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Switching Rule Design for Sector-Bounded Nonlinear Switched Systems ⋆

Tiago Dezuo * **Alexandre Trofino** * **César C. Scharlau** **

* *Department of Automation and Systems Engineering, Federal University of Santa Catarina (UFSC), Florianópolis, SC, Brazil, (e-mail: tiago.dezuo@posgrad.ufsc.br, alexandre.trofino@ufsc.br)*
** *Campus Araranguá, Federal University of Santa Catarina (UFSC), Araranguá, SC, Brazil, (e-mail: cesar.scharlau@ufsc.br)*

Abstract: This paper presents a technique for designing switching rules that drive the state of a class of nonlinear switched system to a desired constant reference. The system may contain state-dependent sector-bounded nonlinear functions. The proposed method considers a switching rule using the 'max' composition of auxiliary functions. The results are given in terms of Linear Matrix Inequalities (LMIs) and they guarantee global asymptotic stability of the closed-loop system even if sliding modes occur on any switching surface of the system. The application of the method is illustrated through a numerical example based on a Photovoltaic (PV) system and important requirements are achieved, such as the Maximum Power Point Tracking (MPPT) and robustness with respect to the uncertain parameters of the PV array.

1. INTRODUCTION

A switched system can be defined as a dynamical system composed by a set of subsystems with continuous time dynamics and a rule that organizes the switching between them (Liberzon and Morse (1999)). The problem of designing switching rules for switched systems has been largely studied and several results are available in the literature (see the survey of DeCarlo et al. (2000), for instance).

Among the switching rule design techniques, some of them are based on Lyapunov functions and Linear Matrix Inequality (LMI) techniques, as for instance in Bolzern and Spinelli (2004) and Trofino et al. (2011). The interest of recasting the problem as LMIs is that it is easy to incorporate new constraints to the problem, provided that these constraints can be also expressed as LMIs, and the availability of powerful computational packages to solve the LMI problems. However, extending the results obtained for the class of linear switched systems to the class of nonlinear switched systems is a difficult task, and the design conditions for general nonlinear systems usually result in conservative LMIs. A possible way to reduce the conservatism is to consider a specific class of nonlinear functions, as for instance the class of sector-bounded functions of the state (see Khalil (2002)).

Renewable energy generation systems, such as the Photovoltaic (PV) systems, can be viewed as a nonlinear switched system due to the power electronic devices employed. For PV systems in particular, one of the biggest challenges for control is the fact that the system presents a highly nonlinear model.

This paper presents an extension of the results in Trofino et al. (2011) to the class of nonlinear switched systems in which the nonlinearities are sector bounded. A multiple Lyapunov function approach is used to design switching rules that guarantee global asymptotic stability of the switched system with convergence to a desired equilibrium point even if sliding motions occur on any switching surface of the system. An extension to the switching rule design based on partial state measurement is presented. It is also shown that, depending on the system structure, it is not necessary to know all the state vector at the desired equilibrium point *a priori* for the design of the switching rule. Motivated by the complexity of the nonlinear function present in PV generation systems, we consider this application to illustrate the design method. As the nonlinear function in PV systems also depends on uncertain parameters, a formula for determining robust sector-bounds for this system is also provided, allowing for the application of Maximum Power Point Tracking (MPPT) algorithms.

The paper is organized as follows. This section ends with the notation used in the paper. The next section presents some preliminary results and definitions. The main results for the switching rule design are presented in the Section 3. Section 4 is devoted to the application of the method to a PV system with MPPT, including numerical simulations. Some concluding remarks end the paper.

Notation. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. $\|.\|$ stands for the euclidean norm of vectors and its induced spectral norm of matrices. Block matrix terms that can be deduced from symmetry are represented by $\bullet$. 0_n and $0_{m\times n}$ are the $n \times n$ and $m \times n$ matrices of zeros. I_n is the $n \times n$ identity matrix. For a real matrix S, S' denotes its transpose and $S > 0$ ($S < 0$) means that S is symmetric and positive-definite (negative-definite). For a set of real numbers $\{v_1, \ldots, v_m\}$ we use $arg\ max\{v_1, \ldots, v_m\}$ to denote a set of indexes that is the subset of $\{1, \ldots, m\}$ associated

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with the maximum element of $\{v_1, \ldots, v_m\}$. The symbol $\otimes$ denotes the Kronecker product and $\vartheta(\theta)$ represents the set of all vertices of a given polytope θ. For a symmetric matrix M, $\lambda_{min}(M)$ denotes its minimum eigenvalue.

2. PRELIMINARIES

Consider a nonlinear switched dynamic system composed of m affine sub-systems as indicated below.

$$\dot{x}(t) = A_i x(t) + b_i + B\phi_x(q_x(x(t))), \quad i \in \mathcal{M} := \{1, \ldots, m\} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $\phi_x : \mathbb{R} \mapsto \mathbb{R}$ is a nonlinear function of the scalar $q_x(x(t)) := C_q x(t)$, $C_q \in \mathbb{R}^{1\times n}$, and $A_i \in \mathbb{R}^{n\times n}$, $b_i \in \mathbb{R}^n$, $B \in \mathbb{R}^n$ are given matrices of structure.

