

Active Fault Diagnosis for Nonlinear Systems with Probabilistic Uncertainties^{*}

Ali Mesbah^{*} Stefan Streif^{**} Rolf Findeisen^{**}
Richard D. Braatz^{*}

^{*} *Massachusetts Institute of Technology, USA;*
e-mail: {amesbah, braatz}@mit.edu.

^{**} *Otto-von-Guericke-University Magdeburg, Germany;*
e-mail: {stefan.streif, rolf.findeisen}@ovgu.de.

Abstract: Stringent requirements on safety and availability of high-performance systems necessitate reliable fault detection and isolation in the event of system failures. This paper investigates active fault diagnosis of nonlinear systems with probabilistic, time-invariant uncertainties of the parameters and initial conditions. A probabilistic model-based approach is presented for the design of auxiliary input signals enhancing fault diagnosability by separation of multiple nonlinear models pertaining to nominal and faulty system operations in the presence of the probabilistic uncertainties. To obtain a computationally tractable formulation, polynomial chaos expansions are used to propagate the probabilistic uncertainties through the system models. The input design problem is formulated in terms of a metric that characterizes the similarity of arbitrarily shaped distributions of the model outputs. An optimal input sequence is generated while considering hard input and state constraints. The simulation results for active diagnosis of multiple faults in a three-tank system indicate the capability of the presented approach to improve fault detectability and isolability under probabilistic uncertainties of the parameters and initial conditions.

1. INTRODUCTION

Fault diagnosis (FD) is crucial for the high-performance operation of complex systems to meet stringent requirements on system availability and safety in the event of component failures. System uncertainties (e.g., due to exogenous disturbances and measurement noise), along with compensatory control actions, often impair reliable fault diagnosis using the measurements acquired during normal system operation (Campbell and Nikoukhah, 2004). Auxiliary input signals can be applied to the system to enhance the detectability and isolability of faults in the system outputs. The so-called *active* fault diagnosis involves designing input signals that ensure the detection or isolation of faults in the presence of uncertainties, while being least intrusive to the system performance (Blanke et al., 2006). Active FD is particularly useful when the reachable output sets of the different faults and the nominal operation overlap due to measurement or process uncertainties, or when the control actions mask the influence of faults.

To this end, most active FD approaches exploit a robust formulation for the input signal design while considering deterministic bounded uncertainties. Typical approaches consider linear or discrete-event systems and minimize the energy of the input signal that can ensure the separation of models pertaining to nominal and faulty system operations (e.g., see (SamPATH et al., 1998; Campbell and Nikoukhah, 2004; Blanke et al., 2006; Ashari et al., 2011), and the references therein). A computationally efficient approach

for designing separating inputs that lead to guaranteed fault diagnosis under deterministic bounds on the disturbances and measurement noise is proposed for linear systems by Scott et al. (2013) using zonotopes. Active FD for nonlinear systems with sufficiently small nonlinearities is considered by Andjelkovic et al. (2008). A set-based framework for nonlinear systems can provide robustness certificates with respect to nonlinearities and bounded uncertainties for given input signals (Streif et al., 2013), and be used to design locally optimal input signals (Paulson et al., 2014) that ensure fault diagnosis.

Alternatively, the active FD problem can be formulated in a probabilistic framework to deal with systems whose uncertainties are characterized in terms of probability distribution functions (Zhang, 1989; Kerestecioglu, 1993). Such a formulation enables taking probabilistic information into account during input design. A probabilistic approach for active FD of mixed discrete-continuous linear systems with additive stochastic uncertainties is proposed by Blackmore et al. (2008). The approach considers an upper bound on the probability of model selection error as the optimization criterion, which is subject to hard input constraints.

This paper addresses the problem of active fault diagnosis for nonlinear systems with probabilistic, time-invariant uncertainties on the parameters and initial conditions (Section 2). A probabilistic approach is presented to design an auxiliary input signal that separates multiple models, which correspond to nominal and faulty system operations. Generalized polynomial chaos (PC) theory (Wiener, 1938; Xiu and Karniadakis, 2002) is employed as a computationally efficient spectral tool for uncertainty analysis and

^{*} Financial support is acknowledged from Novartis Pharma AG, and from the Research Center Dynamic Systems, Saxony-Anhalt, Germany.

propagation of the probability density functions (Section 3). The PC framework enables replacing the implicit mappings between the uncertain variables/parameters and the states (defined in terms of differential algebraic equations) with explicit functions in the form of a series of orthogonal polynomials, whose statistical moments can be readily computed from the expansion coefficients (e.g., see (Fisher and Bhattacharya, 2011; Fagiano and Khammash, 2012; Mesbah et al., 2014), and citations therein for applications of PC expansions). A nonlinear optimization formulation for input design is presented in Section 4. The optimization objective is expressed in terms of the *Bhattacharyya coefficient* (Kailath, 1967), which provides a measure for the similarity of the probability densities of the model outputs. Hard input and state constraints are included into the optimization formulation to reduce violations of system operation requirements (e.g., safety considerations, actuator saturation, etc.) during fault diagnosis. The input signals generated in this manner are expected to be more effective for model separation, and consequently fault diagnosis, than limited power input signals (Blackmore et al., 2008). The proposed probabilistic active FD approach is demonstrated using a three-tank system for multiple fault scenarios under probabilistic parameter uncertainties (Section 5).

