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Output Regulation of Linear Systems with State, Input and Output Delays ⋆

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Abstract: This paper considers the output regulation problem for linear systems in the presence of state, input and output delays. A state feedback output regulation law is constructed from a state predictor, recently developed for systems with state and input delays, and the solution to a pair of regulator equations that transforms the output regulation problem into a stabilization problem. Necessary and sufficient conditions for the existence of a solution to the regulator equations are presented. Numerical examples demonstrate the effectiveness of the developed predictor-based solution.

Keywords: Delay system; output regulation; predictor feedback

1. INTRODUCTION

Output regulation is one of the central problems in control theory. Its objective is to control the plant output such that it tracks a prescribed class of reference signals. The reference signal to track in the output regulation problem, as well as the external disturbance input perturbing the system, are produced by an external generator known as the exosystem. The output regulation problem has been studied extensively since it was formulated by Francis [1977] for linear systems, and by Isidori and Byrnes [1990] for nonlinear systems. In particular, Francis [1977] demonstrated that, for linear systems, the existence of a solution to the output regulation problem is equivalent to the solvability of a pair of linear matrix equations. In the case of nonlinear systems, Isidori and Byrnes [1990] extended the results established by Francis [1977], and demonstrated that the existence of a solution to the output regulation problem is equivalent to solvability of a set of partial differential and algebraic equations. In the case of linear systems with input saturation, Lin et al. [1996] presented necessary and sufficient conditions for the solvability of the output regulation problem in the semi-global framework, and a suitable feedback law was constructed based on the low gain design method in Lin [1998]. The semi-global framework entails that the open loop system is not exponentially unstable. The output regulation problem for general linear systems, including exponentially unstable systems, under constrained control was studied in Hu and Lin [2004].

The stabilization and output regulation problems for systems with time delay have been subject to considerable research in recent years, motivated by the unavoidable presence of delay in most control applications. Surveys of recent results and open problems on the control of delay systems are found in Gu and Niculescu [2003] and Richards [2003]. The output regulation problem for systems with state delays was discussed in Castillo-Toledo and Núñez-Pérez [2003] and Wang et al. [2013], in which the authors derived the necessary and sufficient conditions for the existence of a solution by employing a similar argument as presented in Francis [1977] for the delay-free case. In the nonlinear setting, the conditions for solvability of the output regulation problem presented in Isidori and Byrnes [1990] were extended to systems with state delays in Fridman [2003].

For systems with delays in the input signal, the predictor based control has been studied extensively in the literature. The most common predictor based control laws presented in the literature are derived from the Smith predictor in Smith [1959], the Artstein model reduction technique in Artstein [1982], and the finite spectrum assignment technique in Manitius and Olbrot [1979]. Predictor based control for time-varying delays was treated in Bresch-Pietri and Krstic [2010]. For linear systems with a static input delay and poles in the closed left-half plane, a finite dimensional control law was presented in Lin and Fang [2007]. The authors introduced a low gain design method, in which the stabilizing control law takes the structure of the predictor feedback controller, but the distributed portion of the predictor is truncated. This control law, with the truncated predictor, was later extended to systems with time varying delays in Zhou et al. [2012], and to exponentially unstable systems in Yoon and Lin [2013]. When the delays are in both the state and the input, a predictor feedback controller was recently developed in Yoon and Lin [2014], where the predictor is formulated recursively over the prediction time to guarantee the causality of the resulting predictor equation.

In this paper we consider the output regulation problem for systems with state, input, and output delays. First, the output regulation problem is defined for systems with delays, and conditions for solvability of the regulator problem are derived. Next, an output regulation control

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law is constructed from the predictor based controller developed in Yoon and Lin [2014] for systems with state and input delays. Due to the length limitation, in this paper we limit our discussion to the state feedback output regulation problem. The solution to the error feedback output regulation problem can be obtained from the state feedback solution discussed in this paper, and the output-feedback variation of the predictor presented in Yoon and Lin [2014].

The remainder of this paper is organized as follows. The output regulation problem is defined in Section 2, and a solution to the state feedback output regulation is presented in Section 3. The validity of the derived output regulation controller is demonstrated through numerical simulation in Section 4. Finally, conclusions are drawn in Section 5.