The problem of concern in this paper is to design a switching rule that asymptotically drives the system state to a constant reference $\overline{x}$. In other words, the desired equilibrium point $\overline{x}$ of the (closed-loop) switched system must be asymptotically stable. Given $\overline{x}$, we can represent the tracking error dynamics as a switched system with the following subsystems

$$\dot{e}(t) = A_i e(t) + A_i\overline{x} + b_i + B\phi(q(e(t))), \quad i \in \mathcal{M} \tag{2}$$

where

$$\phi(q(e(t))) := \phi_x(q(e(t)) + C_q\overline{x})) = \phi_x(q_x(x(t))), \tag{3}$$

$$e(t) := x(t) - \overline{x}\ ,\ q(e(t)) := C_q e(t). \tag{4}$$

Now consider the following decomposition of A_i, b_i.

$$A_i = \overline{A}_o + \overline{A}_i \quad , \quad b_i = \overline{b}_o + \overline{b}_i \tag{5}$$

where $\overline{A}_o, \overline{b}_o$ denote the component of A_i, b_i, respectively, that is common for all $i \in \mathcal{M}$ and $\overline{A}_i, \overline{b}_i$ contain the terms that vary according to i. Defining

$$h_o = \overline{A}_o\overline{x} + \overline{b}_o \quad , \quad h_i = \overline{A}_i\overline{x} + \overline{b}_i, \tag{6}$$

we can rewrite $A_i\overline{x} + b_i$ as $h_o + h_i$.

Assuming the sliding mode dynamics of the system can be represented as convex combinations of the subsystems as in Filippov (1988), the global switched system, that includes the subsystem dynamics and the sliding mode dynamics that may eventually occur in any switching surface, is represented by

$$\dot{e}(t) = \sum\nolimits_{i=1}^{m} \theta_i(e(t))(A_i\, e(t) + h_o + h_i + B\phi(q(e(t)))),$$
$$\theta(e(t)) \in \Theta, \tag{7}$$

where $\theta(e(t))$ is a piecewise continuous vector with entries θ_i and $\Theta := \{\theta : \sum_{i=1}^m \theta_i = 1,\ \theta_i \geq 0\}$ is the unitary simplex. A sliding motion may be occurring at a point e if there exists a convex combination of the subsystem vector fields such that $\dot{e}(t)$ is a vector that belongs to the tangent hyperplane of the switching surface at the point e.

In order to achieve the tracking objective, the origin must be an asymptotically stable equilibrium point of (7). Define $\overline{\theta} = \theta(0)$ and $\overline{\phi} = \phi(0)$. Hence, the following lemma is established.

Lemma 1. The origin is an equilibrium point of (7) iff there exists $\overline{\theta} \in \Theta$ such that

$$\sum\nolimits_{i=1}^{m} \overline{\theta}_i\ \left(h_o + h_i + B\overline{\phi}\right) = 0\,. \tag{8}$$

Proof: Set $\dot{e}(t) = 0$ and $e(t) = 0$ in (7). □

Hereafter we assume $\overline{\theta}$ is constant and the dependence of the state on time will be omitted to simplify the presentation. As (8) is a zero identity, we can subtract the left hand side of (8) to (7) and rewrite the error dynamics in the following more convenient form.

$$\dot{e} = A_\theta\, e + k_\theta \quad , \quad \theta(e) \in \Theta, \tag{9}$$

where $A_\theta = \sum_{i=1}^m \theta_i(e)A_i$ and $k_\theta = \sum_{i=1}^m \theta_i(e)k_i$ with

$$k_i = h_i - h_{\overline{\theta}} + B\Delta\phi, \tag{10}$$

where $h_{\overline{\theta}} = \sum_{i=1}^m \overline{\theta}_i h_i$ and $\Delta\phi = \phi - \overline{\phi}$.

Thanks to the error system (9) the problem of concern can be restated as to design a switching rule that asymptotically drives the error system state to the origin. For this purpose consider the switching rule given by

$$\sigma(e) := \arg\max_{i\in\mathcal{M}}\{v_i(e)\},\ \ v_i(e) = e'P_i e + 2e'(S_i - S_{\overline{\theta}}) \tag{11}$$

where $S_{\overline{\theta}} := \sum_{i=1}^m \overline{\theta}_i S_i$ and $P_i \in \mathbb{R}^{n\times n}$ and $S_i \in \mathbb{R}^{n\times 1}$ are matrices to be determined. The set valued signal $\sigma : \mathbb{R}^n \mapsto \mathcal{M}$ is a map specifying the set of subsystems having 'maximum energy'. For instance, $\sigma(e(t_0)) = \{j, k, l\}$ means that at instant $t = t_0$ the error trajectory is at the switching surface defined from the subsystems $\{j, k, l\}$ because $v_j(e(t_0)) = v_k(e(t_0)) = v_l(e(t_0)) = \max_{i\in\mathcal{M}}\{v_i(e(t_0))\}$. Whenever the set $\sigma(e)$ has more than one element, a sliding mode may be occurring at that instant and the elements of convex combination, the entries of the vector $\theta(e)$, are such that $\theta_i(e) = 0$ if $i \notin \sigma(e)$. We refer the reader to (Filippov, 1988, p.50) for details on this point.

Consider the following definition.

Definition 1. (Sector-bounded function). A function $\varphi(q)$: $\mathbb{R} \mapsto \mathbb{R}$, with $\varphi(0) = 0$, is said to be in sector $[l, u]$ if for all $q \in \mathbb{R}$, $p = \varphi(q)$ lies between $p = lq$ and $p = uq$. Then, the inequality

$$(p - uq)(p - lq) \leq 0 \tag{12}$$

holds for all $q,\ p = \varphi(q)$. □

Consider the Definition 1 with the nonlinear function $p = \Delta\phi(q)$, $q = C_q e$, and note that $\Delta\phi = 0$ for $C_q e = 0$. Therefore, it is possible to rewrite (12) as

$$-(\Delta\phi - uC_q e)(\Delta\phi - lC_q e) \geq 0. \tag{13}$$

See (Khalil, 2002, p.232) for more details on sector-bounded nonlinear functions.