2. PROBLEM FORMULATION

For an uncertain system subject to faults, consider discrete-time nonlinear models

$$f^{[i]} : \begin{cases} \mathbf{x}_{k+1}^{[i]} = \mathbf{g}^{[i]}(\mathbf{x}_k^{[i]}, \mathbf{u}_k, \boldsymbol{\theta}^{[i]}) \\ \mathbf{y}_k^{[i]} = \mathbf{h}^{[i]}(\mathbf{x}_k^{[i]}, \mathbf{u}_k, \boldsymbol{\theta}^{[i]}) \end{cases} \quad i \in \mathcal{I} := \{0, \dots, n_f\} \quad (1)$$

where $k \in \mathbb{N}$ is the time index, n_f is the number of faults, the superscripts $^{[i]}$, $i \in \mathcal{I}$, denote the nominal and fault scenarios $\mathcal{F} := \{f^{[0]}, f^{[1]}, \dots, f^{[n_f]}\}$, the algebraic functions $\mathbf{g}^{[i]}$ and $\mathbf{h}^{[i]}$ describe nonlinear system dynamics and the model output, respectively, and $\mathbf{x}_k^{[i]} \in \mathbb{R}^{n_x}$, $\boldsymbol{\theta}^{[i]} \in \mathbb{R}^{n_\theta}$, $\mathbf{u}_k \in \mathbb{R}^{n_u}$, and $\mathbf{y}_k^{[i]} \in \mathbb{R}^{n_y}$ denote the system states, time-invariant parameters, inputs, and outputs available for fault diagnosis, respectively. In what follows, the nominal and fault models (and their associated variables) are represented by the superscripts $^{[0]}$ and $^{[i]}$, $i \in \mathcal{I} \setminus 0$, respectively. Each fault model $f^{[i]}$ in (1) contains its own set of variables $\mathbf{x}_k^{[i]}$, $\mathbf{y}_k^{[i]}$, and parameters $\boldsymbol{\theta}^{[i]}$, whereas the input \mathbf{u}_k is the same for all models. To save space, the superscripts $^{[i]}$ are dropped on the state variables, parameters, and functions $\mathbf{g}^{[i]}$ and $\mathbf{h}^{[i]}$. Note that this work assumes that the (potential) system faults are known a priori and the faulty system dynamics can be described by (1) (e.g., see (Mesbah et al., 2012) and the references therein for performance diagnosis approaches that do not make this assumption).

The method presented in this paper can deal with probabilistic, time-invariant uncertainties on the parameters and of the initial conditions. To simplify the presentation and to conserve space, only uncertainties on the parameters are presented. Probabilistic uncertain initial conditions can be treated straightforwardly and in a similar manner. The parameter vector $\boldsymbol{\theta}$ in the multiple models in (1) consists of independent distributed random variables

θ_j , with known probability density functions (PDFs) F_{θ_j} . Over the support Ω , a probability triple $(\Omega, \mathcal{G}, \mathcal{P})$ is defined in terms of the σ -algebra \mathcal{G} and the probability measure \mathcal{P} on (Ω, \mathcal{G}) . The parameters θ_j belong to the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ of all random variables, whose L_2 -norm is finite (i.e., $\theta_j \in \mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P}) \forall j \in \{1, \dots, n_\theta\}$).

Model-based fault diagnosis consists of two tasks, namely *fault detection* and *fault isolation*. In fault detection, the task is to examine if the nominal scenario $f^{[0]}$ is consistent with the system measurements. Fault isolation, on the other hand, aims to determine which of the fault scenarios $f^{[i]}$, $i = 1, 2, \dots, n_f$ can describe the measurements. The goal is to achieve complete fault isolation, where only one fault scenario is uniquely chosen.

The primary challenge in fault diagnosis is that faults may not be detected and isolated under all operating conditions due to measurement, process, and parameter uncertainties, as well as the potential effect of feedback control in masking the effects of faults. The diagnosability problem results from the overlap of the reachable output sets for the nominal and fault scenarios, which make the scenarios in \mathcal{F} undiagnosable. An approach to circumvent this problem is to design input signals such that any sequence of system measurements is consistent with only one scenario $f^{[i]}$, $i \in \mathcal{I}$.

Due to the parametric uncertainties in the nonlinear systems considered in this paper, the outputs of the multiple models in (1) are random variables. In this situation, the active fault diagnosis problem is to design an auxiliary input sequence that enables the separation of the PDFs of the outputs (denoted by $F_{\mathbf{y}^{[i]}}$) pertaining to the different scenarios in \mathcal{F} .

Problem 1 (Active fault diagnosis under probabilistic parametric uncertainties): Design an input sequence $\mathbf{u} \subseteq \mathcal{U}$ to separate the probability density functions $F_{\mathbf{y}^{[i]}}$, $\forall i \in \mathcal{I}$ at a time instant n_k , while nonlinear inequality constraints on states $\mathbf{x}^{[i]}$ corresponding to the scenarios $f^{[i]}$ are fulfilled (i.e., $\mathcal{H}(\mathbf{x}^{[i]}, \mathbf{u}) \leq 0, \forall i \in \mathcal{I}$).