2. PROBLEM DEFINITION

Consider a linear time-invariant system with multiple delays in the state, input and error,

$$\dot{x}(t) = \sum_{i=0}^{N} A_i x_i(t-\tau_i) + Bu(t-\tau_u) + \mathcal{P}\omega(t), \quad (1a)$$

$$\dot{\omega}(t) = \mathcal{S}\omega(t), \quad (1b)$$

$$e(t) = \sum_{i=0}^{N} C_i x(t-\tau_i) + \mathcal{Q}\omega(t). \quad (1c)$$

The dynamics of the delayed plant are described in (1a), with state $x \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. The time delays in the state and input are represented by the real scalars $\tau_i \geq 0$, $i = 0, 1, \cdots, N$, and $\tau_u > 0$, respectively. Without loss of generality, we will assume that $\tau_0 = 0$, and the delays are ordered such that $\tau_i < \tau_j$ for $i < j$. The plant is subjected to a disturbance in the form of $\mathcal{P}\omega$, which is generated by the exosystem in (1b) with state $\omega \in \mathbb{R}^r$. Finally, the regulated error signal $e \in \mathbb{R}^s$ is given in (1c), which is defined as the difference between the plant output and the reference signal $-\mathcal{Q}\omega$.

We introduce the following notation in order to define the state feedback regulator problem. First, we define the scalar $\tau = \max\{\tau_N, \tau_u\}$, and x_t to be the trajectory of the state $x_t(\gamma) = x(t+\gamma)$ for $\gamma \in [-\tau, 0]$. By the same token, let u_t be the history of the input $u_t(\gamma) = u(t+\gamma)$ for $\gamma \in [-\tau, 0]$.

Problem 1. State Feedback Regulator Problem - Given the state trajectory x_t and input history u_t, find a state feedback control law

$$u(t) = \alpha(x_t, u_t, \omega(t)), \quad (2)$$

under which,

(i) the closed-loop system is asymptotically stable when $\omega \equiv 0$;
(ii) the regulated error $e(t)$ satisfies $\lim_{t\to\infty} e(t) = 0$.

3. STATE FEEDBACK REGULATOR PROBLEM

Some standard assumptions are made on the system (1) that are required for the solvability of the output regulation problem.

A1. The eigenvalues of $\mathcal{S}$ have nonnegative real parts.
A2. The system (1) with $\omega(t) \equiv 0$ is stabilizable by a control law of the form (2).

Assumption A1 does not affect the generality of the problem since the asymptotically stable eigenvalues of $\mathcal{S}$ do not affect the regulation of the output. Assumption A2 is required for the existence of a control law (2) that asymptotically stabilizes system (1), when $\omega = 0$.

3.1 Solution of the Regulator Problem

Lemma 1. Consider the delayed system (1) satisfying Assumptions A1 and A2. Further assume that there is a state feedback control law

$$u(t) = \alpha_s(x_t, u_t),$$

such that the closed-loop system (1) under the above control law and $\omega \equiv 0$ is asymptotically stable. Then, Problem 1 is solvable if and only if there exist matrices $\Pi \in \mathbb{R}^{n\times r}$ and $\Gamma \in \mathbb{R}^{m\times r}$ such that

$$\Pi\mathcal{S} = A_0\Pi + \sum_{i=1}^{N} A_i \Pi e^{-\tau_i \mathcal{S}} + B\Gamma + \mathcal{P}, \quad (3a)$$

$$0 = C_0\Pi + \sum_{i=1}^{N} C_i \Pi e^{-\tau_i \mathcal{S}} + \mathcal{Q}. \quad (3b)$$

Furthermore, the control law that achieves output regulation is given as

$$u(t) = \nu(t) + \Gamma e^{\tau_u \mathcal{S}}\omega(t), \quad (4)$$

where $\nu(t) = \alpha_s(z_t, \nu_t)$, the state $z(t)$ is defined as $z(t) = x(t) - \Pi\omega(t)$, and the state trajectory $z_t(\gamma)$ is defined as $z_t(\gamma) = z(t+\gamma)$ for $\gamma \in [-\tau, 0]$.

Proof. The above lemma is a generalization of the results from Castillo-Toledo and Núñez-Pérez [2003] and Francis [1977] to systems with state, input, and output delays. It was demonstrated in Francis [1977] that a special case of (3) provides the necessary and sufficient conditions for the solvability of the output regulation problem in linear systems without delay. The sufficient condition for the solvability of the output regulation problem was extended in Castillo-Toledo and Núñez-Pérez [2003] to systems with state delays.