3. MAIN RESULTS

Before presenting the theorem for the switching rule design, let us introduce some auxiliary notation. Let $\aleph_\theta : \mathbb{R}^m \mapsto \mathbb{R}^{r\times m}$ be a linear annihilator of θ as in Definition 1 of Trofino et al. (2011), i.e. $\aleph_\theta$ is a linear function of θ with $\aleph_\theta\theta = 0$, $\forall\theta \in \Theta$, let $\alpha_i \in \mathbb{R}$, $i \in \mathcal{M}$, be given positive scalars chosen according to Remark 1 of Trofino et al. (2011) and

$$A = [A_1\ \ldots\ A_m]\,,\ \alpha = [\alpha_1 I_n\ \ldots\ \alpha_m I_n] \tag{14}$$

$$H = [h_1\ \ldots\ h_m]\,,\ P = [P_1\ \ldots\ P_m]\,,\ S = [S_1\ \ldots\ S_m]$$

$$\mathbf{1}_m = [1\ \ldots\ 1] \in \mathbb{R}^{1\times m},\ L_a(\theta) = \sum\nolimits_{i=1}^{m} \theta_i L_i \tag{15}$$

$$C_a = \begin{bmatrix} 0_{(1\times mn)} & \mathbf{1}_m & 0 \end{bmatrix},\ C_b(\theta) = \begin{bmatrix} \aleph_\theta \otimes I_n & 0_{(rn\times m+1)} \end{bmatrix} \tag{16}$$

$$\Gamma = \begin{bmatrix} -I_o'C_q'(u\,l)C_q I_o & \bullet & \bullet \\ 0_{m\times mn} & 0_m & \bullet \\ C_q I_o(u+l)/2 & 0_{1\times m} & -1 \end{bmatrix},\ \ \begin{matrix} I_o = \mathbf{1}_m \otimes I_n \\ P_{\overline{\theta}} = \sum_{i=1}^{m} \overline{\theta}_i P_i \end{matrix} \tag{17}$$

$$\Psi = \begin{bmatrix} A'P+P'A+(P-P_{\overline{\theta}}I_o)'\alpha+\alpha'(P-P_{\overline{\theta}}I_o) & \bullet & \bullet \\ H'P+S'A+2S'\alpha & H'S+S'H & \bullet \\ B'P & B'S & 0 \end{bmatrix} \tag{18}$$

In this paper annihilators are used jointly with the Finsler's Lemma to reduce the conservativeness of parameter dependent LMIs as in Trofino and Dezuo (2013).

Theorem 1. Let $\overline{x}$ be a given constant vector, representing the desired equilibrium point of the system (1), and suppose that $x(t)$ is available online. Consider the error system (9) and let $\overline{\theta} \in \Theta$ be a given constant vector according to Lemma 1. With the auxiliary notation (14)-(18), let L_b, L_i, $i \in \mathcal{M}$, be matrices to be determined with the dimensions of $C_b(\theta)', C_a$, respectively.

Suppose that $\exists\, P, S, \tau, L_b, L_i$, $i \in \mathcal{M}$, solving the following LMI problem.

$$P_{\overline{\theta}} > 0 \tag{19}$$

$$\Psi+\tau\,\Gamma+L_bC_b(\theta)+C_b(\theta)'L_b'+L_a(\theta)C_a+C_a'L_a(\theta)' < 0, \quad \forall\theta \in \vartheta(\Theta) \tag{20}$$

Then the system (9) is globally asymptotically stable with the switching rule (11) and

$$V(e) := \max_{i\in\mathcal{M}}\{v_i(e)\}\ ,\ v_i(e) = e'P_ie + 2e'(S_i - S_{\overline{\theta}}) \tag{21}$$

is a Lyapunov function for the switched system. □

Proof: The proof consists of showing that if the LMIs (19),(20) are satisfied, then the continuous function $V(e)$ defined in (21) satisfies the conditions

$$\phi_1(e) \le V(e) \le \phi_2(e), \tag{22}$$

$$D_{\dot{e}}V(e) \le -\phi_3(e), \tag{23}$$

where $\phi_1(e)$, $\phi_2(e)$, and $\phi_3(e)$, are continuous positive definite functions and $D_hV(e)$ is the Dini's directional derivative of $V(e)$ in the direction h, and is given by (Lasdon, 1970, p.420)

$$D_hV(e) = \max_{i\in\sigma(e)} \nabla v_i(e)h, \tag{24}$$

where $\nabla v_i(e) = 2(e'P_i + S_i' - S_{\overline{\theta}}')$ denotes the gradient of $v_i(e)$. The local asymptotic stability follows from (22), (23) using the same arguments in (Filippov, 1988, p.155).