In Problem 1, the input and state constraints are denoted by compact convex sets \mathcal{U} and \mathcal{X} , respectively. These constraints can be specified so that the designed input sequence is minimally intrusive on the system performance, so that the system safety is not jeopardized during fault diagnosis.

The key issue in solving Problem 1 is the propagation of probabilistic parametric uncertainties through the models (1). The most common approaches for probabilistic analysis of uncertain systems are Monte Carlo and Markov Chain Monte Carlo methods, which can be prohibitively expensive. Next, polynomial chaos expansions are introduced, which provide computationally efficient means for the approximation of random variables that have finite second-order moments.

3. POLYNOMIAL CHAOS EXPANSIONS

A general second-order random variable $y(\boldsymbol{\theta}) \in \mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ can be expressed in terms of a polynomial chaos expansion as (Xiu and Karniadakis, 2002)

$$y(\boldsymbol{\theta}) = \sum_{i=0}^{\infty} a_i \Phi_i(\boldsymbol{\theta}) \quad (2)$$

where a_i denotes the expansion coefficients and $\Phi_i(\boldsymbol{\theta})$ denotes the PC basis functions of degree m with respect to the random variables $\boldsymbol{\theta}$. The basis functions belong to the Askey scheme of polynomials, which encompasses a set of orthogonal polynomials in the Hilbert space defined by the support of the random variables $\boldsymbol{\theta}$ (Ghanem and Spanos, 1991). Hence, the basis functions Φ_i comprise a set of orthogonal polynomials in $\mathcal{L}^2(\Omega, \mathcal{G}, \mathcal{P})$ that satisfy

$$\mathbf{E}[\Phi_i(\boldsymbol{\theta})\Phi_j(\boldsymbol{\theta})] = \mathbf{E}[\Phi_i(\boldsymbol{\theta})^2]\delta_{ij}$$

where δ_{ij} is the Kronecker delta and $\mathbf{E}[\Phi(\boldsymbol{\theta})] := \int_{\Omega} \Phi(\boldsymbol{\theta}) dF_{\boldsymbol{\theta}}$ denotes expectation of a function $\Phi(\boldsymbol{\theta})$. The choice of the orthogonal polynomials is made such that their weight function is the multivariate PDF of $\boldsymbol{\theta}$ ($F_{\boldsymbol{\theta}}$). For example, Hermite polynomials are utilized for Gaussian random variables (Xiu and Karniadakis, 2002).

In practice, the PC expansion (2) is truncated after p terms, which is determined by the dimension of $\boldsymbol{\theta}$ and the order of the orthogonal polynomials (i.e., $p+1 = \frac{(n_{\boldsymbol{\theta}}+m)!}{n_{\boldsymbol{\theta}}!m!}$). The truncated PC expansion takes the form

$$\hat{y}(\boldsymbol{\theta}) := \sum_{i=0}^p a_i \Phi_i(\boldsymbol{\theta}) = \mathbf{a}^{\top} \boldsymbol{\Lambda}(\boldsymbol{\theta}) \quad (3)$$

where $\mathbf{a} := [a_0, \dots, a_p]^{\top}$ and $\boldsymbol{\Lambda}(\boldsymbol{\theta}) := [\Phi_0(\boldsymbol{\theta}), \dots, \Phi_p(\boldsymbol{\theta})]^{\top}$.

Probabilistic collocation methods (e.g., see discussion and citations in (Nagy and Braatz, 2007; Fagiano and Khammash, 2012)) can be used to determine the coefficients \mathbf{a} in (3). Consider the residual

$$R(\mathbf{a}, \boldsymbol{\theta}) = \hat{y}(\boldsymbol{\theta}) - y(\boldsymbol{\theta})$$

for each system output, where $y(\boldsymbol{\theta})$ is computed using the nonlinear model (1). The coefficients \mathbf{a} can be computed by requiring that the residual R be orthogonal to each basis function Φ_i (e.g., (Tatang et al., 1997)),

$$\int_{\Omega} R(\mathbf{a}, \boldsymbol{\theta}) \Phi_i(\boldsymbol{\theta}) dF_{\boldsymbol{\theta}} = 0, \quad i = 0, \dots, p. \quad (4)$$

Equation (4) can be approximated using the Gaussian quadrature method

$$\sum_{j=0}^l v_j F_{\boldsymbol{\theta}}(\boldsymbol{\theta}_j) R(\mathbf{a}, \boldsymbol{\theta}_j) \Phi_i(\boldsymbol{\theta}_j) = 0, \quad i = 0, \dots, p \quad (5)$$

where v_j are the weights of the Gaussian quadrature approximation and $\boldsymbol{\theta}_j$ are the samples of the parameter vector $\boldsymbol{\theta}$ drawn from the multivariate PDF $F_{\boldsymbol{\theta}}$. Since the orthogonal polynomials $\Phi_i(\boldsymbol{\theta}_j)$ are non-zero terms, (5) reduces to

$$F_{\boldsymbol{\theta}}(\boldsymbol{\theta}_j) R(\mathbf{a}, \boldsymbol{\theta}_j) = 0, \quad j = 0, \dots, l. \quad (6)$$

Equation (6) implies that the expansion coefficients \mathbf{a} can be estimated through computing the residual $R(\mathbf{a}, \boldsymbol{\theta}_j)$ at l collocation points (i.e., parameter vectors $\boldsymbol{\theta}_j$ with a non-zero probability $F_{\boldsymbol{\theta}}(\boldsymbol{\theta}_j)$). In this approach, the accuracy of a polynomial chaos expansion is largely affected by the selection of the collocation points. For orthogonal polynomials of degree m , a common choice for collocation points is the roots of polynomials with degree $m+1$ (Tatang et al., 1997).

