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The Synthesis of Invariant Systems on the Base of The Vortex Algorithm ⋆

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Abstract: The solution of the invariance problem for arbitrary external perturbations is proposed for the system output control. It is assumed that perturbations are not measured, and acts through another channels than control inputs (so called unmatched perturbation). The nonlinear control algorithms are developed on the base of discontinuous function. The main idea consists in using relay linearization effect with the help of self damping oscillations of the system output. In such operation mode the theoretically infinite linearization coefficient can be realized, and asymptotical invariance of the system output is provided. The simulation results show the efficiency of the designed algorithms.

1. INTRODUCTION

The theory of invariance is a major field of control theory. At present, there are the following formulations of invariance problems and methods for their solution as applied to linear models of control systems: complete invariance, invariance with prescribed accuracy, and asymptotic invariance. Let us consider each of them in more detail. Complete invariance means that the transition process with respect to output variables is independent of disturbances. In the class of linear controls, the complete invariance problem is reduced to finding a feedback control such that the controllability space of the closed loop system with respect to disturbances belongs to the kernel of the output mapping. In this formulation, the problem was first considered in Schipanov [1956] and was solved (with necessary and sufficient conditions stated) by applying the geometric approach Wonham [1979]. The class of completely invariant systems is expanded if disturbances can be measured. In this case, those of the disturbances that belong to the control space (matched disturbances) Drazenovic [1969] can be compensated for by applying a combination control. Next, the complete invariance problem is solved only for the uncompensated disturbances. By applying the theory of systems with discontinuous controls, this problem is solved under the assumption that the disturbances are non-measurable magnitude bounded functions; more specifically, enforced sliding modes are applied that are invariant under matched disturbances V.A Utkin [2001], V.I Utkin [2009]. Note that a finite time is required for producing sliding modes, after which complete invariance is achieved. Invariance with prescribed accuracy, or ε-invariance Rozonoer [1963] guarantees that the transition process with respect to output variables differs from the undisturbed motion by a prescribed value. This formulation makes use of systems with deep feedback V.A Utkin [2001], in which ε-invariance is provided with increasing feedback factors under the assumption that the disturbances are magnitude bounded functions that belong to the control space. In the case of unmatched disturbances, the class of ε-invariant systems is expanded by applying the block approach V.A. Utkin et al. Drakunov, Izosimov, Luk'yanov, V.I. Utkin [1990] due to a hierarchical choice of local continuous feedbacks. Asymptotic invariance means that the output variables (or their residuals with respect to given values) tend to zero as the time in the transition process tends to infinity irrespective of the disturbances. In asymptotic invariance problems, it is assumed that disturbances are generated by a known dynamic model (model disturbances) with unknown initial conditions. This formulation deals with an extended model of a control system (control system + disturbance model) and the problem is to stabilize the output variables by applying the theory of asymptotic observers or the dynamic compensation method Wonham [1979]. In both cases, the invariance problem is reduced directly or indirectly to the synthesis of a feedback control with the use of estimated components of external disturbances. The well known difficulties in synthesis related to an increase in the dimension of the extended system can be resolved by applying the block approach, in which case the synthesis problem is decomposed into independently solved subproblems of lower dimension V.A Utkin [2001]. Let us note some aspects concerning the use of the above results of invariance theory in practice. The possibility of providing complete invariance is strictly limited by the structural properties of the control system model. Specifically, even if the complete invariance problem has been solved, one needs to satisfy the engineering requirements for the operation of the closed loop system, for example, its stability. In asymptotic invariance problems, the most vulnerable assumption from a practical point of view is that of model disturbances. More important in practice is the ε-invariance problem: the class of such systems is wider than in complete invariance problems and the assumption

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of magnitude-bounded disturbances is more realistic than in the other formulations. Moreover, the formulation of the ε-invariance problem frequently satisfies the imposed engineering requirements. Note that the invariance problem as solved with respect to the whole state vector, which was frequently addressed, especially at an earlier stage of the invariance theory development, is impractical and has a solution only in the case of matched disturbances. In this paper, a new approach to the synthesis of invariant systems is described that is based on nonlinear oscillation modes created in the closed-loop system by applying bang-bang controls in the case of unmatched disturbances of a broad class. In contrast to the synthesis of invariant systems based on sliding modes, where the motion in a sliding mode is described by linear equations, in the proposed approach, the capabilities of nonlinear control theory are used to a full extent. The properties of nonlinear oscillations are frequently employed in control theory, for example, in algorithm with second order sliding modes Levant [1993, 2003], which ensure that second order systems with matched disturbances converge in a finite time. At a physical level, the basic idea of this work is based on the vibrolinearization of relay characteristics by applying a high frequency signal to the relay input. It is well known Mossaheb [1983] that the vibrolinearization coefficient of the relay increases unlimitedly with decreasing amplitude of the high-frequency signal. Based on this fact, damped eigenoscillations with respect to the output coordinate can be generated in the closed-loop system by applying a bang-bang control and, as a consequence, an infinitely large vibrolinearization coefficient can be obtained. As a result, the asymptotic invariance problem for output variables can be solved in the case of unmatched disturbances from a broad class. Note that the unlimited growth of the vibrolinearization coefficient is then caused not by applying an external high-frequency signal Iannelli [2006], Kochetkov [2010], but rather by damped high-frequency eigenoscillations in the closed-loop system.

