Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

S-procedure – an infinite dimensional view ⋆

Z. Szabó *

* *Institute for Computer Science and Control, Hungarian Academy of Sciences, Hungary, (Tel: +36-1-279-6171; e-mail: szabo.zoltan@sztaki.mta.hu).*

Abstract: Variants of the S-procedure provide an important tool in robust stability and robust performance design, mainly in the state space framework based on linear matrix inequalities. The aim of this paper is to formulate three classical results, the basic S-lemma, the Finsler's lemma and a variant of the full-block S-procedure, in the input output framework, i.e., in the infinite dimensional setting. While the presented results widen the scope of the applicability of these fundamental tools, they also provide a link to the theory of indefinite spaces, as an efficient framework to handle robust control problems.

1. INTRODUCTION AND MOTIVATION

Robust stability and robust performance analysis and synthesis of control systems with parameter uncertainties and parameter variations is one of the fundamental issues in system theory. In the most common framework models are augmented with performance specifications and uncertainties while weighting functions are applied to the performance signals to meet performance specifications and guarantee a tradeoff between performances. As a result of this construction a linear fractional transformation (LFT) interconnection structure, which is the basis of control design, is achieved, in which a design problem for robust quadratic performance is formulated.

An efficient solution technique was developed in the state space framework by solving a set of linear matrix inequalities (LMIs). The LMIs are obtained by using some variant of the S-procedure and usually involve a relaxation of an infinite number of conditions to a set of finite number of constraints. The main theoretical and practical tools in this respect are the full-block S-procedure (extended KYP lemma, or robust Finsler's lemma), a variant of the Elimination lemma and some variant of the classical S-procedure, see, e.g., Iwasaki and Hara [1998], Scherer [2001]. The classical S-procedure, Yakubovich [1977] is a relaxation method: it tries to solve a system of quadratic inequalities via a LMI relaxation.

Despite the success of the approach in handling all kinds of control problems there are still some basic issues concerning the design. Apart the basic $\mathcal{H}_\infty$ problem and some special configurations these techniques leads to a certain amount of conservatism in the achieved performance. Much effort has been done in the lossless parametrization of the multipliers associated to a given uncertainty set. One way to reduce the conservatism is the application of an IQC technique, see, e.g., Megretski and Rantzer [1997], with some dynamic multipliers. More difficulty arises in the solution of the qLPV problems, formulated in the solution of quasi linearized nonlinear design tasks.

In Szabó et al. [2013] the authors provide the infinite dimensional counterpart of the robust stability framework based on quadratic separators. Here we are to continue these efforts with those methods that focus on performance issues. Besides the theoretical challenge, one of the main motivations to extend these techniques to infinite dimension, i.e., to the input output setting, is to investigate the effect of the application of the relaxations on a higher level. This issue, however, is well beyond the scope of this paper.

The aim of this paper is to formulate three classical results, the basic S-lemma, the Finsler's lemma and a variant of the full-block S-procedure, in the input output framework, i.e., in the infinite dimensional setting. We consider that the presented results widen the scope of the applicability of these fundamental tools. Moreover, they also provide a link to the theory of indefinite spaces, as an efficient framework to handle robust control problems, see, e.g., Helton [1987], Hassibi et al. [1999]. Thus the presented methods have also an educative value, putting in a different perspective already known approaches and enlightening their role in the solution of the control relevant problems.

Section 2 presents the main tools that lead to the S–lemma. Despite the use of the indefinite techniques, the approach is elementary and it is quite accessible, revealing the ideas behind the result. Section 3 introduces the infinite dimensional version of the Finsler's lemma. This is followed, in Section 4, by an infinite dimensional version of the extended KYP lemma. In the finite dimensional setting this is a special formulation of the full-block S-procedure and it can also be viewed as a generalization of the Finsler's lemma, see Iwasaki and Hara [1998]. For the motivation of labeling this statement as an extended KYP lemma see Szabó et al. [2012]. Our approach to this problem is from a pure analysis, i.e., performance assessment point of view. A possible application of the result in the finite dimensional context is sketched in Section 5.

⋆ The research has been conducted as part of the project TÁMOP-4.2.2.A-11/1/KONV-2012-0012: Basic research for the development of hybrid and electric vehicles. The Project is supported by the Hungarian Government and co-financed by the European Social Fund.

