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Indirect solution for optimal control problems with a pure state constraint

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Abstract: This paper presents an algorithm that provides a regularization for the costate dynamics of state constrained optimal control problems with a scalar constraint under the assumption that the Hamiltonian is convex in the control and the state dynamics equation of the constrained state is monotonically increasing in the control variable. The algorithm is demonstrated with a classical optimal control problem.

1. INTRODUCTION

Let us consider a numerical solution for optimal control problems with a scalar "pure" state inequality constraint, i.e. problems of the type:

$$(\mathcal{P}) \begin{cases} \min\limits_{u\in\mathcal{U}} \Phi(x(t_1)) + \displaystyle\int_{t_0}^{t_1} F(x(t),u(t))\mathrm{d}t, \\ \text{subject to:} \quad \dot{x} = f(x(t),u(t)),\ x(t_0)=x_0, \\ h(x_c(t)) \leq 0. \end{cases}$$

Here, u is the control variable, x the state variable, $\mathcal{U}$ is a closed convex set of admissible controls for every t and x, t_0 the initial time, t_1 the end time, Φ the function describing the final cost, F the integral cost function, f the state dynamics equation, x_0 the initial condition, h the inequality constraint function on the scalar constrained state x_c.

Pontryagin's Minimum Principle (PMP) is a well known tool to solve unconstrained optimal control problems, i.e. problems of type $\mathcal{P}$ without the constraint function h. Solutions obtained through the PMP are often referred to as indirect solutions given the two step procedure of first constructing the Hamiltonian function $H = F + p^\top f$ and deriving necessary conditions of optimality which results in the following boundary value problem:

$$\dot{x}(t) = H_p, \qquad x(t_0) = 0, \tag{1}$$
$$\dot{p}(t) = H_x, \qquad p(t_1) = \Phi_x, \tag{2}$$

in which the control at each time instance is given by $\min_u H$. The boundary value problem can then be solved in a second step using numerical tools, see, e.g., Ascher, et.al. [1988]. This mechanism provides an infinite dimensional solution meaning that quantization of the state and control space is not required.

Extentions of the necessary conditions of optimality for state constrained problems are availiable, see Jacobson, et.al. [1971], and also Maurer [1977], Seierstad and Sydsæter [1987], Hartl, et.al. [1995]. This involves the construction of the Lagrangian $L = H + \nu h$ and in addition to (1) states the following necessary conditions for optimality:

$$\dot{p}(t) = L_x, \qquad p(t_1) = \Phi_x, \tag{3}$$
$$p(\tau^+) = p(\tau^-) + \eta(\tau), \qquad \eta(\tau) = \xi(\tau^-) - \xi(\tau^+), \tag{4}$$
$$\nu(t)h(t) = 0, \qquad \nu(t) \geq 0, \tag{5}$$

where $\nu(t) = \dot{\xi}(t)$, and τ^- and τ^+ indicate the time just before and just after the time instance τ where the costate may have a discontinuity. If H and g are convex in x then (1), (3) to (5) are also sufficient conditions for optimality, see [Seierstad and Sydsæter, 1987, Theorem 1, p. 317]. Nevertheless, information regarding the time instance τ where the optimal state trajectory touches with the constraint cannot be directly derived from the optimality conditions. Therefore, the solution for (1), (3) to (5) cannot be obtained with the standard tools for boundary value problems, see Bonnans [2007].

Direct numerical methods have been developed to solve state constrained optimal control problems, see, e.g., Bock and Plitt [1984], Pytlak [1999], Gerdts and Kunkel [2008], Loxton, et.al. [2009]. Direct methods are based on an a priori quantization of the control, state and/or time space of the continuous time optimal control problem and reformulation to a finite dimensional nonlinear program. The discretization methods are robust in practice. Nevertheless, the computational effort grows at a nonlinear rate with the number of grid points used for the quantization.

In Van Keulen, et.al. [2014], an algorithm is proposed to obtain an infinite dimensional solution for state constrained optimal control problems with a scalar state. It is shown that the time instance where the unconstrained solution exceeds the contraint the most is a contact point with the constraint under the assumption that H is convex in the control and that the state dynamics are monotonically increasing in the control. If the state dynamics are scalar, the boundary value problem can be split at this time instance into two subtrajectories which again can be solved using the same mechanism. In this way the recursive algorithm provides a regularization for the costate dynamics.