2. DISCUSSION OF THE PROBLEM

Consider the invariance problem with respect to the output for the second order linear system

$$\begin{aligned} \dot{z}_1 &= z_2 + f_1(t), \\ \dot{z}_2 &= u + f_2(t), \end{aligned} \tag{1}$$

where $f_1(t),\ f_2(t)$ are external disturbances, $|f_1(t)| \leq F_1 = \text{const} > 0, |f_2(t)| \leq F_2 = \text{const} > 0, |\cdot|$ hereafter denotes the absolute value of a number, $y = z_1$ is the output of the system, and u is a scalar control. It is assumed that the variables z_1 and z_2 can be measured.

A given control accuracy can be ensured by applying a linear control law. Consider a step-by-step pro cedure for choosing feedback factors as based on the block approach V.A. Utkin et al. Drakunov, Izosimov, Luk'yanov, V.I. Utkin [1990]. At the first step, we introduce the new variable $\overline{z}_2 = l_1 z_1 + z_2$, where $l_1 = \text{const} > 0$. In new coordinates, the equations of the system are rewritten as

$$\begin{aligned} \dot{z}_1 &= -l_1 z_1 + \overline{z}_2 + f_1(t), \\ \dot{\overline{z}}_2 &= -l_1^2 z_1 + l_1 \overline{z}_2 + u + l_1 f_1 + f_2(t). \end{aligned}$$

At the second step, we choose a control function of the form

$$u = l_1^2 z_1 - (l_1 + l_2)\overline{z}_2 = -l_1 l_2 z_1 - (l_1 + l_2) z_2, \tag{2}$$

where $l_2 = \text{const} > 0$.

The equations of the closed-loop system are

$$\begin{aligned} \dot{z}_1 &= -l_1 z_1 + \overline{z}_2 + f_1(t), \\ \dot{\overline{z}}_2 &= -l_2 \overline{z}_2 + l_1 f_1 + f_2(t). \end{aligned}$$

The given accuracy with respect to the external disturbances $f_1(t)$ and $f_2(t)$ can be achieved by choosing control parameters l_1 and l_2 satisfying the inequality

$$|z_1| \leq F_1 \left(\frac{1}{l_1} + \frac{1}{l_2} \right) + \frac{F_2}{l_1 l_2}. \tag{3}$$

This synthesis procedure can be extended to the case of an n-dimensional system V.A Utkin [2001]. Note that the control function given by (2) determine two real roots of the characteristic equation of the closed-loop system, and, in the general case, the prescribed accuracy is achieved by choosing two amplification factor l_1 and l_2.

Consider a different method for choosing a control function that leads to complex roots of the characteristic equation:

$$u = -\omega^2 z_1 - \alpha z_2, \quad \omega^2 > \frac{\alpha^2}{4}, \tag{4}$$

where $\omega, \alpha = \text{const} > 0$.

The equations of closed-loop system (1), (4) are

$$\begin{aligned} \dot{z}_1 &= z_2 + f_1(t), \\ \dot{z}_2 &= -\omega^2 z_1 - \alpha z_2 + f_2(t), \end{aligned} \tag{5}$$

As $t \to \infty$ the variables of system (5) tend to a neighborhood of the origin whose size is determined by the inequalities

$$|z_1(t)| \leq \frac{\alpha F_1}{\omega^2} + \frac{F_2}{\omega^2}, \quad |z_2(t)| \leq F_1. \tag{6}$$

In contrast to algorithm (2), in this system, any prescribed accuracy of the control with respect to the output variable can be achieved by choosing the only parameter ω.

Now consider the problem of providing invariance with the help of a nonlinear oscillator of the form

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= -M_1 \text{sign}(z_1), \end{aligned}$$

where $M_1 = \text{const} > 0$.

This system satisfies the energy conservation law

$$E = M_1|z_1| + \frac{z_2^2}{2} = M_1|z_1(0)| + \frac{z_2^2(0)}{2} = \text{const} > 0.$$

However, in contrast to a linear oscillator, the cyclic oscillation frequency of the system depends on the initial conditions (oscillation energy)

$$\omega = \frac{\pi M_1}{2\sqrt{E}}. \tag{7}$$

It can be seen that the lower the energy, the higher the oscillation frequency. Let us consider the so-called "twisting" algorithm Levant [1993] according to which the control of system (1) is taken in the form

$$u = -M_1 \text{sign}(z_1) - M_2 \text{sign}(z_2), \tag{8}$$

where $M_1 = \text{const} > 0,\ M_2 = \text{const} > 0,\ M_1 > M_2 > F_2$.

One can determine the equations of the closed-loop system:

$$\begin{aligned} \dot{z}_1 &= z_2 + f_1(t), \\ \dot{z}_2 &= -M_1 \mathrm{sign}(z_1) - M_2 \mathrm{sign}(z_2) + f_2(t). \end{aligned} \tag{9}$$

As was shown in V.I Utkin [2009], Levant [1993], the energy of nonlinear oscillations for $f_1(t) = 0$ damps in a finite time. According to (6) and (7), this leads to infinite frequency of switchings in the stable mode and, therefore, to full invariance of the variables z_1, z_2 with respect to $f_2(t)$. We note that in contrast to the linear oscillator this mode of the closed-loop system arises under bounded control. In Kochetkov, V.A. Utkin [2013] was shown that, only the output ε-invariance can be provided in the closed-loop system (9). This fact constrains practical application of this approach and the new approach for providing invariance is developed in the paper.

