^i}\mathcal{N}^i(R_k^i), \quad (39)$$

where $G_l^i=\{k|\ R_k^i\subset\Phi_l^i\}$ for any l.

Proof. (i) For any $q\in R_k^i$, we have $\mathrm{Col}_q(L_i)=\delta^k_{2^n}$. From (38), it follows that $\delta^{i_k}_{2^s}=\delta^{j_q}_{2^s}$. Set $\xi_k=i_k$, then $j_q=\xi_k$, which implies $q\in\Phi^i_{\xi_k}$. Therefore, $R_k^i\subset\Phi^i_{\xi_k}$ and consequently $\mathcal{N}^i(R_k^i)\subset\mathcal{N}^i(\Phi^i_{\xi_p})$.

(ii) Since $\{R_k^i\}_{k=1}^{2^s}$ and $\{\Phi_l^i\}_{l=1}^{2^s}$ are two vertex partitions of A, from (i) and Proposition 5, Eq. (39) is derived. □

From the above contents, we know that it is only needed to search PEVP from the vertex partition $\{\Phi_l^i\}_{l=1}^{2^s}$. We illustrate this procedure using the example as follows.

Example 2. Reconsider the system (31), we have

$$R_1=\{2,3,4,5,6,7\},\quad R_2=\{1,8\}, \quad (40)$$

$$R_1^1=\{2,4,5,7\},\quad R_3^1=\{1,3,6,8\}, \quad (41)$$

$$R_4^2=\{1,3,5,7\},\quad R_5^2=\{2,4,6,8\}, \quad (42)$$

where all the other sets in $\{R_l^1\}_{l=1}^8$ and $\{R_l^2\}_{l=1}^8$ are $\emptyset$. By (40) and Proposition 4, we have $h=2$ and $r_0=1$, then

$s \geq n - r_0 = 2$. If the system is decomposable with respect to outputs, then s must be 2. In the following, we try to explore whether there exists an equal vertex partition $\{S_l\}_{l=1}^4 (|S_l| = 2)$ satisfying the conditions of Theorem 2. By (ii) of Theorem 2 and (40), we can let $S_1 = \{1, 8\} \subset R_2$ and $S_2, S_3, S_4 \subset R_1$. From (41) and (42), it follows that

$$S_1 \subset R_3^1, \quad S_1 \subset R_4^2 \cup R_5^2.$$

Since there exists α_1^i such that $S_1 \subset \Phi_{\alpha_1^i}^i$, by (39), we let

$$\Phi_1^1 = R_3^1, \quad \Phi_1^2 = R_4^2 \cup R_5^2.$$

Since $\{4, 5\} = N^2(\Phi_1^2) \subset \Phi_1^2$, we let $S_2 = \{4, 5\}$. Thus by (41) and Theorem 3, we can directly let $S_3 = \{2, 7\}$, $S_4 = \{3, 6\}$. Then the equal vertex partition $\{S_l\}_{l=1}^4$ is obtained, which is a PEVP for both $\mathcal{G}_1$ and $\mathcal{G}_2$, corresponding to L_1 and L_2 respectively. Moreover, (40) implies that the vertices in S_l have the same output. Then, the system is decomposable with respect to outputs. In the future work, we will address the Kalman decomposition without the regularity assumptions.

6. CONCLUSIONS

We have investigated the decomposition with respect to outputs for BCNs, which is a generalization of the observability decomposition of the traditional linear control theory. Our analysis relies on some equivalent algebraic and graphical conditions of the decomposability with respect to outputs. It has been revealed that a BCN is decomposable with respect to outputs if and only if it has an equal vertex partition satisfying some conditions. The main advantage of our results lie in that no any regularity assumption is used and a constructive approach is provided.

REFERENCES

Albert, R. and Othmer, H.G. (2003). The topology of the regulatory interactions predicts the expression pattern of the segment polarity genes in drosophila melanogaster. *Journal of theoretical biology*, 223(1), 1–18.

Chen, H. and Sun, J. (2012). A new approach for global controllability of higher order boolean control network. *Neural Networks*, 39, 12–17.