2. THE S–LEMMA

In the finite dimensional setting the S-lemma is a fundamental result with a wide range of applications, see, e.g., Boyd et al. [1994], Polik and Terlaky [2007]. It has different versions and proofs, mainly based on some convexity arguments.

While the infinite dimensional version of the result was already available, apparently it is not known in the control community. Thus, the result of this section have already appeared in the indefinite space theory in Krein and Shmulyan [1986] and were also included in the books of Bognár [1974] and Azizov and Iokhvidov [1989].

The proofs, however, were provided only for Hilbert spaces over the complex field. Here we give the proofs of the assertions for the general case, i.e., for spaces over the real field, too. For the sake of completeness we include all proofs.

Let $\phi_i : \mathcal{H} \times \mathcal{H} \mapsto \mathbb{C}(\mathbb{R})$ be sesquilinear forms given by

$$\phi_i(x, y) = \langle \Phi_i x, y\rangle$$

defined on the Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$, where Φ_i are bounded linear symmetric operators on $\mathcal{H}$. Let us consider the sets

$$\begin{aligned}\mathcal{H}_i^- &= \{x \,|\, \phi_i(x,x) < 0\}, \quad \mathcal{H}_i^{-0} = \{x \,|\, \phi_i(x,x) \le 0\},\\ \mathcal{H}_i^+ &= \{x \,|\, \phi_i(x,x) > 0\}, \quad \mathcal{H}_i^{+0} = \{x \,|\, \phi_i(x,x) \ge 0\},\\ \mathcal{N}_i &= \{x \,|\, \phi_i(x,x) = 0, x \ne 0\}.\end{aligned}$$

In what follows we suppose that ϕ_1 is indefinite, i.e., there are $y, z \in \mathcal{H}$ such that $\phi_1(y,y) = -1$ and $\phi_1(z,z) = 1$. This is an essential assumption for the validity of the assertions that follows.

Lemma 2.1. *If ϕ_1 is indefinite and $\mathcal{N}_1 \subset \mathcal{H}_2^-$ then either $\mathcal{H}_1^- \subset \mathcal{H}_2^-$ or $\mathcal{H}_1^+ \subset \mathcal{H}_2^-$.*

Proof. Suppose that there are $y, z \in \mathcal{H}$ such that

$$\phi_1(y,y) = -1, \quad \phi_1(z,z) = 1$$

and $\phi_2(y,y) \ge 0$, $\phi_2(z,z) \ge 0$. Take $x_\nu = z + \nu y$ with $\nu \in \mathbb{R}$.

Note, that $\phi_1(x_\nu, x_\nu) = 0$ is equivalent to

$$1 + 2\nu \mathrm{Re}\phi_1(y,z) - \nu^2 = 0,$$

which always has two solutions: $\nu_+ > 0$ and $\nu_- < 0$.

Thus

$$\begin{aligned}\phi_2(x_\nu, x_\nu) &= \phi_2(z,z) + 2\nu\mathrm{Re}\phi_2(y,z) + \nu^2\phi_2(y,y) =\\ &= \phi_2(z,z) + \phi_2(y,y) + 2\nu\mathrm{Re}\phi_2(y,z) + (\nu^2-1)\phi_2(y,y) =\\ &= \phi_2(z,z) + \phi_2(y,y) + 2\nu(\mathrm{Re}\phi_2(y,z) + \phi_2(y,y)\mathrm{Re}\phi_1(y,z))\end{aligned}$$

can be made negative with a suitable ν, which is a contradiction. □

The following lemma is a fundamental result here:

Lemma 2.2. *Let T be a linear relation on $\mathcal{H}$ such that $T\mathcal{N}_1 \subset \mathcal{H}_2^-(\mathcal{H}_2^{-0})$. Then, for all $y \in \mathcal{H}_1^-$ and $z \in \mathcal{H}_1^+$ we have*

$$\frac{\phi_2(Ty,Ty)}{\phi_1(y,y)} > (\ge) \frac{\phi_2(Tz,Tz)}{\phi_1(z,z)}. \tag{1}$$

Moreover, with

$$\begin{aligned}\mu_+(T) &= \sup_{z\in\mathcal{H}_1^+} \frac{\phi_2(Tz,Tz)}{\phi_1(z,z)} =\\ &= \sup_{\phi_1(z,z)=1} \phi_2(Tz,Tz) < \infty,\end{aligned} \tag{2}$$

$$\begin{aligned}\mu_-(T) &= \inf_{y\in\mathcal{H}_1^-} \frac{\phi_2(Ty,Ty)}{\phi_1(y,y)} =\\ &= \inf_{\phi_1(y,y)=-1} -\phi_2(Ty,Ty) > -\infty,\end{aligned} \tag{3}$$

we have

(1) $\mu_+(T) \le \mu_-(T)$
(2) $\phi_2(Tx,Tx) \le \mu\phi_1(x,x)$ *for* $\mu_+(T) \le \mu \le \mu_-(T)$.