3. PROBLEM STATEMENT

Consider a linear time-invariant system represented in the regular form V.A. Utkin et al. Drakunov, Izosimov, Luk'yanov, V.I. Utkin [1990]

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{10}x_0 + Q_1 f(t), \\ \dot{x}_0 &= A_{01}x_1 + A_{00}x_0 + Q_0 f(t) + Bu, \end{aligned} \qquad y = Dx_1, \tag{10}$$

where $x_1 \in \mathbb{R}^{n-p}, x_0 \in \mathbb{R}^p$, $u \in \mathbb{R}^p$ is the vector of control inputs, $f(t) \in \mathbb{R}^q$ is the vector of external disturbances, $y \in \mathbb{R}^m$ is the output of the system, $\mathrm{rank}\, B = \dim\, x_0$, $\mathrm{rank} D = m$, $1 \le m \le n-p$, the pair $\{A_{11}, A_{10}\}$ is controllable, and the whole state space vector can be measured.

Assume that the external disturbance $f(t)$ is restricted by the condition

$$\mathrm{Im}\, Q_1 \subset \mathrm{Im}\, A_{10}, \tag{11}$$

where $\mathrm{Im}(\cdot)$ denotes the image of a matrix.

Condition (11) means that there exists a matrix Λ, such that

$$Q_1 = A_{10}\Lambda. \tag{12}$$

The class of external disturbances is restricted by the inequalities

$$|f_i(t)| \le F_i,\ |\dot{f}_i(t)| \le \overline{F}_i,\ |\ddot{f}_i(t)| \le \widetilde{F}_i,\ i = \overline{1,q}, \tag{13}$$

where $f_i(t) - i$is the ith component of the external disturbance vector, $F_i, \overline{F}_i,\ \widetilde{F}_i$ are constants.

The problem is to provide the asymptotic invariance of the output variables with respect to the external disturbance:

$$\lim_{t\to\infty} |y_i(t)| = 0,\ i = \overline{1,m},$$

where $y_i(t)$ is the ith component of the vector $y(t)$.

4. SYNTHESIS OF A CONTROL ALGORITHM

The procedure for designing a control algorithm is based on the block approach V.A. Utkin et al. Drakunov, Izosimov, Luk'yanov, V.I. Utkin [1990], which is, in fact, a step-by-step procedure. The first step of deriving a block form for system (10) is described as follows. We introduce the notation

$$\mathrm{rank}\, A_{10} = p_1 \le p.$$

It is well known that, by applying a nonsingular coordinate transformation $\overline{x}_1 = T_1 x_1$ and taking into account (12), system (10) can be represented in the form

$$\begin{aligned} \dot{x}_2 &= A_{22}x_2 + A_{21}\widetilde{x}_1, \\ \dot{\widetilde{x}}_1 &= A_{12}x_2 + \widetilde{A}_{11}\widetilde{x}_1 + \overline{A}_{10}x_0 + \overline{Q}_1 f(t), \\ \dot{x}_0 &= A_{02}x_2 + \widetilde{A}_{01}\widetilde{x}_1 + A_{00}x_0 + Q_0 f(t) + Bu, \end{aligned} \tag{14}$$

where $y = DT_1^{-1}\overline{x}_1$ is the output of the system, $\overline{x}_1 = (x_2, \widetilde{x}_1)^{\mathrm{T}}$, $\dim \widetilde{x}_1 = \mathrm{rank}\overline{A}_{10} = p_1$, $\dim x_2 = p_2$.

At the next steps, the first equation is divided in a similar manner into two subsystems such that the dimension of the lower subsystem coincides, as at the first step, with the rank of the matrix multiplying the variable of the next block. After a block controllability form has been derived by applying the block approach, since the pair $\{A_{11}, A_{10}\}$ is controllable (and, hence, so is the pair $\{A_{22}, A_{21}\}$), we can choose a nonsingular coordinate transformation

$$\widetilde{x}_1 = s_1 + Cx_2,$$

such that Eq. (14) is rewritten as

$$\begin{aligned} \dot{x}_2 &= \overline{A}_{22}x_2 + A_{21}s_1, \\ \dot{s}_1 &= \widetilde{A}_{12}x_2 + \overline{A}_{11}s_1 + \overline{A}_{10}x_0 + \overline{Q}_1 f(t), \\ \dot{x}_0 &= \widetilde{A}_{02}x_2 + \widetilde{A}_{01}s_1 + A_{00}x_0 + Q_0 f(t) + Bu, \\ &\quad y = \overline{D}\begin{pmatrix} x_2 \\ s_1 \end{pmatrix} \end{aligned}$$

where the matrix $\overline{A}_{22} = A_{22} + A_{21}C$ has a characteristic equation with the desirable spectrum of roots, $\widetilde{A}_{12} = A_{12} - C(A_{22} + A_{21}C) + \widetilde{A}_{11}C$, $\widetilde{A}_{02} = A_{02} + \widetilde{A}_{01}C$, $\overline{A}_{11} = \widetilde{A}_{11} - CA_{21}$, $\overline{D} = DT_1^{-1}\begin{pmatrix} I_{p_2} & 0 \\ C & I_{p_1} \end{pmatrix}$, I_{p_1}, I_{p_2} are identity matrices of sizes p_1 and p_2.