In this paper, the algorithm of Van Keulen, et.al. [2014] is extented to include nonscalar optimal control problems with a scalar "pure" state constraint. The state and costate dynamics that are not subject to constraints are to be continuous along the time horizon. Hence, the scalar procedure of splitting the state trajectory into subtrajectories is not possible for the nonscalar problem description. Instead, the extended algorithm becomes a multipoint boundary value problem with an additional boundary condition at the time instance where the constraint is exceeded the most in the unconstrained solution. Constraint exceeding of the multipoint boundary value problem solution can again be evaluated. A recursive algorithm is obtained that converges to the constrained optimal solution.

The algorithm has all the advantages of the indirect approach; a priori quantization of control and state space can be avoided. Besides, multiplier information is inherently available in the solution. The algorithm is demonstrated with a classical optimal control problem, see Lasdon, et.al. [1967], Jacobson and Lele [1969], which has been used to demonstrate algorithms for direct state constrained optimal control problems, see Jennings and Fisher [2002], Rutquist and Edvall [2006].

This paper is structured as follows. In the next section, the extentions to the algorithm of Van Keulen, et.al. [2014] are discussed. In Section 3, the example is presented. Finally, in Section 4 a discussion and conclusions are provided.

2. THE ALGORITHM

In this section, the proof from Van Keulen, et.al. [2014] is adjusted to include nonscalar optimal control problems with a scalar inequality constraint.

Lemma 1. Let $H = F + p^\top f$ with u a scalar variable, H a strictly convex function in u, F a convex function in u, f a convex and monotonic increasing function in u, and h a constraint function of first order meaning that the control appears explicitely in the first time derivative of the constraint function. Additionally, assume the differential equation $\dot{p} = \frac{\partial H}{\partial x}$, with $\frac{\partial H}{\partial x}$ locally Lipschitz in p on a domain defined by $\mathcal{U}$, then $u^* = \arg\min_u H$ is a monotonic decreasing function of p and there is a monotonic increasing relation between the initial value of the costate associated to the constrained state $p_c(t_0)$ and the final state $x_c(t_1)$ and between the inverse relation of the final state $x_c(t_1)$ and the initial value of the costate $p_c(t_0)$.

Proof. If H is strictly convex in u, then H has a unique minimum u^* defined by $\frac{\partial H(u)}{\partial u} = 0$ with $\frac{\partial H(u)}{\partial u} < 0$ if $u < u^*$ and $\frac{\partial H(u)}{\partial u} > 0$ if $u > u^*$. If $f(u)$ is convex and monotonic increasing in u, then $\frac{\partial f(u)}{\partial u} > 0$ and also monotonic increasing in u. Therefore, $p_a^\top \frac{\partial f(u)}{\partial u} > p_b^\top \frac{\partial f(u)}{\partial u}$ if $p_a^\top > p_b^\top$ (element wise). Furthermore, since F and f are convex, it follows that $\frac{\partial F(u_a)}{\partial u} \geq \frac{\partial F(u_b)}{\partial u}$ if $u_a > u_b$ and $\frac{\partial f(u_a)}{\partial u} \geq \frac{\partial f(u_b)}{\partial u}$ if $u_a > u_b$.

In the optimum it holds that $\frac{\partial F(u)}{\partial u} + p\frac{\partial f(u)}{\partial u} = 0$, with $p_a^\top \frac{\partial f(u)}{\partial u} > p_b^\top \frac{\partial f(u)}{\partial u}$ if $p_a^\top > p_b^\top$ and using $\frac{\partial F(u_a)}{\partial u} \geq \frac{\partial F(u_b)}{\partial u}$ if $u_a > u_b$ and $\frac{\partial f(u_a)}{\partial u} \geq \frac{\partial f(u_b)}{\partial u}$ if $u_a > u_b$, it follows that the minimum u^* of H is a monotonic decreasing function of p.