Proof. The proof of (1) follows the idea applied already at Lemma 2.1. Suppose that there are $y, z \in \mathcal{H}$ such that $\phi_1(y,y) = -1$, $\phi_1(z,z) = 1$ and

$$\phi_2(Tz,Tz) + \phi_2(Ty,Ty) \ge (>)0.$$

Take $x_\nu = z + \nu y$ with $\nu \in \mathbb{R}$ such that $\phi_1(x_\nu, x_\nu) = 0$. Then

$$\begin{aligned}\phi_2(Tx_\nu, Tx_\nu) &= \phi_2(Tz,Tz) + \phi_2(Ty,Ty) +\\ &+ 2\nu(\mathrm{Re}\phi_2(Ty,Tz) + \phi_2(Ty,Ty)\mathrm{Re}\phi_1(y,z)) < (\le)0.\end{aligned}$$

Thus by a suitable choice of ν we have

$$\phi_2(Tz,Tz) + \phi_2(Ty,Ty) < (\le)0,$$

a contradiction.

From (1) we have $\mu_+(T) \le \mu_-(T)$, thus, with an arbitrary $\phi_1(y_0,y_0) = -1$ and $\phi_1(z_0,z_0) = 1$

$$-\infty \le \phi_2(Tz_0,Tz_0) \le \mu_+(T) \le \mu_-(T) \le \phi_2(Ty_0,Ty_0) \le \infty.$$

If $\mu_+(T) \le \mu \le \mu_-(T)$ then for $x \in \mathcal{H}_1^{+0}$ we have $\phi_2(Tx,Tx) \le \mu\phi_1(x,x)$ while on $x \in \mathcal{H}_1^-$ we have $\mu \le \frac{\phi_2(Tx,Tx)}{\phi_1(x,x)}$, i.e., $\phi_2(Tx,Tx) \le \mu\phi_1(x,x)$. Thus $\phi_2(Tx,Tx) \le \mu\phi_1(x,x)$ on $\mathcal{H}$, as claimed. □

Lemma 2.3. *One has:*

(1) $(\mathcal{H}_1^+ \subset \mathcal{H}_2^{-0}) \Rightarrow (\mathcal{N}_1 \subset \mathcal{H}_2^{-0})$,
(2) $(\mathcal{H}_1^- \subset \mathcal{H}_2^{-0}) \Rightarrow (\mathcal{N}_1 \subset \mathcal{H}_2^{-0})$.

Proof. 1.) Suppose that $\phi_1(x,x) = 0$ and $\phi_2(x,x) = 1$. Let $\phi_1(z,z) = 1$ and take $z_\nu = x + \nu z$, $\nu \in \mathbb{R}$ such that $\mathrm{Re}\phi_1(x,z) \ge 0$. Then $\phi_1(z_\nu, z_\nu) = \nu^2 + 2\mathrm{Re}\phi_1(x,z) > 0$. But $\phi_2(z_\nu,z_\nu) = 1 + 2\mathrm{Re}\phi_2(x,z) + \nu^2\phi_2(z,z)$ is positive for sufficiently small ν, a contradiction. 2.) can be proved analogously. □

Theorem 2.1 (S-lemma). *If $\mathcal{H}_1^- \subset \mathcal{H}_2^-$ then there exists an $\alpha \ge 0$ such that $\phi_2(x,x) \le \alpha\phi_1(x,x)$ for $x \in \mathcal{H}$.*

Proof. Using Lemma 2.3 from $\mathcal{H}_1^- \subset \mathcal{H}_2^-$ follows that $\mathcal{N}_1 \subset \mathcal{H}_2^{-0}$. Then, applying Lemma 2.2 one has $\phi_2(x,x) \le \mu\phi_1(x,x)$.