Introducing the new variables

$$s_2 = \widetilde{A}_{12}x_2 + \overline{A}_{11}s_1 + \overline{A}_{10}x_0,\quad \overline{x}_0 = \overline{T}_2 x_0,$$

where $\mathrm{rank}\, \overline{T}_2 = p - p_1$, $\mathrm{rank}\, \left(\overline{A}_{10}^{\mathrm{T}}, \overline{T}_2^{\mathrm{T}}\right)^{\mathrm{T}} = p$, we can rewrite the last system in new coordinates as

$$\begin{aligned} \dot{x}_2 &= \overline{A}_{22}x_2 + A_{21}s_1, \\ \dot{s}_1 &= s_2 + \overline{Q}_1 f(t), \\ \dot{s}_2 &= A_2x_2 + A_{s_1}s_1 + A_{s_2}s_2 + \overline{A}_{20}\overline{x}_0 + \\ &\quad + \overline{Q}_2 f(t) + B_2 u, \\ \dot{\overline{x}}_0 &= \overline{A}_{02}x_2 + \overline{A}_{01}s_1 + \overline{A}_{02}s_2 + \overline{A}_{00}\overline{x}_0 + \\ &\quad + \overline{Q}_0 f(t) + B_0 u, \\ &\quad y = \overline{D}\begin{pmatrix} x_2 \\ s_1 \end{pmatrix} \end{aligned} \tag{15}$$

where $\dim s_2 = \mathrm{rank}\, B_2 = p_1$, $\dim \overline{x}_0 = \mathrm{rank}\, B_0 = p - p_1$, $Re\, \lambda_i(\overline{A}_{22}) < 0$, $i = \overline{1,p_2}$, $\lambda_i(\overline{A}_{22})$ are the eigenvalues of the matrix $\overline{A}_{22}$. In view of $\mathrm{rank}\, B = p$ according to the basic idea of this work, the control functions are specified as

$$\begin{pmatrix} B_2 \\ B_0 \end{pmatrix} u = \begin{bmatrix} -A_2x_2 - (A_{s_1} + L_\beta)s_1 - A_{s_2}s_2 - \\ -\overline{A}_{20}\overline{x}_0 - M\mathrm{sign}(s_1) \\ \\ -L_\alpha)s_2 - \overline{A}_{02}x_2 - \overline{A}_{01}s_1 - \overline{A}_{00}s_2 - \\ -\overline{A}_{00}\overline{x}_0 - H\mathrm{sign}(\overline{x}_0) \end{bmatrix}, \tag{16}$$

where $\mathrm{sign}(s_1) = [\mathrm{sign}(s_{1i}), ..., \mathrm{sign}(s_{1p_1})]^{\mathrm{T}}$, $\mathrm{sign}(\cdot)$ is the sign function, s_{1i} is the ith component of the vector s_1, $L_\alpha = \mathrm{diag}\,\{\alpha_i + \beta_i\}$, $L_\beta = \mathrm{diag}\,\{\alpha_i\beta_i\}$, $\alpha_i = \mathrm{const} > 0$, $\beta_i = \mathrm{const} > 0$, $M = \mathrm{diag}\{M_i\}$, $M_i = \mathrm{const} > 0$, $i = \overline{1,p_1}$, $H = \mathrm{diag}\{H_j\}$, $H_j = \mathrm{const} > 0$, $j = \overline{1,p-p_1}$

Define

$$\begin{gathered}\overline{s}_2 = s_2 + \overline{Q}_1 f(t),\\ \xi(t) = \overline{Q}_1 \dot{f}(t) + L_\alpha \overline{Q}_1 f(t) + \overline{Q}_2 f(t),\\ \xi_0(t) = \overline{Q}_0 f(t).\end{gathered} \tag{17}$$

According to constraints (13) imposed on the class

$$\begin{gathered}|\xi_i(t)| \le \Sigma_i,\ |\dot{\xi}_i(t)| \le \overline{\Sigma}_i,\ |\xi_{0j}(t)| \le \Sigma_{0j},\\ i = \overline{1, p_1},\ j = \overline{1, p - p_1},\end{gathered} \tag{18}$$

where $\xi_i(t)$ is the ith component of the vector $\xi(t)$, $\xi_{0j}(t)$ is the jth component of the vector $\xi_0(t)$ and the quantities $\Sigma_i = \text{const} > 0$, $\overline{\Sigma}_i = \text{const} > 0$ and $\Sigma_{0j} = \text{const} > 0$ are calculated according to (13) and (17).