Once the coefficients of the PC expansion (3) are obtained, the statistics of the random variable $\hat{y}(\boldsymbol{\theta})$ can be determined efficiently by exploiting the orthogonality property of the polynomials. Next, PC expansions are used to present a tractable formulation for Problem 1.

4. ACTIVE FAULT DIAGNOSIS USING POLYNOMIAL CHAOS EXPANSIONS

4.1 A Metric for Similarity of Two Distributions

The task of fault diagnosis under probabilistic uncertainties involves separating the probability density functions of random variables that pertain to the outputs of the nominal operation and fault scenarios (cf. Problem 1). This paper uses the probability of misclassification (aka the Bayes error) in statistical hypothesis testing (Anderson, 2003) to define a measure for active fault diagnosis. The Bayes error is directly related to the similarity of two probability densities; the larger the Bayes error is, the more similar the distributions are. The active FD problem should therefore be formulated as the design of an input signal that minimizes the Bayes errors associated with the different scenarios in \mathcal{F} .

A measure closely related to the Bayes error is the Bhattacharyya coefficient (Kailath, 1967), which indicates the degree of overlap between two distributions. Properties of the Bhattacharyya coefficient such as its relation to the Fisher information measure, as well as explicit forms for various distributions are established (Kailath, 1967; Djouadi et al., 1990). This paper considers the Bhattacharyya coefficient defined by

$$\mathfrak{B}^{[i,j]}(\boldsymbol{\theta}) := \mathfrak{B}(F_{\mathbf{y}^{[i]}}, F_{\mathbf{y}^{[j]}}) = \int \sqrt{F_{\mathbf{y}^{[i]}} F_{\mathbf{y}^{[j]}}} d\boldsymbol{\theta} \quad (7)$$

where $F_{\mathbf{y}^{[i]}}$ and $F_{\mathbf{y}^{[j]}}$ are the PDFs of the outputs corresponding to scenarios $f^{[i]}$ and $f^{[j]}$ in \mathcal{F} , respectively. To compute $\mathfrak{B}^{[i,j]}$, the probability densities can be obtained by sampling. Hence, the histogram formulation is employed to represent the PDFs $F_{\mathbf{y}^{[i]}}$ and $F_{\mathbf{y}^{[j]}}$ in (7) by the discrete distributions $\hat{F}_{\mathbf{y}^{[i]}} := \{\hat{F}_{\mathbf{y}^{[i]},u}\}_{u=1,\dots,o}$ (with $\sum_{u=1}^o \hat{F}_{\mathbf{y}^{[i]},u} = 1$) and $\hat{F}_{\mathbf{y}^{[j]}} := \{\hat{F}_{\mathbf{y}^{[j]},u}\}_{u=1,\dots,o}$ (with $\sum_{u=1}^o \hat{F}_{\mathbf{y}^{[j]},u} = 1$), respectively, where o denotes the number of partitions in the histograms. The sample estimate of the Bhattacharyya coefficient takes the form

$$\hat{\mathfrak{B}}^{[i,j]}(\boldsymbol{\theta}) = \sum_{u=1}^o \sqrt{\hat{F}_{\mathbf{y}^{[i]},u} \hat{F}_{\mathbf{y}^{[j]},u}}. \quad (8)$$

This expression implies that the Bhattacharyya coefficient will be larger when the distributions have a larger overlap (if the distributions do not overlap, $\mathfrak{B}^{[i,j]}$ will be zero). The choice of the number of partitions o in (8) is critical, as too few partitions will lead to loss of accuracy due to overestimating the overlap; and too many partitions will result in partitions with no members despite being surrounded by populated partitions.

The sample estimate of the Bhattacharyya coefficient is used to define the metric for similarity of two distributions as (Comaniciu et al., 2000)

$$\Delta^{[i,j]}(\boldsymbol{\theta}) = \sqrt{1 - \hat{\mathfrak{B}}^{[i,j]}(\boldsymbol{\theta})}. \quad (9)$$

This expression is also known as the *Hellinger distance*. The use of (9) in this work is motivated by (i) its near optimality due to its relation to the Bayes error, (ii) having a metric structure, and (iii) being valid for any arbitrary distributions. For further discussions on similarity and other distance metrics for PDFs, see (Halder and Bhattacharya, 2012).

4.2 Input Design

The metric $\Delta^{[i,j]}$ is used to formulate the active fault diagnosis problem. Expression (9) implies that smaller overlaps between two distributions (smaller $\hat{\mathfrak{B}}^{[i,j]}$) will lead to larger values of $\Delta^{[i,j]}$, with $\Delta^{[i,j]} = 1$ indicating complete separation of two distributions. Hence, the input signal for fault diagnosis should be designed to maximize the metric $\Delta^{[i,j]}$ for any scenarios $f^{[i]}$ and $f^{[j]}$ in \mathcal{F} .