Since on $\phi_1(y,y) = -1$ we have $\phi_2(y,y) \le 0$ it follows that $\mu_- \ge 0$. Thus $\alpha = \mu$ can be chosen such that $\alpha \ge 0$.

If ϕ_2 is indefinite, them $\alpha > 0$. Indeed, otherwise one has $\phi_2(x,x) \le 0$, a contradiction. In this case if $\mu_+ \ne \mu_-$ the acceptable α values are those with $0 \le \alpha \le \mu_-$. □

3. FINSLER'S LEMMA

For the validity of the S-lemma (Theorem 2.1) it was essential that ϕ_1 is indefinite. It turns out, however, that the results of the previous section can be also used to obtain the infinite dimensional version of the Finsler's lemma.

Theorem 3.1 (Finsler's lemma). *Let ϕ_1 be positive semidefinite, i.e., $\phi_1(x,x) \geq 0$ on $\mathcal{H}$. If $\mathcal{N}_1 \subset \mathcal{H}_2^{-0}$ then there is an $\alpha \geq 0$ such that $\phi_2(x,x) \leq \alpha\phi_1(x,x)$.*

Proof. For $\beta > 0$ set $\psi_\beta = \phi_2 + \beta\phi_1$. Without restricting generality we can consider ψ_β indefinite. Indeed, $\psi_\beta(x,x) \geq 0$ implies $\phi_2(x,x) \geq -\beta\phi_1(x,x)$ on $\mathcal{H}$ which contradicts the assumption. $\psi_\beta(x,x) \leq 0$ implies $\phi_2(x,x) \leq -\beta\phi_1(x,x) \leq 0$ on $\mathcal{H}$, i.e., the assertion if the theorem is fulfilled with $\alpha = 0$.

With ψ_β indefinite, $\psi_\beta(x,x) = 0$ implies $\phi_2(x,x) = -\beta\phi_1(x,x) \leq 0$, i.e., by Lemma 2.2, there exists a μ_β such that $\phi_2(x,x) \leq \mu_\beta\psi_\beta$.

Since $(1-\mu_\beta)\phi_2(x,x) \leq \mu_\beta\beta\phi_1(x,x)$, the assertion is proved if we can chose $0 < \mu_\beta < 1$.

From $\phi_1(x,x) \geq 0$ it follows that $\psi_\beta(x,x) \geq \phi_2(x,x)$ thus

$$\frac{\phi_2(z,z)}{\psi_\beta(z,z)} \leq 1,\ z \in \mathcal{H}^+_{\psi_\beta}, \quad \frac{\phi_2(y,y)}{\psi_\beta(y,y)} \geq 1,\ y \in \mathcal{H}^-_{\psi_\beta}.$$

On $\phi_1(x,x) = 0$ we have $\psi_\beta(x,x) < 0$ thus $\mu_{\beta,-} = 1$.

On the other hand $\psi_\beta(z,z) = 1$ is equivalent to

$$\phi_2(z,z) = 1 - \beta\phi_1(z,z). \tag{4}$$

If $\phi_1(x,x) = 0$, then $\psi_\beta(x,x) = \phi_2(x,x) < 0$, so, by continuity we have $\psi_\beta(x,x) < 0$ on $\phi_1(x,x) \leq \epsilon$, for a sufficiently small $\epsilon > 0$. Thus (4) can hold only on the set $\{x \,|\, \phi_1(x,x) \geq \epsilon\}$.

It follows that

$$\sup_{\psi_\beta(z,z)=1} \phi_2(z,z) \leq \sup_{\{x \,|\, \phi_1(x,x)\geq\epsilon\}} 1 - \beta\epsilon < 1.$$

Thus $\mu_{\beta,+} < \mu_{\beta,-} = 1$ and one can chose μ_β such that $0 < \mu_{beta} < 1$, as desired. □

The more familiar version of the lemma can be obtained by considering $\phi_1(x,x) = x^*V^*Vx$ and $\phi_2(x,x) = x^*Px$. Then $\mathcal{N}_1 = \mathrm{Ker}V$.

In the finite dimensional context usually two "extensions" of the Finsler's lemma are considered. The one is the Projection lemma and the other is the robust Finsler's lemma. Actually the first result is a special version of the Elimination lemma and despite the apparent similarities with the Finsler's lemma its roots are different. The control relevant infinite dimensional extension of the Elimination lemma is not our concern here.