Theorem 1. Let the following conditions hold in system (15), (16)
(i) The external disturbance vector $\xi(t)$ satisfies constraints (18).
(ii) The elements of the matrices L_α, L_β, M and$H$ in (16) are chosen according to the expressions

$$\begin{gathered}M_i > \Sigma_i,\ \ \alpha_i(M_i - \Sigma_i) > \overline{\Sigma}_i,\ \beta_i = \frac{\overline{\alpha}_i}{2(1 + \frac{\Sigma_i}{M_i})},\\ \overline{\alpha}_i = \alpha_i\left(1 - \frac{\Sigma_i}{M_i}\right) - \frac{\overline{\Sigma}_i}{M_i},\ H_i > \Sigma_{0i}.\end{gathered}$$

Then the output $y = \overline{D}\left(x_2^{\mathrm{T}}\ s_1^{\mathrm{T}}\right)^{\mathrm{T}}$ of closed-loop system (15), (16) tends exponentially to zero independently of the external disturbances.

Proof. By substituting the control action (16) in system (15) and taking into consideration notation (17), we establish that

$$\begin{aligned}\dot{x}_2 &= \overline{A}_{22}x_2 + A_{21}s_1,\\ \dot{s}_1 &= \overline{s}_2,\\ \dot{\overline{s}}_2 &= -L_\beta s_1 - L_\alpha \overline{s}_2 - M\text{sign}(s_1) + \xi(t),\\ \dot{\overline{x}}_0 &= \overline{Q}_0 f(t) - H\text{sign}(\overline{x}_0),\end{aligned} \quad y = \overline{D}\begin{pmatrix}x_2\\ s_1\end{pmatrix}.$$

We notice that the last subsystem is feedback autonomous and does not affect the output of system 15), (16). In virtue of the theorems conditions, the sliding mode over the manifold $\overline{x}_0 = 0$ arises in the subsystem during a finite time V.I Utkin [2009], and in what follows the subsystem is disregarded in the proof of exponential convergence.

Let us consider composite Lyapunov function

$$V = \sum_{i=1}^{p_1} V_i, \quad V_i = |s_{1i}| - \frac{\xi_i(t)}{M_i}s_{1i} + \frac{(\alpha_i s_{1i} + \overline{s}_{2i})^2}{2M_i}. \tag{19}$$

In virtue of these equations, we obtain by differentiating the ith component of the Lyapunov function (19) that

$$\begin{aligned}\dot{V}_i = -\alpha_i s_{1i}&\left[\text{sign}(s_{1i}) - \frac{\xi_i(t)}{M_i}\right] - \frac{\dot{\xi}_i(t)}{M_i}s_{1i} -\\ &-\frac{\beta_i}{M_i}\left(\alpha_i s_{1i} + \overline{s}_{2i}\right)^2 \le\\ &\le -\overline{\alpha}_i\left[|s_{1i}| + \frac{\beta_i}{\overline{\alpha}_i M_i}\left(\alpha_i s_{1i} + \overline{s}_{2i}\right)^2\right].\end{aligned}$$

Taking into account constraints (18) on the class of perturbations, the inequality from (19) is given by

$$\left(1 + \frac{\Sigma_i}{M_i}\right)|s_{1i}| + \frac{\left(\alpha_i s_{1i} + \overline{s}_{2i}\right)^2}{2M_i} \ge V_i.$$

From the last inequalities we establish the constraint on the derivative of the ith component of the Lyapunov function

$$\dot{V}_i(t) \le -\gamma_i V_i(t), \qquad \gamma_i = \frac{\overline{\alpha}_i}{1 + \frac{\Sigma_i}{M_i}}.$$

The exponential convergence of the output of the closed-loop system (15), (16) to zero $\forall t \ge t_0$ follows from the fact that the vector $s_1(t)$ converges exponentially and $\overline{A}_{22}$ is a Hurwitz matrix, which proves the theorem.

The following result shows that the system with the control given by

$$\begin{pmatrix}B_2\\ B_0\end{pmatrix}u = \begin{bmatrix}-A_2x_2 - A_{s_1}s_1 - (A_{s_2} + L_\alpha)s_2 -\\ -\overline{A}_{20}\overline{x}_0 - M\text{sign}(s_1)\\ \\ -\overline{A}_{02}x_2 - \overline{A}_{01}s_1 - \overline{A}_{00}s_2 -\\ -\overline{A}_{00}\overline{x}_0 - H\text{sign}(\overline{x}_0)\end{bmatrix}, \tag{20}$$

$$L_\alpha = \text{diag}\{\alpha_i\},\ \alpha_i = \text{const} > 0,\ i = \overline{1, p_1},$$

which differs from (16) in that damping is achieved only with the help of the vector s_2, is asymptotically invariant under disturbances from the same class (18). In view of (17) and (20), the equations of the closed-loop system become

$$\begin{aligned}\dot{x}_2 &= \overline{A}_{22}x_2 + A_{21}s_1,\\ \dot{s}_1 &= \overline{s}_2,\\ \dot{\overline{s}}_2 &= -L_\alpha \overline{s}_2 - M\text{sign}(s_1) + \xi(t),\\ \dot{\overline{x}}_0 &= \overline{Q}_0 f(t) - H\text{sign}(\overline{x}_0),\end{aligned} \quad y = \overline{D}\begin{pmatrix}x_2\\ s_1\end{pmatrix}. \tag{21}$$

Theorem 2. Let the following conditions hold in system (15), (20):
(i) The external disturbance vector $\xi(t)$ satisfies constraints (18).
(ii) The elements of the matrices L_α, M, H in (20) are chosen according to the inequalities

$$M_i > \Sigma_i,\ \ \alpha_i(M_i - \Sigma_i) > \overline{\Sigma}_i,\ H_i > \Sigma_{0i}.$$