Define the state transformation $z(t) = x(t) - \Pi\omega(t)$ for a given matrix Π. Then, $\dot{z}(t)$ can be derived from (1) as

$$\dot{z}(t) = \sum_{i=0}^{N} A_i z(t-\tau_i) + \sum_{i=0}^{N} A_i \Pi\omega(t-\tau_i) + Bu(t-\tau_u) + \mathcal{P}\omega(t) - \Pi\mathcal{S}\omega(t),$$

and the error equation in (1c) can be rewritten as,

$$e(t) = \sum_{i=0}^{N} C_i z(t-\tau_i) + \sum_{i=0}^{N} C_i \Pi\omega(t-\tau_i) + \mathcal{Q}\omega(t).$$

Let the control input to (1) be in the form of (4) for a given matrix Γ. Then, the state equation for $z(t)$ becomes

$$\dot{z}(t) = \sum_{i=0}^{N} A_i z(t-\tau_i) + B\nu(t-\tau_u)$$

$$+\left(\sum_{i=0}^{N} A_i \mathrm{e}^{-\tau_i \mathcal{S}} + B\Gamma + \mathcal{P} - \Pi\mathcal{S}\right)\omega(t), \tag{5a}$$

$$e(t) = \sum_{i=0}^{N} C_i z(t-\tau_i) + \left(\sum_{i=0}^{N} C_i \Pi \mathrm{e}^{-\tau_i \mathcal{S}} + \mathcal{Q}\right)\omega(t), \tag{5b}$$

where we used the fact that $\omega(t) = e^{\sigma S}\omega(t-\sigma)$, for $\sigma \in \mathbb{R}$, which was obtained from the closed-form solution of (1b).

Since the matrices Π and Γ are the solution to (3), the state equation (5) can be simplified to

$$\dot{z}(t) = \sum_{i=0}^{N} A_i z(t-\tau_i) + B\nu(t-\tau_u), \tag{6a}$$

$$e(t) = \sum_{i=0}^{N} C_i z(t-\tau_i). \tag{6b}$$

The above system of equations is equivalent to (1) with $\omega \equiv 0$. Therefore, Assumption 2 yields that that there exists a control law $\nu(t) = \alpha_{\mathrm{s}}(z_t, \nu_t)$, where $\nu_t(\gamma) = \nu(t+\gamma)$ for $\gamma \in [-\tau, 0]$, such that the closed-loop of the delayed system (6a) is asymptotically stable. Furthermore, as the state $z(t)$ asymptotically approaches zero, the limit of the error signal will also converge to zero, or $\lim_{t\to\infty} e(t) = 0$. Therefore, (4) is a solution to the output regulation problem.

On the other hand, assume that the control input $u(t)$ is a solution to the state feedback output regulation problem, and consider the input to be in the form (4). Then, the error signal in (5b) must satisfy the objective of the regulation problem, that is, $\lim_{t\to\infty} e(t) = 0$. Since Assumption A1 states that the ω subsystem in (1b) has nonnegative poles, the objective $\lim_{t\to\infty} e(t) = 0$ from any arbitrary initial conditions $x(t_0)$ and $\omega(t_0)$ is achieved only if $\lim_{t\to\infty} z(t) = 0$ and,

$$\sum_{i=0}^{N} C_i \Pi \mathrm{e}^{-\tau_i \mathcal{S}} + \mathcal{Q} = 0. \tag{7}$$

Furthermore, it was assumed that $\nu(t) = \alpha_s(z_t, \nu_t)$ asymptotically stabilizes (5a) when $\omega \equiv 0$. Then, for $\lim_{t\to\infty} z(t) = 0$ to be true under Assumption 1 and arbitrary initial conditions $x(t_0)$ and $\omega(t_0)$, it is required that

$$\sum_{i=0}^{N} A_i \mathrm{e}^{-\tau_i \mathcal{S}} + B\Gamma + \mathcal{P} - \Pi\mathcal{S} = 0. \tag{8}$$

Finally, (7) and (8) are the same as the conditions in (3). This concludes the proof. □

Remark 1. The conditions for the existence of the solution to (3) have been discussed in detail in Castillo-Toledo and Núñez-Pérez [2003] and Wang et al. [2013]. It was demonstrated that (3) is solvable if and only if

$$\mathrm{rank}\begin{pmatrix} \sum_{i=0}^{N} A_i \mathrm{e}^{-\tau_i \lambda} - \lambda I & B \\ \sum_{i=0}^{N} C_i e^{-\tau_i \lambda} & 0 \end{pmatrix} = n + s,$$

for all λ in the spectrum of S.