PC expansions are used to propagate the parametric uncertainties through the nonlinear multiple models (1). The PC expansions enable efficient construction of the PDFs of the outputs $\mathbf{y}^{[i]}$ available for fault diagnosis. The probability density functions $F_{\mathbf{y}^{[i]}}$ can be approximated by Monte Carlo simulations of the PC expansions, which significantly accelerate Monte Carlo-based uncertainty analysis. Alternatively, the PDFs can be constructed from the moments of PDFs, which are readily computed from the PC coefficients \mathbf{a} (e.g., see (Fisher and Bhattacharya, 2011; Streif et al., 2014)).

This paper obtains the discrete approximations of the probability density functions $F_{\mathbf{y}^{[i]}}$ by Monte Carlo simulations of the PC expansions. Using the metric (9) as a measure for the similarity of any arbitrary distributions, the input design problem for diagnosis of multiple fault scenarios under probabilistic uncertainties (Problem 1) is formulated below.

Problem 2 (Auxiliary input design): Consider the time-index set $\mathcal{T} = \{0, 1, \dots, n_k\}$. The optimal input sequence $\mathbf{u}^* := [\mathbf{u}_0, \dots, \mathbf{u}_{n_k}]^\top$ that facilitates the separation of the scenarios $f^{[i]} \in \mathcal{F}$, $\forall i \in \mathcal{I}$ at time instant n_k is defined by

$$\begin{aligned} \mathbf{u}^* &:= \arg \max_{\mathbf{u}^*} \sum_{i=0}^{n_f-1} \sum_{j=i+1}^{n_f} \Delta_{n_k}^{[i,j]}(\boldsymbol{\theta}) \\ \text{subject to: } &\hat{\mathbf{x}}_k^{[i]} = \bar{\mathbf{a}}_k^{[i]} \bar{\boldsymbol{\Lambda}}(\boldsymbol{\theta}), \quad \forall k \in \mathcal{T}, \quad \forall i \in \mathcal{I} \\ &\mathcal{H}^{nom}(\hat{\mathbf{x}}_k^{[i]}, \mathbf{u}_k) \leq \mathbf{0}, \quad \forall k \in \mathcal{T}, \quad \forall i \in \mathcal{I} \\ &\mathbf{u}_k \in \mathcal{U}, \quad \forall k \in \mathcal{T} \end{aligned} \quad (10)$$

where $\bar{\mathbf{a}}_k^{[i]} := \{\mathbf{a}_{k,q}^{[i]}\}_{q=1, \dots, n_x}$ denotes the vector of coefficients of the PC expansions of states $\mathbf{x}_k^{[i]}$, $\bar{\boldsymbol{\Lambda}}(\boldsymbol{\theta}) := \{\boldsymbol{\Lambda}_q(\boldsymbol{\theta})\}_{q=1, \dots, n_x}$ is defined as in (3), and \mathcal{H}^{nom} represents the nonlinear state inequality constraints, which are enforced about the nominal state trajectory (i.e., $\hat{\mathbf{x}}_k^{[i]}(\boldsymbol{\theta}_{nom})$).

In (10), the similarity of all combinations of the PDFs $F_{\mathbf{y}^{[i]}}$, $\forall i \in \mathcal{I}$ is evaluated only at time instant n_k ($\Delta^{[i,j]}$ is computed at n_k). The hard input constraints and the state inequality constraints are considered in the determination of the designed input sequence \mathbf{u}^* . The following algorithm is used to solve (10).

Algorithm 1. (Active fault diagnosis using polynomial chaos expansions):

- Input: 1) Nominal and fault scenarios \mathcal{F}
2) Initial states $\mathbf{x}_0^{[i]}$
3) Uncertainty description of parameter vector $\boldsymbol{\theta}$
4) Time-index set \mathcal{T}
5) Type and order p of orthogonal polynomials
6) Number of collocation points l
7) Histogram partitions o
8) Input and state constraints
-

At each optimization iteration in (10):

- 1) Use the input sequence \mathbf{u} and $\mathbf{x}_0^{[i]}$ to carry out l simulations of the nonlinear multiple models (1) using l collocation points of the parameter vector $\boldsymbol{\theta}$
 - 2) Determine PC coefficients $\bar{\mathbf{a}}_k^{[i]}$, $\forall k \in \mathcal{T}$, $\forall i \in \mathcal{I}$ using (6)
 - 3) Use the PC expansions to perform Monte Carlo simulations to construct the discrete distributions $\hat{F}_{\mathbf{y}^{[i]}}$, $\forall i \in \mathcal{I}$
 - 4) Compute the metric (9) for all combinations of the scenarios $f^{[i]} \in \mathcal{F}$, $\forall i \in \mathcal{I}$
 - 5) Solve the optimization problem (10) to obtain \mathbf{u}^* . Each optimization iteration entails repeated evaluation of the PC expansions and the metric as in Steps 1 to 4.
-

The use of PC expansions in Algorithm 1 significantly improves the computational efficiency of the input design approach in the presence of probabilistic uncertainties, as the Monte Carlo simulations in Step 3 are performed using the PC expansions instead of the nonlinear models (1). The basis functions $\boldsymbol{\Lambda}(\boldsymbol{\theta})$ of the PC expansions are computed off-line for the different samples of $\boldsymbol{\theta}$.