Then the following assertions are valid:
(i) The output of the closed-loop system (21) is asymptotically invariant under the external disturbance
(ii) There exists a finite time t_r and constants $S_{1i},\ S_{2i}$ and $\overline{\gamma}_i$ such that, for $t \ge t_r$ the output of closed-loop system (21) is exponentially stable and the components $s_1(t)$ and $\overline{s}_2(t)$ of the vectors $s_1(t)$ and $\overline{s}_2(t)$ satisfy the estimates

$$|s_{1i}(t)| \le S_{1i}e^{-\overline{\gamma}_i(t-t_r)}, |\overline{s}_{2i}(t)| \le S_{2i}e^{-\frac{\overline{\gamma}_i}{2}(t-t_r)}, i = \overline{1, p_1}.$$

Proof. To prove assertion (i) of Theorem 2, we consider the composite Lyapunov function (19) whose derivative of the ith component in virtue of the equation of system (21) is given by

$$\begin{aligned}\dot{V}_i = -\alpha_i s_{1i}&\left[\text{sign}(s_{1i}) - \frac{\xi_i(t)}{M_i}\right] - \frac{\dot{\xi}_i(t)}{M_i}s_{1i} \le\\ &\le -\overline{\alpha}_i|s_{1i}|,\end{aligned} \tag{22}$$

where $\overline{\alpha}_i = \alpha_i\left(1 - \frac{\Sigma_i}{M_i}\right) - \frac{\overline{\Sigma}_i}{M_i}$.
According to the conditions of Theorem 2, we obtain that the derivative of the Lyapunov function is nonpositive $\forall\, t \ge t_0$.

The proof of assertion (ii) can be found in Kochetkov, V.A. Utkin [2013].

As was discussed in Section 2, the "twisting"algorithm provides only the desired precision of control of the output variable at the cost of increased relay amplitude. It shown in Kochetkov, V.A. Utkin [2013] that an unmatched external disturbance restricts the switching frequency, which finally leads to only the prescribed accuracy of the system output. To overcome this effect, another control algorithm as applied to (15) in view of (20) is proposed, which makes use of high-frequency external signals

$$\begin{pmatrix} B_2 \\ B_0 \end{pmatrix} u = \begin{bmatrix} -A_2 x_2 - A_{s_1} s_1 - A_{s_2} s_2 - \overline{A}_{20}\overline{x}_0 - \\ -M_2 \operatorname{sign}(\delta(t) + s_2) - M_1 \operatorname{sign}(s_1) \\ \\ -\overline{A}_{02} x_2 - \overline{A}_{01} s_1 - \overline{A}_{00} s_2 - \overline{A}_{00}\overline{x}_0 - \\ -H \operatorname{sign}(\overline{x}_0) \end{bmatrix},$$

$$M_1 = \operatorname{diag}\{M_{1i}\}, M_{1i} = \text{const} > 0, M_2 = \operatorname{diag}\{M_{2i}\},$$
$$M_{2i} = \text{const} > 0, \delta(t) = \operatorname{diag}\{\delta_i(t)\}, \tag{23}$$
$$\delta_i(t) = \begin{cases} \dfrac{2\delta_i \omega}{\pi} t, \ 0 \le t \le \dfrac{\pi}{2\omega}; \\ 2\delta_i - \dfrac{2\delta_i \omega}{\pi} t, \ \dfrac{\pi}{2\omega} \le t \le \dfrac{3\pi}{2\omega}; \qquad i = \overline{1, p_1}, \\ 4\delta_i + \dfrac{2\delta_i \omega}{\pi} t, \ \dfrac{3\pi}{2\omega} \le t \le \dfrac{4\pi}{\omega}, \end{cases}$$

where ω is the frequency of vibration signals and $\delta_i = \text{const} > 0$ is the amplitude of the ith vibration signal.

Substituting (23) into (15), we obtain the following equations of the closed-loop system:

$$\begin{aligned} \dot{x}_2 &= \overline{A}_{22} x_2 + A_{21} s_1, \\ \dot{s}_1 &= \overline{s}_2, \\ \dot{\overline{s}}_2 &= -M_2 \operatorname{sign}(\delta(t) + s_2) - M_1 \operatorname{sign}(s_1) + \xi(t), \\ \dot{\overline{x}}_0 &= \overline{Q}_0 f(t) - H \operatorname{sign}(\overline{x}_0), \\ y &= \overline{D} \begin{pmatrix} x_2 \\ s_1 \end{pmatrix}. \end{aligned} \tag{24}$$

According to Kochetkov [2010], Iannelli [2006], with a certain choice of the parameters $\delta_i(t)$, the action $M_{2i}\operatorname{sign}(\delta_i(t)+s_{2i})$ of the ith relay on the system is equivalent to the linear feedback $\dfrac{M_{2i}}{\delta_i} s_{2i}$. Based on Theorem 2 and the results of Kochetkov [2010], Iannelli [2006], the following theorem is stated without proof.