The robust Finsler's lemma, a special formulation of the full block S-procedure, however, has its roots in the S-lemma. In the finite dimension case this fact was revealed in Szabó et al. [2013]. The particularities of the infinite dimensional case are presented in the next section.

4. EXTENDED KYP LEMMA

In order to put the result in its proper context it is necessary to reveal first the linear structures behind the linear fractional transforms (LFTs), as a general framework to include the rational dependencies that occur in the formulation of robust feedback control problems.

If P is partitioned as $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ then a lower and an upper LFT is defined as

$$\mathfrak{F}_l(P,K) = P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, \tag{5}$$

$$\mathfrak{F}_u(P,\Delta) = P_{22} + P_{21}\Delta(I - P_{11}\Delta)^{-1}P_{12}, \tag{6}$$

provided that the inverse $(I - P_{22}K)^{-1}$ and $(I - P_{11}\Delta)^{-1}$, respectively, exists. Then P is called the coefficient matrix of the LFT.

There is an intimate relationship between linear relations and LFTs, revealed by the concept of transformers, introduced in Shmulyan [1976, 1980].

4.1 Linear relations and LFTs

If X and Y are linear spaces, a linear relation T, defined as a set of pairs $(x,y) \in T$, is a linear subspace of $X \oplus Y$. If $x \in dom(T)$ then $T(x) = \{y \in Y : (x,y) \in T\}$ and correspondingly if $y \in ran(T)$, then $T^{-1}(y) = \{x \in X : (x,y) \in T\}$. Let $T \subset X \times Y$ and $R \subset Y \times Z$ be linear relations. Then the product $RT \subset X \times Z$ is the linear relation defined by

$$RT = \{\{x,z\} \in X \times Z : \{x,y\} \in T, \{y,z\} \in R\}.$$

The product of relations is clearly associative. $\lambda T = \{\{x,\lambda y\} : \{x,y\} \in T\}$. These definitions agree with the usual ones that correspond to operators. A linear operator $P : X \mapsto Y$ is equivalent to a special relation defined by a graph subspace $\mathcal{G}_P = \mathrm{Im}\begin{pmatrix} I \\ P \end{pmatrix}$, i.e., the graph of the operator. For details see, e.g., Arens [1961].

Möbius transformations, which are defined as

$$Z' = \mathfrak{M}_S(Z) = (C + DZ)(A + BZ)^{-1}, \tag{7}$$

relate two graph subspaces, $\mathcal{G}_Z$ and $\mathcal{G}_{Z'}$, through the invertible linear operator $S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, i.e., $\mathcal{G}_{Z'} = S\mathcal{G}_Z$. Moreover, it turns out that the Möbius transformation inherits the group structure of the linear operators, i.e.,

$$\mathfrak{M}_P \circ \mathfrak{M}_Q = \mathfrak{M}_{PQ},$$

for details see, e.g., Szabó et al. [2012].

It turns out that LFTs can be obtained in the same way as the Möbius transformations, by performing some interchange in the signal spaces and by considering linear relations instead of the linear operators.

Given the linear spaces $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ consider $\mathcal{L} = \mathcal{X} \oplus \mathcal{Y}$ and $\tilde{\mathcal{L}} = (\mathcal{X}_2 \oplus \mathcal{Y}_1) \oplus (\mathcal{X}_1 \oplus \mathcal{Y}_2)$. Observe that we have $\tilde{\mathcal{L}} = S_p\mathcal{L}$ with the permutation matrix $S_p = \begin{pmatrix} 0 & 0 & I & 0 \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$.

Every linear operator $T : (\mathcal{X}_2 \oplus \mathcal{Y}_1) \mapsto (\mathcal{X}_1 \oplus \mathcal{Y}_2)$ induces a relation $\mathcal{R}_T \subset \mathcal{L}$ through its graph subspace, i.e.,

$$\mathcal{R}_T = S_p \mathcal{G}_T \sim \begin{pmatrix} T_{11} & T_{12} \\ I & 0 \\ \hline 0 & I \\ T_{21} & T_{22} \end{pmatrix}. \tag{8}$$

It turns out that evaluating this relation on the graph subspaces $\mathcal{G}_Z$, i.e., on the linear operators $Z : \mathcal{X}_1 \mapsto \mathcal{X}_2$, we obtain a graph subspace $\mathcal{G}_{Z'} = \mathcal{R}_T \mathcal{G}_Z$, corresponding to the linear operator $Z' : \mathcal{Y}_1 \mapsto \mathcal{Y}_2$, provided that $I - T_{11}Z$is boundedly invertible.