First, it will be demonstrated that the condition (22) is satisfied. As $\theta_i(e) = 0$ for $i \notin \sigma(e)$ and $V(e) = v_i(e)$, $\forall i \in \sigma(e)$, we get the identities below.

$$\sum\nolimits_{i=1}^{m} \theta_i(e) = \sum\nolimits_{i\in\sigma(e)} \theta_i(e) = 1 \tag{25}$$

$$\sum_{i=1}^{m} \theta_i(e)\,v_i(e) = \sum_{i\in\sigma(e)} \theta_i(e)\,v_i(e) = \sum_{i\in\sigma(e)} \theta_i(e)\,V(e) = V(e) \tag{26}$$

Therefore, the following is true.

$$V(e) := \max_{i\in\mathcal{M}}\{v_i(e)\} = \sum\nolimits_{i=1}^{m} \theta_i(e)\,v_i(e) \tag{27}$$

Keeping in mind that $\sum_{i=1}^{m} \overline{\theta}_iS_i = S_{\overline{\theta}}$, we get that

$$\sum\nolimits_{i=1}^{m} \overline{\theta}_i(S_i - S_{\overline{\theta}}) = S_{\overline{\theta}} - S_{\overline{\theta}} = 0 \tag{28}$$

and from (11),(17),(28) it follows that

$$\sum_{i=1}^{m} \overline{\theta}_i\,v_i(e) = e'(\sum_{i=1}^{m} \overline{\theta}_iP_i)e + 2e'\sum_{i=1}^{m} \overline{\theta}_i(S_i - S_{\overline{\theta}}) = e'P_{\overline{\theta}}e. \tag{29}$$

Note that the maximum element of a finite set of real numbers is always greater or equal to any convex combination of the elements of the set. Therefore, we can conclude from (19),(27),(29) that $\forall e \neq 0$ we have

$$V(e) \ge \sum\nolimits_{i=1}^{m} \overline{\theta}_i\,v_i(e) = e'P_{\overline{\theta}}e > 0. \tag{30}$$

Thus, $V(e)$ is positive definite and radially unbounded, because $e'P_{\overline{\theta}}e$ is a positive quadratic form in view of (19). Besides, $v_i(e) \le \beta_i(\|e\|)$ where $\beta_i(\|e\|) := \|P_i\|\|e\|^2 + 2\|S_i - S_{\overline{\theta}}\|\|e\|$. Hence, (22) is satisfied with

$$\phi_1(e) = \lambda_{min}(P_{\overline{\theta}})\|e\|^2 \quad , \quad \phi_2(e) = \max_{i\in\mathcal{M}}\{\beta_i(\|e\|)\} \tag{31}$$

where the lower and upper limits are class $\mathcal{K}_\infty$ functions.

Next, we show that the condition (23) is satisfied, i.e. $V(e)$ decreases along any system trajectory for any $\Delta\phi$ belonging to a given sector $[l, u]$. Note that for any point e and direction h, the directional derivative of $V(e)$ exists and is given by (24). In order to take all possible sliding modes into account (Filippov, 1988, p.155), let us take the directional derivative in the direction $h = \dot{e}$, where $\dot{e} = f_\theta(e)$ characterizes the switched system (9), i.e.

$$f_\theta(e) = A_\theta\, e + k_\theta. \tag{32}$$

First, consider a point e on a switching surface under sliding motion. In this case $\sigma(e)$ has more than one element and does not change on an infinitesimal increment of time $t^+ - t > 0$ and thus

$$v_i(e(t)) = V(e(t))\ ,\ \forall i \in \sigma(e(t)) = \sigma(e(t^+)), \tag{33}$$

$$\nabla v_i(e)\,f_\theta(e) = \nabla v_j(e)\,f_\theta(e) \quad , \quad \forall i, j \in \sigma(e). \tag{34}$$

While (33) represents the property of maximum energy of $v_i(e)$ associated to $\sigma(e)$, the condition (34) implies that $\sigma(e)$ does not change with an incremental step in the direction $f_\theta(e)$ and then a sliding mode involving the subsystems specified by $\sigma(e)$ is occurring. Observe that (34) can be rewritten as $\nabla v_{ij}(e)\,f_\theta(e) = 0$, where $v_{ij}(e) = v_i(e) - v_j(e)$, showing that the vector field $f_\theta(e)$ belongs to the hyperplane tangent to the switching surface given by $\mathcal{F} = \{e :\ v_{ij}(e) = 0\}$. From (24) and (34) with $h = f(e) = \dot{e}$, it follows that

$$D_{\dot{e}}V(e) = \nabla v_i(e)\,f_\theta(e) \quad , \quad \forall i \in \sigma(e). \tag{35}$$

Since $\theta(e) \in \Theta$ and $\theta_i(e) = 0$, $\forall i \notin \sigma(e)$, it follows from (34) that (35) can be rewritten as

$$D_{\dot{e}}V(e) = \sum\nolimits_{i=1}^{m} \theta_i(e)\,\nabla v_i(e)\,f_\theta(e) =: \Omega(\theta(e)). \tag{36}$$

The above expression for $D_{\dot{e}}V(e)$ was derived for the points where $\sigma(e)$ is not singleton and a sliding motion is taking place, as indicated in (33) and (34). However, (36) is still valid when $\sigma(e)$ is singleton, case in which $\theta_i(e) = 1$ for $i \in \sigma(e)$ and $\theta_i(e) = 0$ for $i \notin \sigma(e)$. The points where $\sigma(e)$ is not singleton but no sliding mode will occur from these points, i.e. (34) is not satisfied and thus the trajectory is leaving a switching surface, then the expression (36) must be satisfied $\forall\theta_i(e) \in \Theta$ with $\theta_i(e) = 0$ for $i \notin \sigma(e)$ as indicated in (24). However this type of condition is difficult to be checked as $D_{\dot{e}}V(e)$ in (36) is trajectory dependent. To overcome this difficulty the idea is to use a more conservative condition where $\theta(e)$ is replaced with an arbitrary time varying parameter θ in the unity simplex Θ. To reduce the conservativeness associated with this relaxation of the problem, we can apply the S-Procedure to the condition (23) and take into account the constraint

(30) that represents the "max" composition. Therefore, we replace (23) with the following condition.

$$\Omega(\theta) + 2\alpha_\theta \left(V(e) - e' P_{\overline{\theta}} e\right) < -\phi_3(e) \ , \ \forall \theta \in \Theta, \quad (37)$$

where $\Omega(\theta)$ is the function indicated in (36) with $\theta(e)$ replaced by an arbitrary time-varying parameter θ, $V(e)$ is indicated in (27) and $\alpha_\theta := \sum_{i=1}^m \alpha_i \theta_i > 0$ is a scaling factor with given positive constants α_i. Note that (37) implies from (36) that $\Omega(\theta(e)) = D_{\dot{e}} V(e) < -\phi_3(e)$ because $2\alpha_\theta\big(V(e) - e' P_{\overline{\theta}} e\big)$ is non-negative from (30) and $\theta(e) \in \Theta$.