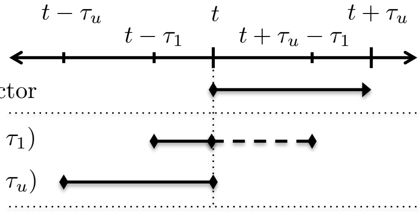

Fig. 1. A diagram of the predictor at time t for the state at $t + \tau_u$, and the required information of the delayed state and input. The dashed line illustrates the non-causal part of the predictor.

3.2 Prediction Based Control Law

The results in Lemma 1 allow us to transform the output regulation problem into the stabilization of a state and input delayed system. The predictor feedback approach, in which the future state is predicted in order to compensate for the effects of the delay, has been studied extensively in the control of input delayed systems. In the absence of state delays, a causal predictor can be constructed from the closed-form solution of the delayed state space equation. When a state delay is present in the system equations, as in (6), the closed-form solution of the state space equation becomes

$$z(t+\tau_u) = \mathrm{e}^{A_0\tau_u} z(t) + \sum_{i=0}^{N} \int_t^{t+\tau_u} \mathrm{e}^{A_i(t+\tau_u-\sigma)} A_i z(\sigma-\tau_i)\mathrm{d}\sigma$$
$$+ \int_t^{t+\tau_u} \mathrm{e}^{A_0(t+\tau_u-\sigma)} B\nu(\sigma-\tau_u)\mathrm{d}\sigma. \tag{9}$$

A prediction of the future state may be obtained from the above solution, but the calculation of the state prediction may require future state information if $\tau_1 < \tau_u$.

To illustrate the non-causality of the predictor equation (9), consider the simple linear time-invariant system with a single state delay τ_1 and an input delay τ_u,

$$\dot{z}(t) = A_0 z(t) + A_1 z(t-\tau_1) + B\nu(t-\tau_u). \tag{10}$$

The predictor feedback law computes the input at time $t + \tau_u$ in order to cancel the effect of the input delay in the system. For a state feedback law, this requires the estimate of the state $z(t + \tau_u)$. We consider the state prediction as in (9) with $N = 1$. The diagram in Fig. 1 illustrates the prediction of $z(t + \tau_u)$ on the time line. It is observed from the integration limits in (9) that the prediction requires the input history between $[t-\tau_u, t]$ and the state trajectory between $[t-\tau_1, t+\tau_u-\tau_1]$. For $\tau_u > \tau_1$, the prediction becomes non-causal since it requires future information of the states as shown in Fig. 1. The following proposition discusses how to build a causal predictor for a linear system with both input and state delays.

Proposition 2. Consider the state and input delayed system

$$\dot{z}(t) = \sum_{i=0}^{N} A_i z(t-\tau_i) + B\nu(t-\tau_u), \tag{11}$$

with the state vector $z(t) \in \mathbb{R}^n$ and input vector $\nu(t) \in \mathbb{R}^m$. Let the positive integer k be such that $k\tau_1 < \tau_u \le (k+1)\tau_1$. Define the index sets $\mathcal{N}_j$ and $\bar{\mathcal{N}}_j$ for a positive integer j as $\mathcal{N}_j = \{i : \tau_i \ge j\tau_1\}$ and $\bar{\mathcal{N}}_j = \{i : \tau_i < j\tau_1\}$, respectively. We also define the index sets $\mathcal{N}_u = \{i : \tau_i \geq \tau_u\}$ and $\bar{\mathcal{N}}_u = \{i : \tau_i < \tau_u\}$. Then, there is a $\bar{\tau}_u > 0$ such that $\hat{z}_{\tau_u}(t) = z(t + \tau_u)$ for all $t > \bar{\tau}_u$, where the predictor $\hat{z}_{\tau_u}$ is computed as