This map is given by the (upper) LFT

$$Z' = \mathfrak{F}_u(T, Z) = T_{22} + T_{21}Z(I - T_{11}Z)^{-1}T_{12}.$$

Analogously, by a slight modification of the permutation matrix S_p, i.e., by considering $\tilde{\mathcal{L}} = (\mathcal{Y}_1 \oplus \mathcal{X}_2) \oplus (\mathcal{Y}_2 \oplus \mathcal{X}_1)$ one can obtain the expression of the (lower) LFT $Z' = \mathfrak{F}_l(T, Z) = T_{11} + T_{12}Z(I - T_{22}Z)^{-1}T_{21}$, too.

Nested LFTs corresponds to the composition of the associated linear relations. The group structure on the representants is also present, however, the familiar matrix product should be changed to the less accessible Redheffer (star) product, see, e.g., Zhou and Doyle [1999].

However, if invertibility conditions holds for the matrix $\begin{pmatrix} T_{11} & T_{12} \\ I & 0 \end{pmatrix}$ then one has $\mathfrak{F}_u(T, Z) = \mathfrak{M}_{\hat{T}}(Z)$ with

$$\hat{T} = \begin{pmatrix} 0 & I \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} \\ I & 0 \end{pmatrix}^{-1} = \begin{pmatrix} T_{12}^{-1} & -T_{12}^{-1}T_{11} \\ T_{22}T_{12}^{-1} & T_{21} - T_{22}T_{12}^{-1}T_{11} \end{pmatrix}.$$

The transformation $T \mapsto \hat{T}$ is called Potapov-Ginsbourg transformation. This relation between an LFT and a Möbius transformations has the advantage to use the matrix product instead of the star product. This fact was widely exploited in the solution of the robust control problems, see, e.g., Ball et al. [1991], Kimura [1997].

The main motivation of introducing this construction is the fact that it provides a natural framework to introduce indefinite spaces, see, e.g., Bognár [1974], Azizov and Iokhvidov [1989].

To illustrate the idea let us consider the linear spaces $\mathcal{X} = \mathcal{X}_1 \oplus \mathcal{X}_2$ and $\mathcal{Y} = \mathcal{Y}_1 \oplus \mathcal{Y}_2$ as indefinite spaces with inner products $[\cdot,\cdot]_X = \langle \mathcal{J}_X \cdot, \cdot \rangle$ and $[\cdot,\cdot]_Y = \langle \mathcal{J}_Y \cdot, \cdot \rangle$. If we endow the space $\mathcal{L} = \mathcal{X} \oplus \mathcal{Y}$ with the inner product $[\cdot,\cdot] = -[\cdot,\cdot]_X + [\cdot,\cdot]_Y$ then for $\tilde{\mathcal{L}} = (\mathcal{X}_2 \oplus \mathcal{Y}_1) \oplus (\mathcal{X}_1 \oplus \mathcal{Y}_2)$ it can be expressed $[\cdot,\cdot] = \langle \mathcal{J}_{\tilde{\mathcal{L}}} \cdot, \cdot \rangle$, where $\mathcal{J}_H = \begin{pmatrix} -I_{H_1} & 0 \\ 0 & I_{H_2} \end{pmatrix}$ for $H = H_1 \oplus H_2$.

Maximal negative subspaces of $\tilde{\mathcal{L}}$ are obviously maximal negative subspaces of $\mathcal{L}$. On one hand side these subspaces are parametrized by contractions T, on the other hand we have [1]

$$(\star)^*(-\mathcal{J}_\mathcal{X}) \begin{pmatrix} T_{11} & T_{12} \\ I & 0 \end{pmatrix} + (\star)^* \mathcal{J}_\mathcal{Y} \begin{pmatrix} 0 & I \\ T_{21} & T_{22} \end{pmatrix} < 0.$$

Thus if T is a contraction then $Z' = \mathfrak{F}_u(T, Z)$ maps the contractive ball to the contractive ball. It turns out, that conversely, if $Z' = \mathfrak{F}_u(T, Z)$ has this property, then the matrix $T_\alpha = \begin{pmatrix} T_{11} & \alpha T_{12} \\ \alpha^{-1}T_{21} & T_{22} \end{pmatrix}$ is a contraction for a suitable $\alpha > 0$, see Shmulyan [1978].