If $\frac{\partial H}{\partial x}$ is Lipschitz continuous in p, it follows that p is a unique solution of $\dot{p}$, see [Khalil, 2002, Theorem 3.1], hence $p_a(t_0) > p_b(t_0)$ implies $p_a^0(t) > p_b^0(t)$ (element wise), where $p^0(t)$ denotes the trajectory resulting from $p(t_0)$. Using that the minimum u^* of H is a monotonic decreasing function of p, it follows that $u_a^0(t) \leq u_b^0(t)$ if $p_a^0(t) > p_b^0(t)$, where u^0 denotes the control trajectories resulting from $p(t_0)$. Finally, given the state dynamics $\dot{x} = f$ with f monotonic increasing in u and u appearing in the first time derivative of h [1], a monotonic increasing relation is found between the costate associated to the constrained state $p_c(t_0)$ and $x_c(t_1)$ and likewise between $x_c(t_1)$ and $p_c(t_0)$. □

Next, a method is proposed to find *each* boundary interval or contact time with the state constraint $h(x_c(t))$ based on the times where the constraint is exceeded the *most* in the unconstrained optimal trajectory. A new multipoint boundary value problem is obtained with a touching point of the constrained state with the constraint at the time where the constraint is exceeded the most.

This procedure is repeated with a recursive scheme until the state constraints are met for all $t \in [t_0\ t_1]$, hereby increasing the complexity of the boundary value problem with one boundary condition for each iteration.

Algorithm 2. The optimal multiplier $p^*(t)$ and state $x^*(t)$ trajectories for the state constrained optimal control problem with scalar state constrained are found by the following sequence:

- compute the unconstrained optimal solution defined by the initial value of the costate $p(t_0)$, i.e., solve the two point boundary value problem as in (1) and (2). If a state constraint is violated, i.e., if $h(x_c(t)) > 0$:

repeat
- find the time instance τ_i where the state boundary is exceeded the most,
$$\tau_i = \arg\max_{t\in[t_0\ \ t_1]}(h(x_c(t))), \tag{6}$$
where i is the ith iteration of this recursive scheme,
- extend the boundary value problem with an additional boundary condition $x_c(\tau_i) = h(x_c(\tau_i))$, then solve the extended boundary value problem, i.e., solve (1) and (2) defined by the boundary conditions $p(t_0)$ and $[p(\tau_1) \dots p(\tau_i)]$,
until $\max_{t\in[t_0\ t_1]} h(x_c(t)) \leq e_h$, where e_h is an allowed constraint exceeding.

The proof for optimality for these three situations is in line with the proof of Van Keulen, et.al. [2014]. The proof of the algorithm is based on the following two observations:

- from the jump conditions of the PMP, the following proporties of the optimal solution are obtained: $p(\tau_i^+) \leq p(\tau_i^-)$ if the upper constraint is reached, $p(\tau_i^+) \geq p(\tau_i^-)$ if the lower constraint is reached,
- using the invertible monotonic increasing relation between $p_c(t_1)$ and $x_c(\tau)$ described in Lemma 1 it follows that the time at which the boundary is exceeded the most is a contact point or part of the boundary interval, and, therefore, part of the optimal constrained solution.

[1] Algorithm 2 fails if the control does not appear explicitely in the first time derivative, see, e.g., the third order problem in Jacobson, et.al. [1971] where the maximum constraint exceeding of the unconstrained solution is not part of the boundary interval of the contstrained optimal trajectory.

3. EXAMPLE

Consider the minimization of the following objective function:

$$J = \int_0^1 x_1^2(t) + x_2^2(t) + \tfrac{1}{200}u^2(t)\mathrm{d}t, \tag{7}$$

subject to the state dynamics and boundary conditions:

$$\dot{x}_1(t) = x_2(t), \qquad x_1(0) = 0, \tag{8}$$

$$\dot{x}_2(t) = -x_2(t) + u(t), \qquad x_2(0) = -1, \tag{9}$$

and the continuous inequality constraint:

$$x_2(t) - 8\left(t - \tfrac{1}{2}\right)^2 + \tfrac{1}{2} \le 0, \tag{10}$$

in which t is the time, x_1 and x_2 are the state variables and u is the control variable. Note that, the functions are continuously differentiable in x and u. Moreover, the integrand of J is a strictly convex function in u, $\dot{x}_2$ is a strictly monotonic increasing function in u, and the control appears explicitely in the first time derivative of (10).

The problem is given in Lasdon, et.al. [1967]. A solution of the problem using an approximation of the constraint with penalty functions is presented in Jacobson and Lele [1969]. A numerical solution for the problem using a direct solution method can be found in Jennings and Fisher [2002] or Rutquist and Edvall [2006].