$$\begin{aligned}
\hat{z}_{\tau_1}(t) &= \mathrm{e}^{A_0\tau_1} z(t) + \sum_{i=1}^{N} \Phi_i(t, \tau_1) + \Phi_\nu(t, \tau_1), \\
\hat{z}_{2\tau_1}(t) &= \mathrm{e}^{2A_0\tau_1} z(t) + \sum_{i\in\mathcal{N}_2} \Phi_i(t, 2\tau_1) + \Phi_\nu(t, 2\tau_1) \\
&\quad + \sum_{i\in\bar{\mathcal{N}}_2} \Psi_i(t, 2\tau_1, 1), \\
&\ \vdots \\
\hat{z}_{k\tau_1}(t) &= \mathrm{e}^{kA_0\tau_1} x(t) + \sum_{i\in\mathcal{N}_k} \Phi_i(t, k\tau_1) + \Phi_\nu(t, k\tau_1) \\
&\quad + \sum_{i\in\bar{\mathcal{N}}_k} \Psi_i(t, k\tau_1, (k-1)), \\
\hat{z}_{\tau_u}(t) &= \mathrm{e}^{A_0\tau_u} z(t) + \sum_{i\in\mathcal{N}_u} \Phi_i(t, \tau_u) + \Phi_\nu(t, \tau_u) \\
&\quad + \sum_{i\in\bar{\mathcal{N}}_u} \Psi_i(t, \tau_u, k). \qquad (12)
\end{aligned}$$

The functions Φ_i, Φ_ν and Ψ_i are respectively defined as

$$\Phi_i(t, \tau) = \int_t^{t+\tau} \mathrm{e}^{A_0(t+\tau-\sigma)} A_i z(\sigma - \tau_i) \mathrm{d}\sigma, \qquad (13)$$

$$\Phi_\nu(t, \tau) = \int_t^{t+\tau} \mathrm{e}^{A_0(t+\tau-\sigma)} B\nu(\sigma - \tau_u) \mathrm{d}\sigma, \qquad (14)$$

and

$$\Psi_i(t, \tau, j) = \int_t^{t+\tau} \mathrm{e}^{A_0(t+\tau-\sigma)} A_i \hat{z}_{j\tau_1}(\sigma - j\tau_1 - \tau_i) \mathrm{d}\sigma. \qquad (15)$$

Proof. The details on the construction of the predictor $\hat{z}_{\tau_u}$ have been discussed in Yoon and Lin [2014]. In what follows we present a sketch of the proof.

Consider the function $\hat{z}_{\tau_1}(t)$ in (12). Since this function is a special case of the predictor (9), in which the prediction time step is equal to the smallest state delay, we conclude that

$$\hat{z}_{\tau_1}(t) = z(t + \tau_1) \text{ for } t \geq 0. \qquad (16)$$

Next, consider $\hat{z}_{2\tau_1}(t)$ in (12). The calculation of this function requires information about $\hat{z}_{\tau_1}(t)$. From the equality in (16) and the integration limits in (15), we observe that $\hat{z}_{2\tau_1}(t)$ also becomes a special case of (9) for $t \geq 5\tau_1$. Therefore, with $\bar{\tau}_2 = 5\tau_1$,

$$\hat{z}_{2\tau_1}(t) = z(t + 2\tau_1) \text{ for } t \geq \bar{\tau}_2. \qquad (17)$$

Finally, assume that

$$\hat{z}_{k\tau_1}(t) = z(t + k\tau_1) \text{ for } t \geq \bar{\tau}_k, \qquad (18)$$

for some $\bar{\tau}_k > 0$. Referring to the predictor equation in (12), we observe that the calculation of $\hat{z}_{\tau_u}$ requires the past information of the predicted state $\hat{z}_{k\tau_1}(t)$. In view of the equality in (18) and the predictor equation (12)-(15), we obtain that

$$\hat{z}_{\tau_u}(t) = z(t + \tau_u) \text{ for } t \geq \bar{\tau}_u, \qquad (19)$$

where $\bar{\tau}_u = \bar{\tau}_k + k\tau_1 + 2\tau_u$. This concludes the proof. □

Remark 2. The predictor in (12) is formulated recursively over the prediction time, by partitioning the input delay into sections equal to or smaller than the state delays, and constructing an auxiliary predictor for each of these time sections. This way, the resulting predictor is guaranteed to be causal, only depending on the past system information and the predictor output history.

Next, we put together the results of Lemma 1 and Proposition 2 in order to construct a solution to the state feedback output regulation problem described in Problem 1. The resulting feedback law combines a feedforward term to counteract the action of the exosystem, and a predictor based controller to stabilize the delayed system in the absence of external disturbances.