Next we show that (20) implies (37) for a suitable positive definite function $\phi_3(e)$ to be specified later. Consider the notation $P_\theta := \sum_{i=1}^m \theta_i P_i$, $S_\theta := \sum_{i=1}^m \theta_i S_i$ and $S_{\theta - \overline{\theta}} := S_\theta - S_{\overline{\theta}}$. Let us rewrite the left hand-side of (37) as

$$\begin{bmatrix} e \\ 1 \end{bmatrix}' \begin{bmatrix} A'_\theta P_\theta + P_\theta A_\theta + 2\alpha_\theta (P_\theta - P_{\overline{\theta}}) & \bullet \\ k'_\theta P_\theta + S'_{\theta-\overline{\theta}} A_\theta + 2 S'_{\theta-\overline{\theta}} \alpha_\theta & k'_\theta S_{\theta-\overline{\theta}} + S'_{\theta-\overline{\theta}} k_\theta \end{bmatrix} \begin{bmatrix} e \\ 1 \end{bmatrix} < 0. \quad (38)$$

Note that $S_{\theta-\overline{\theta}} = S(\theta - \overline{\theta})$ and $k_\theta = h_\theta - h_{\overline{\theta}} + B\Delta\phi = H(\theta - \overline{\theta}) + B\Delta\phi$. Therefore, it is possible to rewrite (38) using the auxiliary notation (14)-(18) as

$$\Omega(\theta) + 2\alpha_\theta \left(V(e) - e' P_{\overline{\theta}} e\right) = \xi' \, \Psi \, \xi < 0, \quad (39)$$

$$\xi = \begin{bmatrix} e_\theta \\ \theta - \overline{\theta} \\ \Delta\phi \end{bmatrix} \ , \ e_\theta = \begin{bmatrix} \theta_1 e \\ \vdots \\ \theta_m e \end{bmatrix} \in \mathbb{R}^{mn}. \quad (40)$$

Now it is possible to incorporate to the condition (39) the fact that $\Delta\phi$ is a sector-bounded function of the error. Using the notation (17), we can rewrite (13) as

$$\xi' \Gamma \xi \geq 0. \quad (41)$$

The inequality (39) must be satisfied whenever (41) is satisfied. By using the (lossless) S-Procedure, this occurs if there exists a scalar $\tau \geq 0$ such that

$$M := \xi' (\Psi + \tau \, \Gamma) \xi < 0. \quad (42)$$

With C_a and $C_b(\theta)$ from (16), it follows that $C_a \xi = 0$ and $C_b(\theta)\xi = 0$. From the Finsler's Lemma, (42) is satisfied if there exist scaling matrices $L_b \in \mathbb{R}^{nm+m+1 \times rn}$, $L_i \in \mathbb{R}^{nm+m+1 \times 1}$, $\forall i \in \mathcal{M}$, and $L_a(\theta)$ defined in (15) such that

$$U(\theta) < 0, \quad \forall \theta \in \Theta \quad (43)$$

where

$U(\theta) := \Psi + \tau \, \Gamma + L_b C_b(\theta) + C_b(\theta)' L'_b + L_a(\theta) C_a + C'_a L_a(\theta)'$.

The expression (43) shows that if (20) is satisfied then $M < 0$ which in turn implies (39). Note that (20) implies $\tau > 0$. Now define the positive constants

$$\epsilon_0 = \min_{\theta \in \Theta} (\theta' \theta) \quad , \quad \epsilon_3 = \min_{\theta \in \Theta} \lambda_{min} \left(-U(\theta)\right). \quad (44)$$

By multiplying the inequality (20) by ξ to the right and by its transpose to the left and keeping in mind that $C_a \xi = 0$ and $C_b(\theta)\xi = 0$, we get

$$\xi' (\Psi + \tau \Gamma) \xi \leq -\epsilon_3 \|\xi\|^2. \quad (45)$$

As $\|\xi\|^2 = \|e_\theta\|^2 + \|\theta - \overline{\theta}\|^2 + \|\Delta\phi\|^2$ and $\|e_\theta\|^2 = \|\theta\|^2 \|e\|^2$, we conclude that $\|\xi\|^2 \geq \|e_\theta\|^2 \geq \epsilon_0 \|e\|^2$, which implies

$$\xi' (\Psi + \tau \Gamma) \xi \leq -\epsilon_3 \epsilon_0 \|e\|^2. \quad (46)$$

Using $\phi_3(e) = \epsilon_3 \epsilon_0 \|e\|^2$ we have shown that the LMI (20) is a sufficient condition for (42), thus for (37) whenever $\Delta\phi \in [l, u]$, and finally for (23). Thus, global asymptotic stability follows from (Filippov, 1988, p.155). □