5. ACTIVE FAULT DIAGNOSIS OF A THREE-TANK SYSTEM UNDER PROBABILISTIC UNCERTAINTIES

Consider the three-tank system shown in Fig. 1, which is a well-known benchmark for fault detection and isolation (Zhang et al., 2002). The tanks T_1 , T_2 , and T_3 are cylinders with identical cross-sectional area A . The three tanks are connected with connection pipes, whose cross-sectional area is denoted by S_p , and with u_1 and u_2 representing the liquid flow rates supplied to tanks T_1 and T_2 (by pumps P_1 and P_2), respectively. The liquid levels in the three tanks are denoted by x_1 , x_2 , and x_3 .

Using the mass conservation law in conjunction with the Torricelli's law, the liquid level dynamics in the three tanks are described by

$$\begin{aligned} \dot{x}_1 &= \frac{-c_1 S_p \text{sign}(x_1 - x_3) \sqrt{2g|x_1 - x_3|} + u_1}{A} \\ \dot{x}_2 &= \frac{-c_3 S_p \text{sign}(x_2 - x_3) \sqrt{2g|x_2 - x_3|} - c_2 S_p \sqrt{2gx_2} - q_f + u_2}{A} \\ \dot{x}_3 &= \frac{c_1 S_p \text{sign}(x_1 - x_3) \sqrt{2g|x_1 - x_3|} - c_3 S_p \text{sign}(x_3 - x_2) \sqrt{2g|x_3 - x_2|}}{A} \end{aligned} \quad (11)$$

where q_f is the outflow rate from tank T_2 due to leakage ($q_f = 0$ during nominal operation), g denotes the gravitational acceleration, and c_1 , c_2 , and c_3 are the nondimensional outflow coefficients, which are considered to be random variables distributed normally with mean μ and variance σ^2 ($c_i \sim \mathcal{N}(\mu, \sigma^2)$, $i = 1, 2, 3$). In (11), the system states and inputs are $\mathbf{x} := [x_1 \ x_2 \ x_3]^\top$ and $\mathbf{u} := [u_1 \ u_2]^\top$, respectively. The only system output available for fault

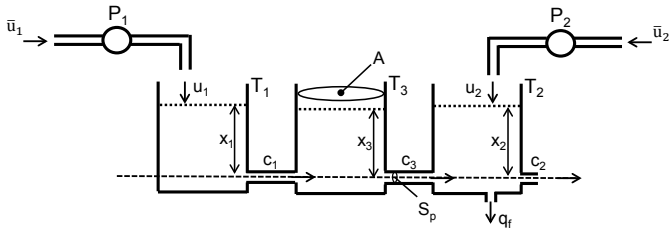


Fig. 1. The three-tank system (Zhang et al., 2002).

diagnosis is the liquid level in tank T_3 (i.e., $y^{[i]} = x_3^{[i]}, \forall i \in \mathcal{I}$). To meet the system operation requirements, the states and inputs are constrained as $0 \leq x_i \leq 0.75$ m, $i = 1, 2, 3$ and $0 \leq u_j \leq 10^{-4}$ m³/s, $j = 1, 2$, respectively. The model parameters are listed in Table 1.

Two fault scenarios are considered:

$f^{[1]}$: **Fault Scenario A:** A multiplicative actuator fault in pump P_1 is defined by letting $u_1 = \bar{u}_1 + (\alpha - 1)\bar{u}_1$, where \bar{u}_1 is the liquid flow rate during nominal operation and α is an uncertain parameter characterizing the magnitude of fault. The parameter α is defined by $\alpha \sim \mathcal{N}(0.6, 4 \times 10^{-4})$.

$f^{[2]}$: **Fault Scenario B:** Tank T_2 has a circular leak with uncertain radius $r \sim \mathcal{N}(0.002, 10^{-6})$, which leads to the outflow rate $q_f = c_2 \pi r^2 \sqrt{2g x_2}$ in (11).

The set \mathcal{F} consists of three operation scenarios, namely, the nominal scenario $f^{[0]}$ and the nonlinear fault scenarios $f^{[1]}$ and $f^{[2]}$.

Active fault diagnosis is performed to detect and isolate the fault scenarios. Due to the probabilistic parametric uncertainties in (11), the output $y^{[i]} = x_3^{[i]}$ of the nonlinear models pertaining to the scenarios $f^{[i]}, i = 0, 1, 2$ are random variables. The active input design formulation (10) is exploited to design the optimal input sequences \mathbf{u}^* that separate the PDFs $F_{y^{[i]}}, i = 0, 1, 2$ at time instant 3000 s ($n_k = 60$ in (10)).