Theorem 3. Consider the delayed system (1) satisfying Assumptions A1 and A2, and let Π and Γ be the solution to (3). Then, for matrices $Q > 0$, $S > 0$ and $\bar{K}_i$ satisfying the matrix inequality

$$\begin{bmatrix} \Theta & H_1 & H_2 & \cdots & H_N \\ H_1^{\mathrm{T}} & -S_1 & 0 & \cdots & 0 \\ H_2^{\mathrm{T}} & 0 & -S_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ H_N^{\mathrm{T}} & 0 & 0 & \cdots & -S_N \end{bmatrix} < 0, \qquad (20)$$

where $H_i = A_i Q + B\bar{K}_i$ and $\Theta = H_0 + H_0^{\mathrm{T}} + \sum_{i=1}^{N} S_i$ for $i = 0, 1, \cdots, N$, a solution to the state feedback output regulation problem is given by the feedback law

$$u(t) = \sum_{i=0}^{N} K_i \hat{z}_{\tau_u}(t - \tau_i) + \Gamma \mathrm{e}^{\tau_u \mathcal{S}} \omega(t), \qquad (21)$$

with feedback gains $K_i = \bar{K}_i Q^{-1}$, $i = 0, 1, 2, \cdots, N$.

Proof. Let Π and Γ be the solution to (3). Then, with the input in the form of $u(t) = \nu(t) + \Gamma \mathrm{e}^{\tau_u \mathcal{S}} \omega(t)$, it follows from Lemma 1 that the output regulation problem in (1) is equivalent to the stabilization of

$$\dot{z}(t) = \sum_{i=0}^{N} A_i z(t - \tau_i) + B\nu(t - \tau_u). \qquad (22)$$

Thus, the objectives of Problem 1 are reached by achieving $\lim_{t\to\infty} z(t) = 0$. Let the control $\nu(t)$ be in the form

$$\nu(t) = \sum_{i=0}^{N} K_i \hat{z}_{\tau_u}(t - \tau_i), \qquad (23)$$

for some feedback gain $K_i \in \mathbb{R}^{m\times n}$. Then, by Proposition 2, there exists a $\bar{\tau}_u > 0$ such that the closed-loop system (22) under the control law (23) becomes

$$\dot{z}(t) = \sum_{i=0}^{N} A_i z(t - \tau_i) + B \sum_{i=0}^{N} K_i \hat{z}_{\tau_u}(t - \tau_u - \tau_i), \qquad (24a)$$

$$\hat{z}_{\tau_u}(t) = z(t + \tau_u), \qquad (24b)$$

for $t \geq \bar{\tau}_u$. The characteristic equation of the above system is found to be $\det \Delta(s)$, where

$$\Delta(s) = \begin{bmatrix} sI - \sum_{i=0}^{N} A_i \mathrm{e}^{-\tau_i s} & -B \sum_{i=0}^{N} K_i \mathrm{e}^{-(\tau_u+\tau_i)s} \\ -\mathrm{e}^{\tau_u s} I & I \end{bmatrix}. \qquad (25)$$

By the properties of the determinant,

$$\det\Delta(s) = \det\left(sI - \sum_{i=0}^{N}\bar{A}_i e^{-\tau_i s}\right), \qquad (26)$$

where $\bar{A}_i = A_i + BK_i$. The spectrum of the closed-loop system (24) is then equal to the roots of the polynomial (26). This implies that the closed-loop system in (24) is asymptotically stable if

$$\dot{z}(t) = \sum_{i=0}^{N}\bar{A}_i z(t-\tau_i) \qquad (27)$$

is asymptotically stable.

The stability condition for (27) is derived by employing the Lyapunov-Krasovskii Stability Theorem with the functional from Gu et al. [2003] as,

$$V(t) = z^{\mathrm{T}}(t)Pz(t) + \sum_{i=1}^{N}\int_{-\tau_i}^{0} z^{\mathrm{T}}(t+\sigma)R_i z(t+\sigma)\mathrm{d}\sigma,$$

for some real matrices $P > 0$ and $R_i > 0$. The derivative of $V(t)$ along the trajectory of (27) is found to be

$$\dot{V}(t) = z^{\mathrm{T}}(t)\left(P\bar{A}_0 + \bar{A}_0^{\mathrm{T}}P + \sum_{i=1}^{N}R_i\right)z(t) + 2z^{\mathrm{T}}(t)P \times \sum_{i=1}^{N}\bar{A}_i z(t-\tau_i) - \sum_{i=1}^{N} z^{\mathrm{T}}(t-\tau_i)R_i z(t-\tau_i). \qquad (28)$$