3.1 Partial state measurement

Consider the system (1) with measurement vector $y(t) = C_i x(t) \in \mathbb{R}^{g_i}$, and $C_i \in \mathbb{R}^{g_i \times n}$, $i \in \mathcal{M}$, given matrices. Suppose the output tracking error $\varepsilon(t) := y(t) - C_i \overline{x} = C_i e$ is available online and assume that the auxiliary functions $v_i(e)$ have the structure (11) with P_i, S_i redefined as

$$P_i := P_0 + C'_i Q_i C_i \ , \quad S_i := S_0 + C'_i R_i \ , \quad i \in \mathcal{M} \quad (47)$$

where $P_0 = P'_0 \in \mathbb{R}^{n \times n}$, $S_0 \in \mathbb{R}^n$, $Q_i = Q'_i \in \mathbb{R}^{g_i \times g_i}$, $R_i \in \mathbb{R}^{g_i}$, are matrices to be determined. The Theorem 1 can be directly applied to cope with the case of partial state information by introducing the constraint (47), which makes the switching rule (11) a function of the output error (see Trofino et al. (2011) for details)

$$\sigma(\varepsilon) = \arg \max_{i \in \mathcal{M}} \{\varepsilon' Q_i \varepsilon + 2\varepsilon' R_i\}. \quad (48)$$

3.2 Design independent of the equilibrium point

In this section we show that it is possible to have the LMIs in Theorem 1 independent of the equilibrium $\overline{x}, \overline{\theta}$ if the matrices A_i and P_i have particular structures. First, note from (6),(14) that if the matrix $\overline{A}_i$, from the decomposition (5), is zero (i.e. A_i is the same for all $i \in \mathcal{M}$), then the LMIs in Theorem 1 are independent of equilibrium point $\overline{x}$.

Now consider $v_i(e)$ with the structure (11),(47) and recall that the full state feedback case is recovered with $C_i = I_n$, $\forall i \in \mathcal{M}$. It is possible to get the LMIs in Theorem 1 also independent of $\overline{\theta}$ by forcing $Q_i = 0, \forall i \in \mathcal{M}$. In this case, we have $P_\theta = \sum_{i=1}^m \theta_i P_i = P_0$ and $P_{\overline{\theta}} = \sum_{i=1}^m \overline{\theta}_i P_i = P_0$. Therefore, the LMI (19) can be replaced by $P_0 > 0$ and the term $(P - P_{\overline{\theta}} I_o)' \alpha + \alpha' (P - P_{\overline{\theta}} I_o)$ is eliminated from (20).

This allows for the Theorem 1 to be applied even if some entries of the desired operation point $\overline{x}$ are not known *a priori* or change with time. In the later case, the changes are possible provided that they can be represented by piecewise constant vectors $\bar{x}, \overline{\theta}$ varying slowly enough when compared to the system dynamics.

4. APPLICATION FOR PV SYSTEMS

In this section the following symbols are used.

Symbols. R_s - Series resistance of the PV module; R_p - Shunt resistance of the PV module; ϵ - Electron charge (1.6×10^{-19} C); η - Diode quality factor; κ - Boltzmann constant (1.38×10^{-23} J/K); T_r - Standard Test Conditions (STC) temperature (298K, i.e. 25°C); G_r - STC Irradiation (1000 W/m^2); I_{sc} - PV module short-circuit current at STC; V_{oc} - PV module open-circuit voltage at STC; γ - Temperature coefficient of I_{sc}; E_g - Band gap for silicon (1.1 eV); M_p - Number of modules in parallel; M_s - Number of modules in series; N_s - Number of PV cells in series in each module.

A PV array can be modeled as a current source, where the output current I_{pv} of the array is a nonlinear function of the voltage V_{pv} in the terminals of the array, given by

$$I_{pv} = M_p I_{ph} - M_p I_r \left(\exp\left(\frac{\epsilon}{\eta \kappa T} \left(\frac{V_{pv}}{M_s N_s} + \frac{R_s I_{pv}}{M_p} \right) \right) - 1 \right) \quad (49)$$

where

$$I_r = I_{rr} \left(\frac{T}{T_r} \right)^3 \exp\left(\frac{\epsilon E_g}{\eta \kappa} \left(\frac{1}{T_r} - \frac{1}{T} \right) \right),$$

$$I_{rr} = \frac{I_{sc} - V_{oc}/R_p}{\exp\left(\frac{\epsilon V_{oc}}{\eta \kappa T_r}\right) - 1} \quad , \quad I_{ph} = \left[I_{sc} + \gamma \left(T - T_r\right)\right] \frac{G}{G_r}.$$

where T is the operating temperature $[K]$ of the array and G is the solar irradiation $[W/m^2]$ received by the array. Note in (49) that it is not possible to isolate I_{pv} to determine its value algebraically. Moreover, (49) is a nonlinear function of the uncertain input parameters T and G.

In this paper we consider the PV array connected to a Boost converter with fixed output voltage as shown in Fig. 1. This is the case for stand-alone systems with a battery bank or grid-connected systems with a constant DC link voltage. The objective is to extract the maximum power of the array even under variations in T and G.

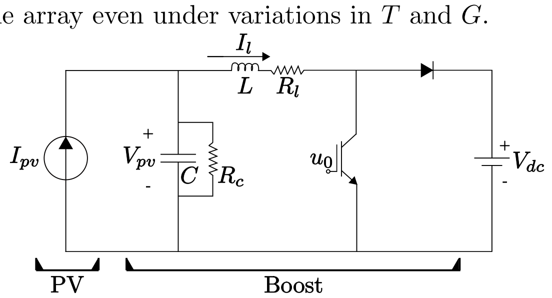

Fig. 1. Topology of PV system considered.