3.1 Necessary conditions of optimality

Pontryagin's Minimum Principle can be applied to derive necessary conditions of optimality for the problem described above. A first step is to construct the Hamiltonian function:

$$H = x_1^2(t) + x_2^2(t) + \tfrac{1}{200}u^2(t) + p_1(t)x_2(t) + p_2(t)\left(-x_2(t) + u(t)\right), \tag{11}$$

in which the state dynamics are adjoined to the integrand of the objective function with the introduction of costate variables p_1 and p_2. Here, the trivial solution $[p_1,\ p_2] = [0,\ 0]$ on the interval $[0,\ 1)$ is excluded from the candidate solution trajectories. The Hamiltonian H is strictly convex in u. The state inequality constraint (10) can be adjoined to the Hamiltonian to form the following Lagrangian:

$$L = H + \nu(t)\left(x_2(t) - 8\left(t - \tfrac{1}{2}\right)^2 + \tfrac{1}{2}\right), \tag{12}$$

where ν is a Lagrange multiplier. The Minimum Principle provides the following necessary conditions for optimality:

- the differential condition on the costate

$$\dot{p}_1(t) = -\frac{\partial L}{\partial x_1} = -2x_1(t), \tag{13}$$

$$\dot{p}_2(t) = -\frac{\partial L}{\partial x_2} = -2x_2(t) - p_1(t) + p_2(t) - \nu(t). \tag{14}$$

  Note that, $\dot{p}$ is continously differentiable in p. Hence the Lipschitz condition of Lemma 1 holds,
- the jump condition at junction time τ

$$p(\tau*) = p(\tau^-) + \eta(\tau), \tag{15}$$

  with $\eta \ge 0$. Under the assumption that ξ is allowed to have a piecewise continuous derivative, it is possible to set $\nu(t) = \dot{\xi}(t)$ for every t for which ξ exist and $\eta(\tau) = \xi(\tau^-) - \xi(\tau^+)$, for all $\tau \in [t_0, t_1]$ where ξ is not differentiable,
- the complementary condition

$$\nu(t) \ge 0, \tag{16}$$

$$\nu(t)\left(x_2(t) - 8\left(t - \tfrac{1}{2}\right)^2 + \tfrac{1}{2}\right) = 0, \tag{17}$$

- the minimum condition

$$u^*(t) = \arg\min_u H(t, x_1^*, x_2^*, u, p_1^*, p_2^*). \tag{18}$$

  Since H is strictly convex in u, the first derivative of the Hamiltonian to the control can be set to zero in order to obtain an analytical expression for the control:

$$\frac{\partial H}{\partial u} = 0 \Rightarrow u^*(t) = -100p_2(t). \tag{19}$$

- the transversality condition at terminal time $t_1 = 1$

$$p_1(1) = \frac{\partial \Phi}{\partial x_1} = 0, \tag{20}$$

$$p_2(1) = \frac{\partial \Phi}{\partial x_2} = 0. \tag{21}$$

3.2 The unconstrained solution

Using the necessary conditions of optimality outlined in Section 3.1, a boundary value problem is obtained, with the objective to find initial conditions for the costate $p_1(0)$ and $p_2(0)$, together with the prescribed boundary conditions on the state $x_1(0)$ and $x_2(0)$, such that the boundary conditions on $p_1(1) = 0$ and $p_2(1) = 0$ are met. The unconstrained optimal solution for problem (7) to (9) can found by solving the system of linear ordinary differential equations with boundary conditions described by (8), (9), (13), (14) where $\nu = 0$, (20) and (21).

A matrix differential equation of the form $\dot{y}(t) = Ay(t)$ is obtained with the following general solution

$$y(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3 t} v_3 + c_4 e^{\lambda_4 t} v_4, \tag{22}$$

where $y = [x;\ p]^\top$, and $\Lambda = [\lambda_1 \ldots \lambda_4]$ and $V = [v_1 \ldots v_4]^\top$ are the eigenvalues and eigenvectors of A, respectively, i.e., the solution of $V^{-1}AV = \Lambda$. Parameters $C = [c_1 \ldots c_4]$ are constants given by:

$$C = V^{-1}y_0, \tag{23}$$

where $y_0 = [x_1(0);\ x_2(0);\ p_1(0);\ p_2(0)]$. The eigenvalues are $\sqrt{200}$, 1, -1, $-\sqrt{200}$, so the associated initial value problem is unstable ($\lambda_1 > 0$ and $\lambda_2 > 0$). Hence, small changes in $p_1(0)$ or $p_2(0)$ lead to large changes in the (co)state at $t = 1$. Nevertheless, the boundary value problem is stable on the specified boundary interval. The solution can be summarized with: solve (22) with y_0 such that the following boundary conditions are met