Define $Z(t) = [z^{\mathrm{T}}(t),\, z^{\mathrm{T}}(t-\tau_1), \cdots, z^{\mathrm{T}}(t-\tau_N)]^{\mathrm{T}}$. Then, the time derivative of $V(t)$ can be rewritten as $\dot{V}(t) = Z^{\mathrm{T}}(t)\Pi Z(t)$, where

$$\Pi = \begin{bmatrix} \Pi_{11} & P\bar{A}_1 & P\bar{A}_2 & \cdots & P\bar{A}_N \\ \bar{A}_1^{\mathrm{T}}P & -R_1 & 0 & \cdots & 0 \\ \bar{A}_2^{\mathrm{T}}P & 0 & -R_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \bar{A}_N^{\mathrm{T}}P & 0 & 0 & \cdots & -R_N \end{bmatrix},$$

with $\Pi_{11} = P\bar{A}_0 + \bar{A}_0^{\mathrm{T}}P + \sum_{i=1}^{N}R_i$. Moreover, we obtain that $\dot{V}(t) < 0$ and the closed-loop system (27) is asymptotically stable if $\Pi < 0$. Finally, let $Q = P^{-1}$, $S_i = QR_iQ$, and $\bar{K}_i = K_iQ$. Then, we observe that the left-hand side of (20) and Π are congruent matrices, and the closed-loop system (27) is asymptotically stable if (20) is satisfied.

Finally, the state feedback control law achieving output regulation is constructed by combining (21) and (12), where $K_i = \bar{K}_iQ^{-1}$. This concludes the proof. □

Remark 3. A sufficient condition for the existence of the controller gains K_i in Theorem 3 is given by the solvability of the matrix inequality (20). Therefore, any requirement on the controllability of the the delay system (24) is considered in the stability condition of Theorem 3. Detailed discussions on the controllability and observability of linear time delay systems can be found in Salamon [1984], Yi et al. [2008], and the references therein.

4. A NUMERICAL EXAMPLE

In this section we present a numerical example to verify the effectiveness of our proposed solution to the output regulation problem. Consider the perturbed balancing beam system studied in Yoon et al. [2013]. The dynamics of the system is described by equation (1) with

$$A_0 = \begin{bmatrix} 0 & 1 \\ 9248 & -1.635 \end{bmatrix},\ A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 10 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$C_0 = [\,281.9\ \ 0.133\,],\ C_1 = [\,0\ \ 0\,],$$

$$\mathcal{S} = \begin{bmatrix} 0 & 10 \\ -10 & 0 \end{bmatrix},\ \mathcal{P} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ \mathcal{Q} = [\,1\ \ 0\,].$$

Let the time delays be $\tau_1 = 0.01\,s$ and $\tau_u = 0.02\,s$. The initial conditions of the above system are given as $x(\theta) = [-0.02\ \ 0.01]^{\mathrm{T}}$ and $\omega(\theta) = [0\ \ 10]^{\mathrm{T}}$ for $\theta \in [-\tau, 0]$. The balancing beam test rig was designed in Lin et al. [2008] to represent an active magnetic bearing (AMB) system subject to rotor unbalance forces, and has previously served as a test rig for the output regulation problem. For the above system matrices, we find that the feedback gains $K_0 = [-9250\ -0.7275]$ and $K_1 = [0\ -10]$ satisfy the matrix inequality (20) in Theorem 3.

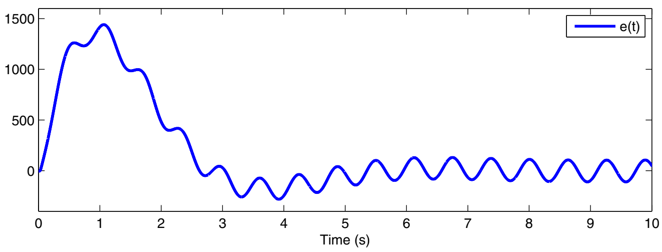

Fig. 2. Tracking error under a stabilizing feedback law

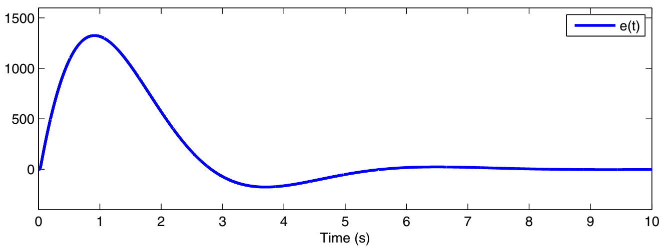

Fig. 3. Tracking error under the output regulation control

We first consider the stabilization problem. More specifically, the perturbed balancing beam with equation (1) is controlled by the stabilizing state feedback law