The PV system from Fig. 1 has only one switching device (u_0) and, therefore, it is composed of two different subsystems ($m = 2$). Consider the state vector $x = [I_l \quad V_{pv}]'$, where I_l is the electric current through the inductor L and V_{pv} is the voltage over the capacitor C. Define the nonlinear function $\phi_x(q_x(x(t))) = I_{pv}(V_{pv})$. Therefore, $q_x(x(t)) = C_q x(t) = V_{pv}$ with $C_q = [0 \ 1]$.

Considering the decomposition (5) and the definition (6), the system dynamics can be represented as in (9) with

$$\overline{A}_0 = \begin{bmatrix} -R_l/L & 1/L \\ -1/C & -1/(R_c C) \end{bmatrix}, \quad \overline{A}_1 = \overline{A}_2 = 0_2,$$
$$\overline{b}_0 = \overline{b}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \overline{b}_2 = \begin{bmatrix} -V_{dc}/L \\ 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1/C \end{bmatrix}. \tag{50}$$

Taking into account the Lemma 1, the equilibrium is defined by $\sum_{i=1}^{m} \overline{\theta}_i (h_o + h_i + B\overline{\phi}) = 0$, from which we get the following system of equations.

$$\begin{cases} \overline{I}_l = -\overline{V}_{pv}/R_c + \overline{I}_{pv} \approx \overline{I}_{pv} \\ \overline{V}_{pv} = R_l \overline{I}_{pv} + \overline{\theta}_2 V_{dc} \end{cases} \tag{51}$$

The approximation in (51) takes into account that $1/R_c$ (the dielectric conductance of C) has very small values (around nΩ^{-1}). Therefore, in practice, the influence of the term $-\overline{V}_{pv}/R_c$ is negligible when compared to $\overline{I}_{pv}$.

Consider the objective of maximizing the power extracted from the PV array and note that we have $\overline{x} = [\overline{I}_{pv} \ \overline{V}_{pv}]'$, according to (51). However, the values of $\overline{I}_{pv}$ and $\overline{V}_{pv}$ are coupled by the nonlinear equation (49), which depends on the uncertain parameters T and G, not known in real time. It means that if $\overline{I}_{pv}$ is fixed, it is not possible to calculate $\overline{V}_{pv}$, and vice versa.

To overcome this difficulty we design the switching rule based on output feedback (Section 3.1), which requires only one of the references in real time. In the case presented in this paper, the output considered is the current state x_1. Therefore, $y(t) = C_i x(t)$ with $C_i = [1 \ 0]$, $\forall i \in \mathcal{M}$.

In order to perform the MPPT, we consider the value of $\overline{I}_{pv}$ as the output of an MPPT algorithm, such as the simple Perturb and Observe (P&O) algorithm. See Tan et al. (2005) for details on the P&O algorithm considered. The algorithm will perform changes in the value of $\overline{I}_{pv}$ to get as close as possible to the maximum power point even in case of changes in T and G. According to the Section 3.2, $\overline{I}_{pv}$ is allowed to change as a slowly varying piecewise constant function.

According to (3), in the system representation (9) we have $\Delta\phi = I_{pv}(q + \overline{V}_{pv}) - \overline{I}_{pv}$, where $q = C_q e = V_{pv} - \overline{V}_{pv}$ and $\overline{I}_{pv} = I_{pv}(\overline{V}_{pv})$. Note that $\Delta\phi = 0$ for $q = 0$, therefore we are able to represent $\Delta\phi$ as a sector-bounded function of q.

Fig. 2 shows the I-V characteristic curve of the array, obtained by plotting (49) for fixed values of T and G. Fig. 2 also shows the axis for the sector-bounded function $\Delta\phi$ as a function of q for given values of $\overline{I}_{pv}$ and $\overline{V}_{pv}$ as well as sector-bounding lines satisfying the sector condition.

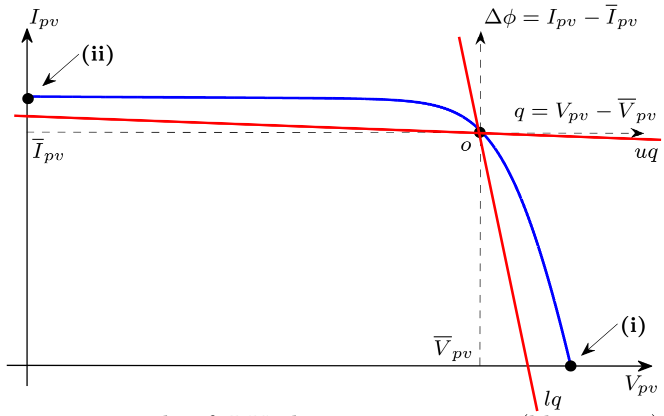

Fig. 2. Example of I-V characteristic curve (blue curve) and sector bounds (red lines).

Note that the equilibrium point $(\overline{I}_{pv}, \overline{V}_{pv})$ can be any point on the I-V characteristic curve. In order to find sector-bounds $[l, u]$ for the curve $\Delta\phi(q)$ for any value of $(\overline{I}_{pv}, \overline{V}_{pv})$ we must consider the following two worst case scenarios. **(i)** When $\overline{I}_{pv} = 0$ all the points of the curve are located in the second quadrant and the curve is limited above by a line with slope $l = \frac{d\Delta\phi}{dq}$ evaluated at the origin ($\Delta\phi = q = 0$ with $\overline{I}_{pv} = 0$). The value of $\frac{d\Delta\phi}{dq}$ at this point characterizes the most negative slope of the curve $\Delta\phi(q)$, as it can be seen in Fig. 2. **(ii)** Analogously, when $\overline{V}_{pv} = 0$, $\frac{d\Delta\phi}{dq}$ at this point characterizes the least negative slope of the curve $\Delta\phi(q)$.