$$x_1(0) = 0, \qquad x_1(1) = free, \tag{24}$$

$$x_2(0) = -1, \qquad x_2(1) = free, \tag{25}$$

$$p_1(0) = ??, \qquad p_1(1) = 0, \tag{26}$$

$$p_2(0) = ??, \qquad p_2(1) = 0. \tag{27}$$

By Lemma 1 it follows that, since the state dynamics are monotonic increasing in u, a monotonic increasing relation is found between $p_2(0)$ and $x_2(1)$.

From the initial value problem (22) with $t = 1$, a set of 2 (nonlinear) algebraic equations for the unknown y_0 is obtained.

$$z(q) \equiv g(q, y(y_0(q))) = 0. \tag{28}$$

Here $q = [p_1(0)\ p_2(0)]$. The Jacobian $\frac{\partial z}{\partial q}$ is constant such that a one step Newton iteration results in the optimum. The unconstrained solution is found with

$$q^* = q^0 - \left(\frac{\partial z}{\partial q}\right)^{-1} z(q^0), \tag{29}$$

where q^* is the optimal initial condition, and q^0 an arbitrary guess for the initial condition.

The unconstrained optimal solution is presented in Fig. 1. It can be seen that both $p_1(0) < 0$ and $p_2(0) < 0$. Clearly, the constraint function is violated. In the next subsection, Algorithm 2 is applied to derive the constrained optimal solution.

3.3 The constrained solution

The first step in Algorithm 2 is to find the time instance where the constraint is exceeded the most, i.e. to solve

$$\tau_1 = \max_{t\in[0\ 1]} x_2(t) - 8\left(t - \tfrac{1}{2}\right)^2 + \tfrac{1}{2}, \tag{30}$$

where $x_2(t)$ is a solution of (22). Optimization (30) is concave if x_2 is concave in the domain $t \in [t_0\ t_1]$. State x_2 is strictly concave in t iff $\ddot{x}_2 < 0$. Differentiating (9) and using (14), (19) and again (9) leads to the following condition:

$$201x_2(t) + 100p_1(t) < 0. \tag{31}$$

It can be verified that the condition (31) holds for $t \in [t_0\ t_1]$ in the unconstrained solution. Hence, it can be concluded that (30) is a concave optimization problem.

Following step 2 from the algorithm, time instance τ_1 is a contact point on the constraint where the state x_2 and the control u are prescribed:

$$x_2(\tau_1) \equiv 8\left(\tau_1 - \tfrac{1}{2}\right)^2 - \tfrac{1}{2}. \tag{32}$$

Next, a multipoint boundary value problem is obtained with the following boundary conditions:

$$x_1(0) = 0, \qquad x_1(1) = free, \tag{33}$$
$$x_2(0) = -1, \qquad x_2(1) = free, \tag{34}$$
$$p_1(0) = ??, \qquad p_1(1) = 0, \tag{35}$$
$$p_2(0) = ??, \qquad p_2(1) = 0, \tag{36}$$
$$p_2(\tau_1) = ??, \qquad x_2(\tau_1) = 8\left(\tau_1 - \tfrac{1}{2}\right)^2 - \tfrac{1}{2}. \tag{37}$$

The objective is then to find the three unknown costate values $p_1(0)$, $p_2(0)$, and $p_2(\tau_1)$. The multipoint boundary value problem can be solved in a similar way as the unconstrained solution with the difference that the number of boundary conditions of problem (28) has grown with one, so vector q has length 3. Nevertheless, the Jacobian in (29) remains constant and a one step Newton iteration results in the solution of the multipoint boundary value problem.

The resulting trajectories that solve the multipoint boundary value problem are shown in Fig. 2. Note that the jump direction of the costate p_2 meets the jump condition, i.e. $p_2(\tau_1^-) > p_2(\tau_1^+)$ and $\eta(\tau_1) < 0$ which is in line with (15) and (16). As can be seen, the constraint is still exceeded.