Third-order expansions of Hermite polynomials are utilized to propagate the parametric uncertainties through the system model in each scenario. The Hermite polynomial is specified by the normal distribution of the uncertain parameters (Xiu and Karniadakis, 2002). In the nominal scenario $f^{[0]}$, the PC expansions have 20 terms (i.e., $p = 19$ in (3)). The PC expansions in the scenarios $f^{[1]}$ and $f^{[2]}$, however, consist of 35 terms, as their respective models have 4 uncertain parameters. Algorithm 1 is applied to solve (10), where the number of histogram partitions and the number of collocation points used for estimating the expansion coefficients are chosen as $o = 80$ and $l = 60$, respectively. The CPU time to solve the optimization problem (10) is approximately 15 s on a regular personal computer (Core i7, 2.90 GHz, 8.00 GB).

Fig. 2 shows the liquid level profiles in tank T_3 for all scenarios $f^{[i]}, i = 0, 1, 2$. The profiles indicate that the ac-

$A = 0.0154$ m ²	$c_1 \sim \mathcal{N}(1.0, 0.0025)$
$S_p = 5 \times 10^{-5}$ m ²	$c_2 \sim \mathcal{N}(0.8, 0.0025)$
$g = 9.81$ m/s ²	$c_3 \sim \mathcal{N}(1.0, 0.0025)$

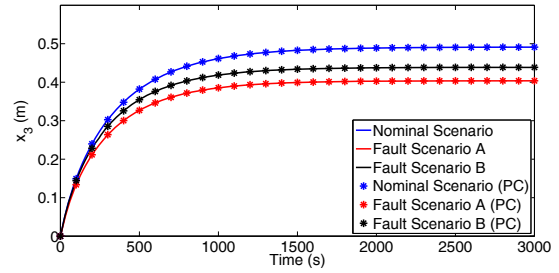
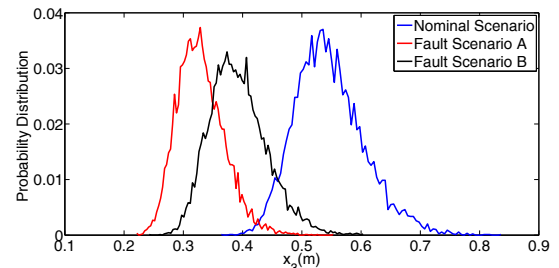


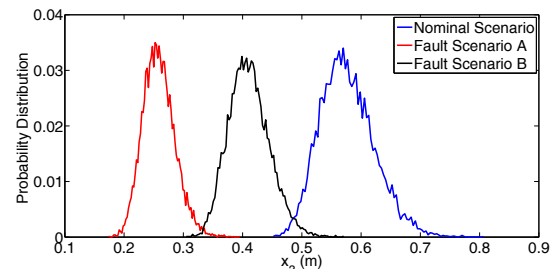
Fig. 2. Comparison between the liquid level profiles in Tank 3 for the nominal and fault scenarios. The profiles are predicted by the nonlinear model and the third-order PC expansions computed using the nominal parameter values.

tuator fault in pump P_1 and the leakage in tank T_2 would lead to a notable change in the liquid level in tank T_3 (the output available for fault diagnosis). Fig. 2 also suggests that the third-order PC expansions, whose coefficients are estimated using 60 collocation points, provide adequate approximation of the liquid level profiles.

Estimated probability density functions of the liquid level in tank T_3 ($F_{y^{[i]}}, i = 0, 1, 2$) for the nominal input sequences are shown in Fig. 3a. The PDFs are constructed based on 10,000 simulations of the three-tank system in the presence of parametric uncertainties. The faults cannot be effectively detected and isolated under the nominal operating conditions, due to overlap of the distributions $F_{y^{[i]}}$, which results from the probabilistic parametric uncertainties. On the other hand, Fig. 3b indicates that applying the optimal input sequences computed using (10) to the three-tank system would lead to better separation of the distributions and, therefore, enhanced fault diagnosability. The optimal input sequences were obtained while



(a) Nominal input sequences



(b) Optimal input sequences computed by (10)

Fig. 3. Probability density functions (based on 10,000 Monte Carlo simulations) of the liquid level in tank T_3 for the nominal and fault scenarios at 3000 s.

considering input and state constraints on the nominal system. However, Fig. 3b shows that the liquid level x_3 may violate its upper bound ($0 \leq x_3 \leq 0.75$) due to parametric uncertainties. Constraint violations under probabilistic uncertainties can be mitigated by incorporating chance constraints into Problem 2 (e.g., see (Mesbah et al., 2014)). The simulation results indicate the capability of the proposed active fault diagnosis approach to improve fault detectability and isolability in the presence of probabilistic uncertainties through the optimal design of input sequences.

6. CONCLUSIONS

A probabilistic approach for active fault diagnosis of nonlinear systems with probabilistic, time-invariant uncertainties is presented. The approach uses polynomial chaos expansions for uncertainty propagation, which leads to a computationally tractable formulation. The Bhattacharyya coefficient is used to define a metric that quantifies the degree of overlap between output distributions of random variables pertaining to multiple fault scenarios. The auxiliary input design problem is formulated in terms of a nonlinear optimization that considers hard input and state constraints to promote safe system operation during fault diagnosis.