However, the slopes of the curve $\Delta\phi(q)$ in relation to q for the worst cases are not known *a priori* because they depend on the uncertain parameters T and G, as shown in the sequence. Note that

$$\frac{d\Delta\phi}{dq} = \frac{dI_{pv}}{dV_{pv}} = -\frac{M_p}{M_s N_s R_s \left(1 + f(T, G, V_{pv}, I_{pv})\right)}, \tag{52}$$

$$f(T,G,V_{pv},I_{pv}) :=$$
$$\eta\kappa T / \left(\epsilon R_s I_r \left(\exp\left(\frac{\epsilon}{\eta\kappa T}\left(\frac{V_{pv}}{M_s N_s} + \frac{R_s I_{pv}}{M_p}\right)\right) - 1\right)\right). \quad (53)$$

In order to find a robust sector $[l,u]$, note that f is always positive, independently of the values of (T,G,V_{pv},I_{pv}). Thus, the most negative (l) and the least negative (u) values of $\frac{d\Delta\phi}{dq}$ can be extracted from (52) as follows.

$$l = \lim_{f\to 0} \frac{d\Delta\phi}{dq} = -\frac{M_p}{M_s N_s R_s} \quad , \quad u = \lim_{f\to\infty} \frac{d\Delta\phi}{dq} = 0 \quad (54)$$

Therefore, the sector $[l,u]$ given by (54) is robust in relation to T and G (it depends only on constant parameters of the system) and it is guaranteed to contain $\Delta\phi(q)$ for any reference $(\overline{I}_{pv}, \overline{V}_{pv})$ because these bounds contemplate the worst case scenarios **(i)** and **(ii)**.

4.1 Numerical simulations

In the sequel we have used SeDuMi with Yalmip interface (Löfberg (2004)) to solve the LMIs and Simulink to perform the simulation of the switched system.

Consider a PV array consisting of 20 Kyocera's KC200GT modules (arranged with $M_p = 2$, $M_s = 10$) connected to a Boost converter with fixed output voltage as shown in Fig. 1. The system parameters considered are shown in Table 1.

Table 1. System parameters

PV module parameters	Value
V_{oc}	32.9V
I_{sc}	8.21A
γ	3.18×10^{-3}A/°C
η	1.2
R_s	5mΩ
R_p	7Ω
N_s	54

Circuit parameters	Value
C	100μF
R_c	1GΩ
L	50mH
R_l	10mΩ
V_{dc}	350V

Consider the sector $[l,u] = [-0.7407, 0]$ obtained with (54) and the matrices P_i, S_i defined as in (47) with $Q_i = 0$ (according to Section 3.2) and $C_i = [1\ 0], \forall i \in \mathcal{M}$. The LMIs of Theorem 1 are satisfied and as a result we obtain the matrices R_i, $i \in \mathcal{M}$, used to compute the switching rule (48).

The simulation is initiated with $T = 10$°C and $G = 1000$ W/m^2 and zero initial conditions for the states. To show the robustness of the technique with respect to variations on these input parameters, T is changed to 25°C in $t = 0.3$s, and G is changed to 1200 W/m^2 in $t = 0.4$s.

Fig. 3(a) presents the power generated by the array (P_{pv}). Note that the Maximum Power Point (MPP) is achieved for all the different values of T and G. Fig. 3(b) shows that the curve of I_{pv} converges to $\overline{I}_{pv}$ (discretized output of an P&O algorithm) due to the convergence of the state I_l to $\overline{I}_{pv}$ as in (51). Fig. 3(c) shows the convergence of the nonmeasured state V_{pv} to its reference (not known *a priori*). □

5. CONCLUDING REMARKS

The switching rule design technique proposed in this paper can be extended in many directions, as to include performance requirements such as guaranteed cost and H_∞ attenuation, for instance. The extension for grid-connected PV systems will be presented in a future work. In the last case, a potential difficulty is that the system model contains time-based nonlinearities (sinusoids), which are not sector-bounded functions of the states.

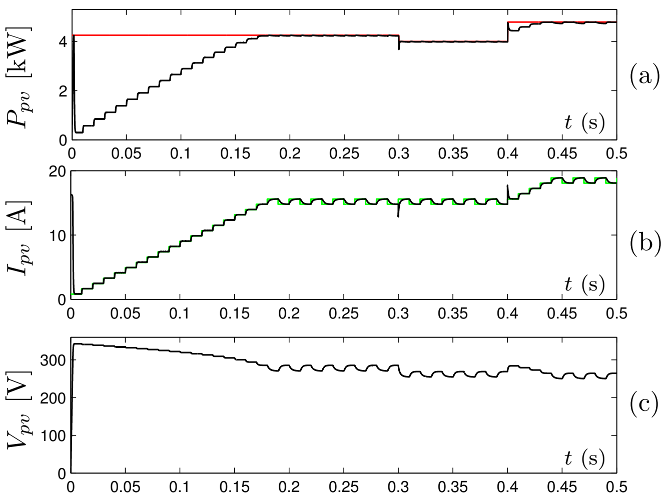

Fig. 3. (a) $P_{pv} = V_{pv}I_{pv}$ (black curve) and the MPP for each values of T and G (red lines). (b) I_{pv} (black curve) and its reference $\overline{I}_{pv}$ (green lines). (c) V_{pv}.

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