Observe that

$$Z' = \mathfrak{F}_u(T, Z) = \mathfrak{F}_u(T_\alpha, Z). \tag{9}$$

We can put this result in a slightly modified form: if $\|\mathfrak{F}_u(T, Z)\| < 1$ holds for $\|Z\| < 1$ then there exists $\alpha > 0$ such that

$$(\star)^*(-\alpha\mathcal{J}_\mathcal{X}) \begin{pmatrix} T_{11} & T_{12} \\ I & 0 \end{pmatrix} + (\star)^* \mathcal{J}_\mathcal{Y} \begin{pmatrix} 0 & I \\ T_{21} & T_{22} \end{pmatrix} < 0,$$

i.e., the graph subspace $\mathcal{G}_{T_\alpha}$ is a maximal negative graph subspace in $\tilde{\mathcal{L}}$.

4.2 The extended KYP lemma

The comparison of the construction with the claim of the robust Finsler's lemma reveals that there is a possibility for a specific, robust control relevant, interpretation of this result. Moreover, it suggests a general formulation of the claim by replacing the special J-scalar products with more general ones. In the finite dimensional setting this was done in Szabó et al. [2013], and can be formulated as:

Proposition 4.1 (Extended KYP lemma)**.** *Consider the set $\mathbf{\Delta}_a$ defined by the inequality*

$$\begin{pmatrix} \delta \\ I \end{pmatrix}^* P \begin{pmatrix} \delta \\ I \end{pmatrix} > 0,$$

where $P \in \mathcal{M}_A$. Then

$$\begin{pmatrix} I \\ F(\delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ F(\delta) \end{pmatrix} < 0, \quad \forall \delta \in \mathbf{\Delta}_a$$

where $F(\delta) = D + C\delta(I - A\delta)^{-1}B$ if and only if

$$\begin{pmatrix} I & 0 \\ A & B \end{pmatrix}^* (\alpha P) \begin{pmatrix} I & 0 \\ A & B \end{pmatrix} + \begin{pmatrix} 0 & I \\ C & D \end{pmatrix}^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \leq 0,$$

for some $\alpha > 0$.

Here $\mathcal{M}_A = \{P \,|\, \mathbf{\Delta}_P \subset \mathcal{D}_A\}$, where

$$\mathbf{\Delta}_P = \{\delta \,|\, \begin{pmatrix} \delta \\ I \end{pmatrix}^* P \begin{pmatrix} \delta \\ I \end{pmatrix} > 0\}$$

and $\mathcal{D}_A = \{I - A\delta \text{ is nonsingular}\}$. Condition $P \in \mathcal{M}_A$ usually can be relaxed.

If we are going to formulate the infinite dimensional result the meaning of the symbols in Proposition 4.1 can be changed in a straightforward way.

We consider the quadratic forms $\Phi(x, x) = \langle Px, x \rangle$ and $\Phi_p(x, x) = \langle P_p x, x \rangle$, where P and P_p are bounded symmetric operators and we use the notation $\langle Px, x \rangle = x^* P x$. Thus, after a slight modification, i.e., replacing the J-spaces with the indefinite spaces defined by $[\cdot,\cdot]_X = \langle \bar{P} \cdot, \cdot \rangle$ and $[\cdot,\cdot]_Y = \langle P_p \cdot, \cdot \rangle$, where

$$\bar{P} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* (-P) \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \tag{10}$$

the condition of the assertion, i.e., the original implication reads as

$$\begin{pmatrix} I \\ \delta \end{pmatrix}^* \bar{P} \begin{pmatrix} I \\ \delta \end{pmatrix} < 0 \quad \Rightarrow \quad \begin{pmatrix} I \\ F(\delta) \end{pmatrix}^* P_p \begin{pmatrix} I \\ F(\delta) \end{pmatrix} < 0,$$

and the equivalent condition is

$$(\star)^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \leq (\star)^* \alpha \bar{P} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix}.$$

[1] To save space, here and in what follows we use the notation $(\star)^* Pv$ for $v^* Pv$.