REFERENCES

- Anderson, T.W. (2003). *An Introduction to Multivariate Statistical Analysis*. John Wiley, New York.
- Andjelkovic, I., Sweetingham, K., and Campbell, S.L. (2008). Active fault detection in nonlinear systems using auxiliary signals. In *Proc. of the American Control Conference*, 2142–2147. Seattle, Washington.
- Ashari, A.E., Nikoukhah, R., and Campbell, S.L. (2011). Auxiliary signal design for robust active fault detection of linear discrete-time systems. *Automatica*, 47, 1887–1895.
- Blackmore, L., Rajamanohran, S., and Williams, B.C. (2008). Active estimation for jump Markov linear systems. *IEEE Transactions on Automatic Control*, 53, 2223–2236.
- Blanke, M., Kinnaert, M., Lunze, J., and Staroswiecki, M. (2006). *Diagnosis and Fault-Tolerant Control*. Springer, Berlin.
- Campbell, S.L. and Nikoukhah, R. (2004). *Auxiliary Signal Design for Failure Detection*. Princeton University Press.
- Comaniciu, D., Ramesh, V., and Meer, P. (2000). Real-time tracking of non-rigid objects using mean shift. In *Proc. of the IEEE Conference on Computer Vision and Pattern Recognition*, 142–149. Hilton Head Island, South Carolina.
- Djouadi, A., Snorrason, O., and Garber, F.D. (1990). The quality of training-sample estimates of the Bhattacharyya coefficient. *IEEE Transactions on Pattern Analysis and Machine Intelligence*, 12, 92–97.
- Fagiano, L. and Khammash, M. (2012). Nonlinear stochastic model predictive control via regularized polynomial chaos expansions. In *Proc. of the IEEE Conference on Decision and Control*, 142–147. Maui, Hawaii.
- Fisher, J. and Bhattacharya, R. (2011). Optimal trajectory generation with probabilistic system uncertainty using polynomial chaos. *Journal of Dynamic Systems, Measurement, and Control*, 133, 014501–1 to 6.
- Ghanem, R. and Spanos, P. (1991). *Stochastic Finite Elements - A Spectral Approach*. Springer, New York.
- Halder, A. and Bhattacharya, R. (2012). Further results on probabilistic model validation in Wasserstein metric. In *Proc. of the IEEE Conference on Decision and Control*, 5542–5547. Maui, Hawaii.
- Kailath, T. (1967). The divergence and Bhattacharyya distance measures in signal selection. *IEEE Transactions on Communication Technology*, COM-15, 52–60.
- Kerestecioglu, F. (1993). *Change Detection and Input Design in Dynamical Systems*. John Wiley, New York.
- Mesbah, A., Bombois, X., Forgone, M., Ludlage, J.H.A., Modén, P.E., Hjalmarsson, H., and den Hof, P.M.J.V. (2012). A unified experiment design framework for detection and identification in closed-loop performance diagnosis. In *Proc. of the 51st IEEE Conference on Decision and Control*, 2152–2157. Maui, Hawaii.
- Mesbah, A., Streif, S., Findeisen, R., and Braatz, R.D. (2014). Stochastic nonlinear model predictive control with probabilistic constraints. In *Proc. of the American Control Conference*. Portland, Oregon.
- Nagy, Z.K. and Braatz, R.D. (2007). Distributional uncertainty analysis using power series and polynomial chaos expansions. *Journal of Process Control*, 17, 229–240.
- Paulson, J.A., Streif, S., Raimondo, D.M., Braatz, R.D., and Findeisen, R. (2014). Robust active fault diagnosis for nonlinear uncertain systems. In *Proc. of the European Control Conference*. Strasbourg.
- Sampath, M., Lafortune, S., and Teneketzis, D. (1998). Active diagnosis of discrete-event systems. *IEEE Transactions on Automatic Control*, 43, 908–929.
- Scott, J.K., Findeisen, R., Braatz, R.D., and Raimondo, D.M. (2013). Design of active inputs for set based fault diagnosis. In *Proc. of the American Control Conference*, 3567–3572. Washington D.C.
- Streif, S., Hast, D., Braatz, R.D., and Findeisen, R. (2013). Certifying robustness of separating inputs and outputs in active fault diagnosis for uncertain nonlinear systems. In *Proc. of International Symposium on Dynamics and Control of Process Systems*, 837–842. Mumbai.
- Streif, S., Petzke, F., Mesbah, A., Findeisen, R., and Braatz, R.D. (2014). Optimal experiment design for probabilistic model discrimination using polynomial chaos. In *Proc. of the IFAC World Congress*. Cape Town.
- Tatang, M.A., Pan, W., Prinn, R.G., and McRae, G.J. (1997). An efficient method for parametric uncertainty analysis of numerical geophysical models. *Journal of Geophysical Research*, 102, 925–932.
- Wiener, N. (1938). The homogeneous chaos. *American Journal of Mathematics*, 60, 897–936.
- Xiu, D. and Karniadakis, G.E. (2002). The Wiener-Askey polynomial chaos for stochastic differential equations. *SIAM Journal of Scientific Computation*, 24, 619–644.
- Zhang, X., Polycarpou, M.M., and Parisini, T. (2002). A robust detection and isolation scheme for abrupt and incipient faults in nonlinear systems. *IEEE Transactions on Automatic Control*, 47, 576–593.
- Zhang, X.J. (1989). *Auxiliary Signal Design in Fault Detection and Diagnosis*. Springer, Berlin.