Another iteration can be performed to find the time instance, both before and after τ_1, where the constraint is exceeded the most, i.e. to solve

$$\tau_L = \max_{t\in[0\ \tau_1]} x_2(t) - 8\left(t - \tfrac{1}{2}\right)^2 + \tfrac{1}{2}, \tag{38}$$

and

$$\tau_R = \max_{t\in[\tau_1\ 1]} x_2(t) - 8\left(t - \tfrac{1}{2}\right)^2 + \tfrac{1}{2}, \tag{39}$$

here, τ_L is the time instance before τ_1 and τ_R the time instance after τ_1. Again, using condition (31), it can be verified that optimization (38) and (39) are concave. In the general application of Algorithm 2 the number of boundary conditions will grow with each iteration. Following Algorithm 2 this results in a multipoint boundary value problem with boundary conditions at τ_L, τ_R, and τ_1 in addition to (33) to (37). However, for this particular example, it can be shown that the subtrajectory inbetween τ_L and τ_R is part of the boundary interval such that the number of unknown boundary conditions remains limited to three.

The reasoning hinges on concavity of x_2 in t. If x_2 for $t \in [\tau_L\ \tau_R]$ is concave than it exceeds the constraint for all $t \in [\tau_L\ \tau_R]$. Concavity of x_2, of the solution with a boundary condition at τ_1, can be checked with (31). Given Lemma 1, it follows that if x_2 is concave in the first iteration, x_2 will also be concave for all further iterations of the algorithm. Therefore, if x_2 is concave for $t \in [\tau_L\ \tau_R]$ than x_2 for $t \in [\tau_L\ \tau_R]$ must be a boundary interval.

Along the boundary interval the system of ordinary differential equations is completely prescribed and an analytical solution can be obtained. From (10)

$$x_2(t) \equiv 8t^2 - 8t + \tfrac{3}{2},\ \forall t \in [\tau_L\ \tau_R]. \tag{40}$$

Differentiating (40) gives

$$\dot{x}_2(t) \equiv 16t - 8,\ \forall t \in [\tau_L\ \tau_R]. \tag{41}$$

Hence, using (3), a condition for the control is obtained

$$u \equiv 8t^2 + 8t - \tfrac{13}{2},\ \forall t \in [\tau_L\ \tau_R]. \tag{42}$$

Furthermore, using (8) and (40) conditions for the other states and costates follow:

$$x_1(t) = x_1(\tau_L^-) + \int_{\tau_L}^{\tau_R} 8\left(t - \tfrac{1}{2}\right)^2 - \tfrac{1}{2}\mathrm{d}t,\ \forall t \in [\tau_L\ \tau_R]. \tag{43}$$

Here, τ_L^- indicates the time instance just before τ_L. The integral can be solved analytically providing a solution for x_1 at the constraint exit time τ_R:

$$x_1(\tau_R) = \tfrac{8}{3}\left(\tau_R^3 - \tau_L^3\right) - 4\left(\tau_R^2 - \tau_L^2\right) + \tfrac{3}{2}\left(\tau_R - \tau_L\right) + x_1(\tau_L^-). \tag{44}$$

Using (13) we obtain

$$p_1(t) = p_1(\tau_L^-) - 2\int_{\tau_L}^{\tau_R} \tfrac{8}{3}\left(t^3 - \tau_L^3\right) - 4\left(t^2 - \tau_L^2\right) + \tfrac{3}{2}\left(t - \tau_L\right) + x_1(\tau_L^-)\mathrm{d}t,\ \forall t \in [\tau_L\ \tau_R]. \tag{45}$$

Again, the integral can be solved analytically providing a solution for p_1 at the constraint exit time τ_R:

$$p_1(\tau_R) = -\tfrac{4}{3}\left(\tau_R^4 - \tau_L^4\right) + \tfrac{8}{3}\left(\tau_R^3 - \tau_L^3\right) - \tfrac{3}{2}\left(\tau_R^2 - \tau_L^2\right) - 2\left(-\tfrac{8}{3}\tau_L^3 + 4\tau_L^2 - \tfrac{3}{2}\tau_L + x_1(\tau_L^-)\right)\left(\tau_R - \tau_L\right) + p_1(\tau_L^-) \tag{46}$$

From (19) and (42)

$$p_2(t) = -\tfrac{8}{100}t^2 - \tfrac{8}{100}t + \tfrac{13}{200}, \forall t \in [\tau_L\ \tau_R]. \tag{47}$$