This formulation reveals the intimate relation of the assertion with the S-lemma. In both the finite and infinite dimensional case at the heart of the result we have the fact that the implication

$$(\star)^* \bar{P} \begin{pmatrix} A & B \\ I & 0 \end{pmatrix} \begin{pmatrix} \eta \\ w \end{pmatrix} < 0 \quad \Rightarrow$$
$$(\star)^* P_p \begin{pmatrix} 0 & I \\ C & D \end{pmatrix} \begin{pmatrix} \eta \\ w \end{pmatrix} < 0,$$

which holds on a set $\{(\eta, w)\}$ constrained by the feedback connection (LFT), can be lifted (extended) to the unconstrained set, for which the common S-lemma can be applied.

In the infinite dimensional case, however, some restrictions on the possible quadratic forms has to be imposed in order to prove this assertion. Besides $P \in \mathcal{M}_A$ we have to impose the technical requirement that $P = M^* J M$ where M is a boundedly invertible operator such that $M^{-1}\mathcal{H}_\phi^+$ should contain the inverse graph subspaces that correspond to the full operator unit ball, i.e., the set of contractions $\mathcal{K} = \{K \mid \|K\| \le 1\}$.

These conditions are not very restrictive. Among the sets $\mathbf{\Delta}_a$ that correspond to such Ps are all the operator balls $U + V\mathcal{K}W$, which are relevant in the control applications as models of the bounded perturbations. It is also assumed that $P_p = M_p^* J M_p$ where M_p is a boundedly invertible operator.

Under these assumptions the proof of the assertion follows the main steps of the proof presented in Shmulyan [1978] for the basic case from the previous section. The technical details are involved and left out for brevity.

We conclude this section with a related result, when the inclusion condition holds only on a smaller set. This asserts that if the Möbius transform generated by a nondegenerate meromorphic (on the right half plane) function W maps the constant strict contractive matrices into the ball of contractive operators then there is a scalar meromorphic function ρ such that ρW is a J-contractive function. For details see Chapter 4 in Dym [1989].

5. A POSSIBLE APPLICATION

While at a formal level the analysis oriented version of the extended KYP lemma bears essentially the same information that the classical version, it represents a different view on the topic. To illustrate a potential application of this new viewpoint let us consider a finite dimensional setting where the nontrivial task is to find a common solution X for the finite set of LMIs

$$\begin{pmatrix} I \\ D_i + C_i X B_i \end{pmatrix}^T Q_i \begin{pmatrix} I \\ D_i + C_i X B_i \end{pmatrix} < 0,$$

where the matrices Q_i, D_i, B_i, C_i are given. For a motivation of this problem see, e.g., de Oliveira [2005], Vesely et al. [2009].

It is known that the solution sets of the individual inequalities are either empty or a set obtained as an image of the contractive ball through a Möbius transform, see Szabó et al. [2012]. Thus, if the problem is solvable, there always exists a matrix ellipsoid $\begin{pmatrix} X \\ I \end{pmatrix}^T P \begin{pmatrix} X \\ I \end{pmatrix} > 0$ formed entirely by solutions of the inequality. It follows that applying Proposition 4.1 we can formulate the following result:

Lemma 5.1. *The given set of inequalities has a common solution if and only if there is a multiplier P and constants $\beta_i > 0$ such that*

$$\begin{pmatrix} I & 0 \\ 0 & B_i \end{pmatrix}^* P \begin{pmatrix} I & 0 \\ 0 & B_i \end{pmatrix} + \begin{pmatrix} 0 & I \\ C_i & D_i \end{pmatrix}^* (\beta_i Q_i) \begin{pmatrix} 0 & I \\ C_i & D_i \end{pmatrix} \le 0.$$

By considering matrix ellipsoids, i.e., by imposing suitable sign constraints on the block diagonal matrices of P one can relax the implicit nonlinear condition for the inertia for a multiplier P imposed by the solvability of

$$\begin{pmatrix} X \\ I \end{pmatrix}^T P \begin{pmatrix} X \\ I \end{pmatrix} > 0.$$

Thus we can reduce the problem to a set of LMIs that can be efficiently handled.

6. CONCLUSIONS

The paper formulate three classical results, the basic S-lemma, the Finsler's lemma and a variant of the full-block S-procedure (extended KYP lemma), in the input output framework, i.e., in the infinite dimensional setting.

The presented results widen the scope of the applicability of these fundamental tools and they also provide a link to the theory of indefinite spaces, as an efficient framework to handle robust control problems. The presented methods have also an educative value, putting in a different perspective already known approaches and enlightening their role in the solution of the control relevant problems.

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