$$u(t) = K_0\hat{x}_{\tau_u}(t) + K_1\hat{x}_{\tau_u}(t-\tau_1), \qquad (29)$$

where $\hat{x}_{\tau_u}$ is a predictor equivalent to (12) for the state $x(t)$ in (1a) and $\omega = 0$. Figure 2 shows the response of the error signal $e(t)$ for the closed-loop system, and the the effect of the perturbation $\omega(t)$ is clearly visible.

In comparison to the simulation result presented in Fig. 2, the response of $e(t)$ for the closed-loop system under the output regulation control (12) and (21) is presented in Fig. 3. The simulation shows that the effect of the perturbation $\omega(t)$ from the exosystem (1b) is eliminated by the proposed feedback law, and $e(t)$ now approaches zero asymptotically.

Finally, the response of the system state $x(t)$ and the prediction error $\tilde{z}_{\tau_u}(t) = z(t+\tau_u) - \hat{z}_{\tau_u}(t)$ corresponding to the example with the output regulation feedback law is presented in Figs. 4 and 5, respectively. Figure 5 shows that the prediction error becomes zero over time.

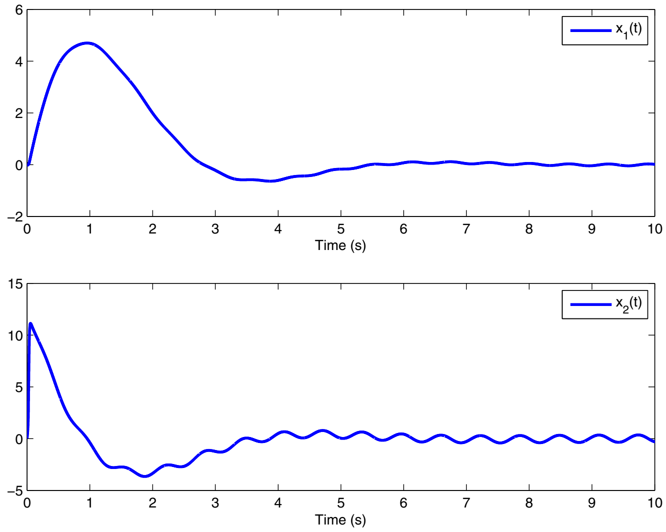

Fig. 4. State response of the closed-loop system under the output regulation control

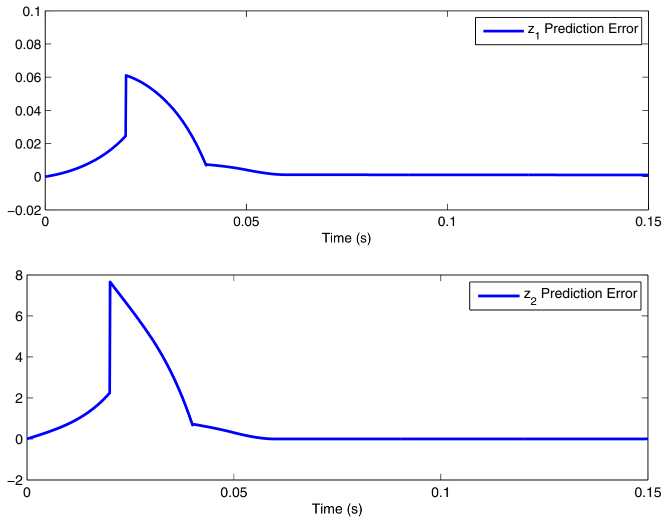

Fig. 5. Prediction error $\tilde{z}_{\tau_u}$ of the closed-loop system under the output regulation control

5. CONCLUSIONS

Based on previous work on output regulation of state delayed systems, the sufficient and necessary conditions for the solvability of the output regulation problem was extended to linear systems with state, input and output delays. A feedback law achieving output regulation was constructed from a predictor based control law that was recently developed for linear systems with state and input delays. Finally, a numerical example was presented to illustrate the results developed in this paper.

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