Theorem 3. Let the following conditions hold in the closed-loop system (15), (23):

(i) The external disturbance vector $\xi(t)$ satisfies constraints (18), where$L_\alpha = \operatorname{diag}\left\{\dfrac{M_{2i}}{\delta_i}\right\}$ in notation (17).

(ii) The elements of the matrices M_1, M_2 and H and the parameters δ_i in (23) satisfy the inequalities

$$M_{1i} > M_{2i} > \Sigma_i, \quad \frac{M_{2i}}{\delta_i}(M_{1i} - \Sigma_i) > \overline{\Sigma}_i,$$
$$\delta_i > 2L_{1i}, \ H_i > \Sigma_{0i},$$

where $\overline{f}_1(t) = \overline{Q}_1 f(t)$, according to (13) $|\overline{f}_{1i}(t)| \le L_{1i}$, $\overline{f}_{1i}(t)$ is the ith component of the vector $\overline{f}_1(t)$, $L_{1i} = \text{const} > 0$, $i = \overline{1, p_1}$.
Then there exist $N_i = \text{const} > 0$ ($i = \overline{1, m}$) and $\omega_0 = \text{const} > 0$ such that, for $\omega > \omega_0$ it is true that

$$\lim_{t\to\infty} |y_i(t)| \le \frac{N_i}{\omega}, \ i = \overline{1, m},$$

where $y_i(t)$ is the ith component of the output of closed-loop system (15), (23).

Theorem 3 implies that, with the frequency of the vibration signal increasing unlimitedly $\omega \to \infty$ control algorithm (23) ensures asymptotic invariance with respect to output variables.

5. SIMULATION RESULTS

Let us consider the sixth-order system

$$\begin{aligned} \dot{x}_1 &= \begin{pmatrix} 55.18 & 27.98 & 21.54 & -90.71 \\ -83.75 & -43.82 & -42.65 & 139.94 \\ 11.68 & 6.24 & 7.21 & -19.39 \\ 7.91 & 4.09 & 4.12 & -12.06 \end{pmatrix} x_1 + \\ &+ \begin{pmatrix} 2.36 & 0.28 \\ -3.4 & -0.47 \\ 0.55 & 0.01 \\ 0.41 & 0.17 \end{pmatrix} x_0 + \begin{pmatrix} -4.57 & 7.91 \\ 6.57 & -11.63 \\ -1.1 & 1.69 \\ -0.74 & 1.75 \end{pmatrix} f(t), \\ \dot{x}_0 &= \begin{pmatrix} -2.8 & 0.6 & 10.5 & 11.7 \\ -3.1 & -0.1 & 9.85 & 6.1 \end{pmatrix} x_1 + \begin{pmatrix} -4 & 6 \\ 2 & 1 \end{pmatrix} x_0 \\ &+ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u + \begin{pmatrix} -1 & 5 \\ 0.3 & -2 \end{pmatrix} f(t), \\ y &= (-2.9 \ 0.5 \ 8.5 \ 11.7)\, x_1. \end{aligned}$$

We present a procedure to solve the formulated problem relying on Theorem 2. After the nonsingular transformation of the coordinates

$$\begin{pmatrix} x_1 \\ \widetilde{x}_0 \end{pmatrix} = T_1 x_1, \ T_1 = \begin{pmatrix} 15.5 & -3.7999 & 4 & 2 & 0 & 0 \\ -0.025031 & 3.8 & -1 & 0.7 & 0 & 0 \\ 22.934 & -6.3665 & 6 & 2 & 0 & 0 \\ 0.58334 & 3.3333 & -0.5 & 7 & 0 & 0 \\ 4.1673 & -1.1327 & 10 & -1 & 1 & 0 \\ -2.1085 & 0.66679 & 4.25 & 4.5 & 0 & 1 \end{pmatrix}$$

this system is representable in the form:

$$\begin{aligned} \dot{x}_2 &= \begin{pmatrix} -5 & 0 \\ -4 & -3 \end{pmatrix} x_2 + \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} s_1, \\ \dot{s}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} s_2 + \begin{pmatrix} -2 & 3 \\ 0,5 & 3 \end{pmatrix} f(t), \\ \dot{s}_2 &= \begin{pmatrix} 102,77 & -4,13 \\ 104,86 & -2,67 \end{pmatrix} x_2 + \begin{pmatrix} -27,95 & -41,7 \\ -22,65 & -40 \end{pmatrix} s_1 + \\ &+ \begin{pmatrix} 6 & 5 \\ 3,25 & 6,5 \end{pmatrix} s_2 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} u + \begin{pmatrix} -21,5 & 32 \\ -5,95 & 24,25 \end{pmatrix} f(t), \\ y &= (-3,5833 \ 1,6667 \ 1,0000 \ 1,0000) \left(x_2^{\mathrm{T}} \ s_1^{\mathrm{T}}\right)^{\mathrm{T}}. \end{aligned}$$