Using (14) and the results above, it follows that the Lagrange multiplier is described with the function:

$$\nu(t) = 2x_2(t) - p_1(t) + p_2(t) + \tfrac{16}{100}t + \tfrac{8}{100},\ \forall t \in [\tau_L\ \tau_R]. \tag{48}$$

To summarize, the constrained solution consists of a multi-point boundary value problem with a boundary condition at the touching point of the constrained path.

$$x_1(0) = 0, \qquad x_1(1) = free, \tag{49}$$
$$x_2(0) = -1, \qquad x_2(1) = free, \tag{50}$$
$$p_1(0) = ??, \qquad p_1(1) = 0, \tag{51}$$
$$p_2(0) = ??, \qquad p_2(1) = 0, \tag{52}$$
$$p_2(\tau_R) = ??, \qquad x_2(\tau_L) = h(\tau_L). \tag{53}$$

The Jacobian in (29) for this problem is also constant. The results of four iterations of the algorithm are shown in Figs. 3 to 6. It can be seen that the necessary conditions of optimality, including the jump conditions, are met. The error in constraint exceeding after five iterations $e_h < 5 \cdot 10^{-5}$ indicating that the algorithm converges fast to the constrained optimal solution. The only numerical operation for this example is the concave maximization to find the time instance where the constraint is exceeded the most.

Since x_2 is concave in t it follows also that at the touching points of the constraint the time derivative of the constraint and x_2 are identical. Moreover, p_2 is continues but nonsmooth at touching points τ_L and τ_R. The exact solution of the problem is described with τ_L and τ_R by the following equalities.

$$\frac{dh(\tau_L)}{dt} - \dot{x}_2(\tau_L) = 0, \tag{54}$$
$$\frac{dh(\tau_R)}{dt} - \dot{x}_2(\tau_R) = 0. \tag{55}$$

4. CONCLUSION AND DISCUSSION

This article presents an infinite dimensional numerical solution for optimal control problems with a scalar pure state constraint based on the necessary conditions for optimality.

The algorithm is demonstrated with a classical optimal control problem which has often been applied to evaluate direct numerical solutions for state constrained optimal control problems. It is shown that in just five iterations the algorithm converges to the optimal solution.

This example has computational advantages that do not apply to general optimal control problems of the type $\mathcal{P}$. In general, an analytical solution for the initial value problem $\dot{x}(x_0)$, $L_x(p_0)$ can not be found. Discretization of the time space is required and the boundary value problem requires a numerical solution.

Also, in the example, the constraint state is a concave function of time which gives two numerical advantages. First, finding the maximum exceeding of the constraint becomes a concave optimization (see optimization (6)). Second, the number of boundary conditions in the example remained limited for all iterations. In the general application of the algorithm the number of boundary conditions grows with one with each iteration. This could lead to a more involved multipoint boundary value problem.

Nonetheless, the algorithm presented in this paper does not require a priori assumptions on the structure of the costate dynamics and the advantages of indirect solutions that are known for unconstrained optimal control problems are available; quantization of the state and control space can be avoided.

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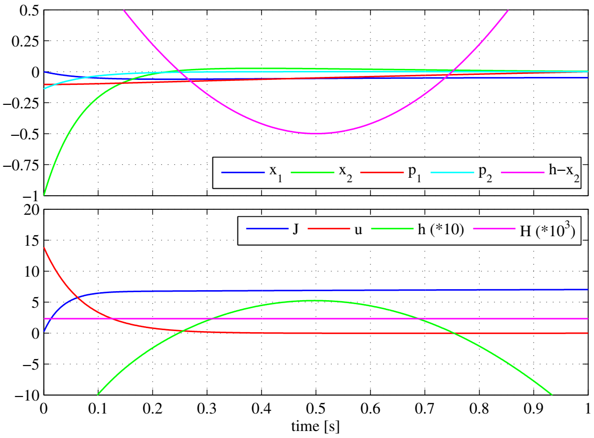

Fig. 1. Unconstrained solution.