Cheng, D. (2011). Disturbance decoupling of boolean control networks. *IEEE Trans. on Automatic Control*, 56(1), 2–10.

Cheng, D., Li, Z., and Qi, H. (2010). Realization of boolean control networks. *Automatica*, 46(1), 62–69.

Cheng, D. and Qi, H. (2009). Controllability and observability of boolean control networks. *Automatica*, 45(7), 1659–1667.

Cheng, D. and Qi, H. (2010). A linear representation of dynamics of boolean networks. *IEEE Trans. on Automatic Control*, 55(10), 2251–2258.

Cheng, D., Qi, H., and Li, Z. (2011a). *Analysis and control of Boolean networks: a semi-tensor product approach*. Springer.

Cheng, D., Qi, H., Li, Z., and Liu, J.B. (2011b). Stability and stabilization of boolean networks. *International Journal of Robust and Nonlinear Control*, 21(2), 134–156.

Cheng, D. and Zhao, Y. (2011). Identification of boolean control networks. *Automatica*, 47(4), 702–710.

Datta, A., Choudhary, A., Bittner, M.L., and Dougherty, E.R. (2004). External control in markovian genetic regulatory networks: the imperfect information case. *Bioinformatics*, 20(6), 924–930.

Faure, A., Naldi, A., Chaouiya, C., and Thieffry, D. (2006). Dynamical analysis of a generic boolean model for the control of the mammalian cell cycle. *Bioinformatics*, 22(14), e124–e131.

Feng, J., Yao, J., and Cui, P. (2012). Singular boolean networks: Semi-tensor product approach. *Science China Information Sciences*, 56(11), 1–14.

Fornasini, E. and Valcher, M. (2013a). Observability, reconstructibility and state observers of boolean control networks. *IEEE Trans. on Automatic Control*, 58(6), 1390–1401.

Fornasini, E. and Valcher, M.E. (2013b). On the periodic trajectories of boolean control networks. *Automatica*, 49(5), 1506–1509.

Genoud, T. and Metraux, J.P. (1999). Crosstalk in plant cell signaling: structure and function of the genetic network. *Trends in plant science*, 4(12), 503–507.

Laschov, D. and Margaliot, M. (2012). Controllability of boolean control networks via the perron–frobenius theory. *Automatica*, 48(6), 1218–1223.

Laschov, D., Margaliot, M., and Even, G. (2013). Observability of boolean networks: a graph-theoretic approach. *Automatica*, 48(8), 2351–2362.

Li, F. and Sun, J. (2012). Controllability of higher order boolean control networks. *Applied Mathematics and Computation*, 219(1), 158–169.

Li, H. and Wang, Y. (2012). On reachability and controllability of switched boolean control networks. *Automatica*, 48(11), 2917–2922.

Li, R. and Chu, T. (2012). Complete synchronization of boolean networks. *IEEE Trans. on Neural Networks and Learning Systems*, 23(5), 840–846.

Pal, R., Datta, A., Bittner, M.L., and Dougherty, E.R. (2005). Intervention in context-sensitive probabilistic boolean networks. *Bioinformatics*, 21(7), 1211–1218.

Shmulevich, I. and Kauffman, S.A. (2004). Activities and sensitivities in boolean network models. *Physical Review Letters*, 93(4), 048701.

Snoussi, E.H. (1989). Qualitative dynamics of piecewise-linear differential equations: a discrete mapping approach. *Dynamics and stability of Systems*, 4(3-4), 565–583.

Zhang, L., Feng, J., and Yao, J. (2012). Controllability and observability of switched boolean control networks. *Control Theory & Applications, IET*, 6(16), 2477–2484.

Zhao, Y., Kim, J., and Filippone, M. (2013). Aggregation algorithm towards large-scale boolean network analysis. *IEEE Trans. on Automatic Control*, 56(8), 1976–1985.

Zhao, Y., Li, Z., and Cheng, D. (2011). Optimal control of logical control networks. *IEEE Trans. on Automatic Control*, 56(8), 1766–1776.

Zou, Y. and Zhu, J. (2014). System decomposition with respect to inputs for Boolean control networks. *Automatica*, DOI: 10.1016/j.automatica.2014.02.039.