By taking control in the form (20) we establish the equations of the closed-loop system:

$$\begin{aligned} \dot{x}_2 &= \begin{pmatrix} -5 & 0 \\ -4 & -3 \end{pmatrix} x_2 + \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} s_1, \\ \dot{s}_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} s_2 + \begin{pmatrix} -2 & 3 \\ 0,5 & 3 \end{pmatrix} f(t), \\ \dot{s}_2 &= -\begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix} s_2 - \begin{pmatrix} M_1 & 0 \\ 0 & M_2 \end{pmatrix} \begin{bmatrix} \operatorname{sign}(s_{11}) \\ \operatorname{sign}(s_{12}) \end{bmatrix} + \\ &+ \begin{pmatrix} -21,5 & 32 \\ -5,95 & 24,25 \end{pmatrix} f(t), \\ y &= (-3,5833 \ 1,6667 \ 1,0000 \ 1,0000) \left(x_2^{\mathrm{T}} \ s_1^{\mathrm{T}}\right)^{\mathrm{T}}. \end{aligned}$$

The harmonic perturbations

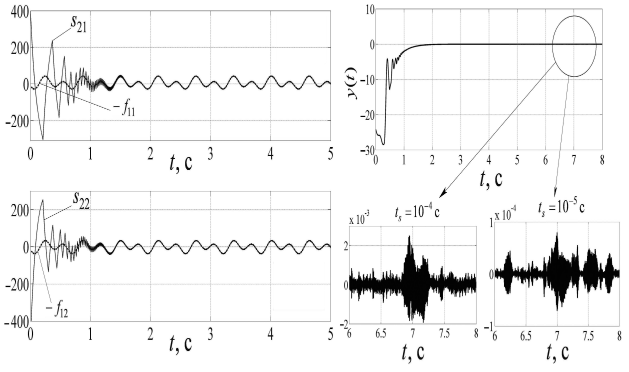

Fig. 1. The simulation results

$$f(t) = \begin{bmatrix} 2 + 3\sin(10t) \\ -5\cos(10t) + 8\sin(20t) \end{bmatrix}$$

are used for simulation. By denoting

$$f_1(t) = \begin{pmatrix} -2 & 3 \\ 0,5 & 3 \end{pmatrix} f(t), \quad f_2(t) = \begin{pmatrix} -21,5 & 32 \\ -5,95 & 24,25 \end{pmatrix} f(t),$$

we get the constraints for the components $f_{ij}(t)$ $(i = 1,2;\ j = 1,2)$ of the vectors $f_1(t)$ $f_2(t)$:

$$\begin{pmatrix} |f_{11}(t)| \\ |f_{12}(t)| \end{pmatrix} \leq \begin{bmatrix} 44,16 \\ 40,1 \end{bmatrix}, \quad \begin{pmatrix} |f_{21}(t)| \\ |f_{22}(t)| \end{pmatrix} \leq \begin{bmatrix} 426,6 \\ 328,5 \end{bmatrix},$$

$$\begin{pmatrix} |\dot{f}_{11}(t)| \\ |\dot{f}_{12}(t)| \end{pmatrix} \leq \begin{bmatrix} 641,55 \\ 630,75 \end{bmatrix}, \quad \begin{pmatrix} |\dot{f}_{21}(t)| \\ |\dot{f}_{22}(t)| \end{pmatrix} \leq \begin{bmatrix} 6845,1 \\ 5105,6 \end{bmatrix},$$

$$\begin{pmatrix} |\ddot{f}_{11}(t)| \\ |\ddot{f}_{12}(t)| \end{pmatrix} \leq \begin{bmatrix} 11216 \\ 11107 \end{bmatrix},$$

where the inequalities are understood in the component-wise sense.

Using the last expressions, we write the constraints in the form of (18):

$$|\xi_1(t)| \leq 44,16\alpha_1 + 1068,2; \ |\xi_2(t)| \leq 40,1\alpha_2 + 959,25;$$

$$|\dot{\xi}_1(t)| \leq 641,55\alpha_1 + 18061,1;$$

$$|\dot{\xi}_2(t)| \leq 630,75\alpha_2 + 16213.$$

One can readily verify that the conditions of Theorem 2 are satisfied for the following values of the elements of the matrices L_α M:

$$\alpha_1 = \alpha_2 = 10, \ M_1 = 4000, M_2 = 3620.$$

Figure 1 depicts the results of modeling in the MATLAB/Simulink environment. The Dorman Prince (ode5) method was used for numerical integration. The right side of Fig. 1 shows for the output variable the error at different steps of integration: $t_s = 10^{-4}$ to the left and $t_s = 10^{-5}$ to the right. As can be seen, the smaller the step of integration, the smaller the stationary error. It is quite clear why the theoretical result of Theorem 2 is valid only for the infinite frequency of relay switching ($t_s = 0$). One also can see from the results of modeling that the components $s_{21}(t)$ and $s_{22}(t)$ of the vector $s_2(t)$ follow the external perturbations $f_{11}(t)$ and $f_{12}(t)$.

6. CONCLUSION

The present paper solved the problem of providing invariance of the output to a wide class of external non-coordinated perturbations on the basis of the proposed relay vortex algorithms. The developed nonlinear control algorithms provide asymptotic convergence of the output variables to zero for arbitrary initial conditions under the magnitude-bounded control actions maintaining oscillations in the closed-loop system with an unlimited growth of frequency and asymptotic tendency of the oscillation amplitude to zero. For the case where the control actions are of knowingly key nature, a method of realization of the relay vortex algorithms with the use of the method of vibrolinearization of the relay elements was proposed. The above algorithms can be used to solve a wide range of the application problems Kochetkov [2011], V.A. Utkin [1998].

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