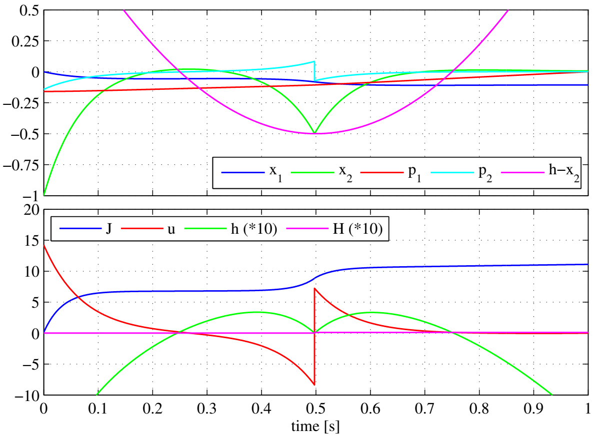

Fig. 2. Constrained solution after the first iteration.

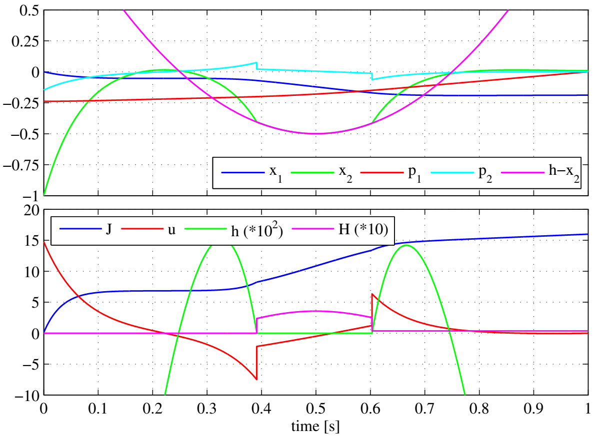

Fig. 3. Constrained solution after the second iteration.

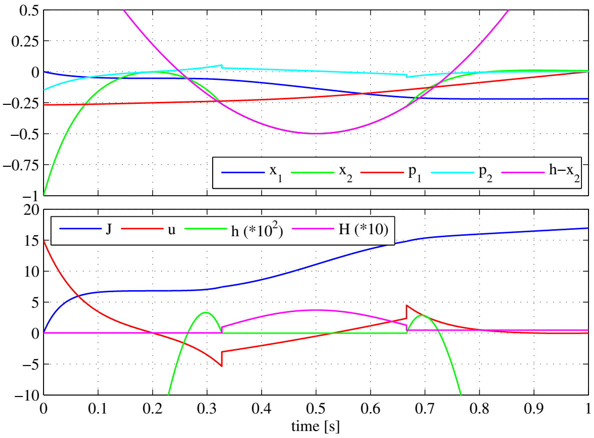

Fig. 4. Constrained solution after the third iteration.

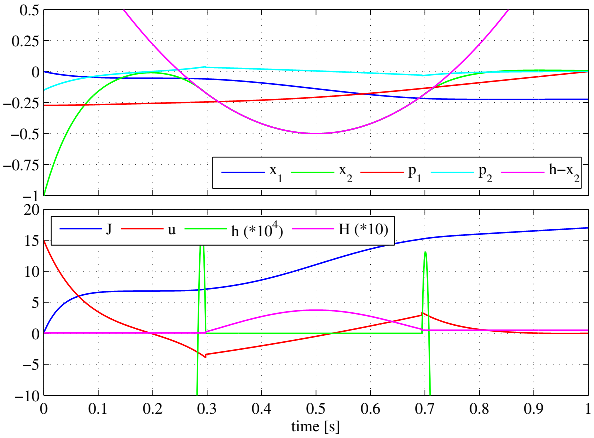

Fig. 5. Constrained solution after the fourth iteration.

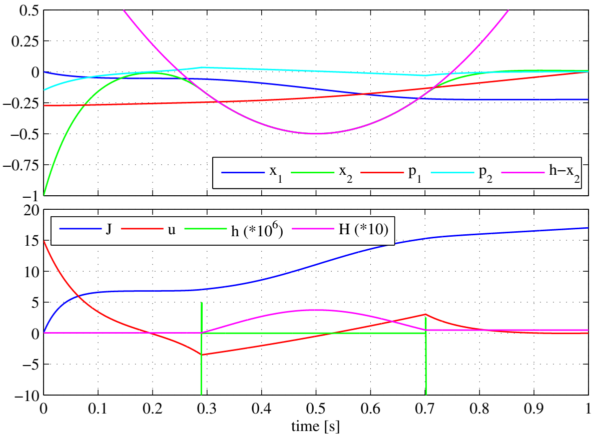

Fig. 6. Constrained solution after the fifth iteration.