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The Information Structure of Feedforward/Preview Control Using Forecast Data

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Abstract: Preview control using a fedforward imperfect forecast measurement of a disturbance signal is investigated in the context of discrete-time linear quadratic Gaussian (LQG) control. A new approach for incorporating such forecast measurements is built directly on established preview control models and results. The calculation of the optimal control gain, for which an efficient computation has already been derived, is found to be independent of the stochastic forecast measurements, implying that the optimal state estimator is where performance improvements in this problem set-up occur. Most significantly, the forecast data model is shown to equip the problem with a nested information structure whereby any forecast feedforward control problem of a fixed horizon length is always equivalent to a problem with a longer horizon and infinitely unreliable forecast measurements beyond the smaller horizon length. A numerical example illustrates the effect of forecast horizon length and data quality on the closed-loop system performance.

Keywords: Feedforward control; preview control; LQG; state estimation; forecasting

1. INTRODUCTION

We formulate the feedforward problem using the (by now) traditional technique of including a delay line, fed by future disturbance values, into an augmented plant structure. This approach dates back, at least, to Tomizuka and Whitney [1975].

We then pose a standard linear quadratic Gaussian (LQG) control problem using this structure, which incorporates the feedforward into the state feedback. This follows closely the $\mathscr{H}_2$-optimal method of Hazell and Limebeer [2010] and yields an LQ feedback gain and a state estimator, along with a closed-loop performance calculation. The separation theorem shows that the LQ gain and the state estimator are designed separately.

The central novelty of the paper rests in the incorporation of separate measurement noises into the feedforward signal, which consists of the entire state of the delay block and not just of its input, as in earlier treatments. This is able to capture, via the associated measurement noise covariances, the forecast phenomenon of diminishing reliability with preview horizon. This is our method for addressing the information structure.

From the linear Gaussian formulation, we prove a well-known (but perhaps unproven) feature that the absence of data can be accommodated through taking infinite variance of that data's measurement noise. This is used to prove, by construction, that the N-step-ahead feedforward controller with forecast information can be precisely embedded within the $(N + k)$-step-ahead feedforward (for $k \geq 0$) with the final k steps having infinite variance measurement noise. This approach also simply addresses the presence of both previewed and unpreviewed disturbances acting on a controlled system.

1.1 Literature

Feedforward or preview control deals with the application of measurements in advance of a disturbance process impinging on a regulated system. These advance measurements are incorporated into the feedback control signal to aid in the rejection of the effects of the disturbance. At its core, feedforward deals with information in control. In this paper we explore this information structure in detail for the case of discrete-time Linear Quadratic Gaussian (LQG) or $\mathscr{H}_2$ control. The study is motivated by control issues in the so-called *Smart Grid*, such as demand response and consumption forecasting, where data from the grid and/or from the external environment (such as weather and irradiance) provide information regarding the demand. A feature of this data is that its quality often varies with horizon of availability. Thus, one-hour-in-advance weather predictions are inherently less reliable than five-minute-in-advance values. Our analysis seeks to explore how such data quality issues can be incorporated into the calculation of feedforward control and, more importantly, how their quality (or lack thereof) affects eventual regulation performance. In this fashion, the results should prove useful for examining the possible impact of capital expenditure on improving the quality of measurements in advance.

Technically, the paper demonstrates that, for LQG control, the information aspects are captured by the state estimator and hence both the feedforward control horizon and the data quality can be divorced from the state feedback gain calculation entirely. This separates the consideration of the informational data properties into just the development of the appropriate Kalman filter. Our approach is to demonstrate that the LQG feedforward control signal with horizon N can be constructed as that of horizon $M \geq N$ and with a related but distinct information structure; the state estimator changes, but all the feedback gains except the M^{th} value remain fixed and this terminal value (as noted by Hazell and Limebeer [2010]) tends to zero exponentially with M. Once this is established, the analysis of the effect of data quality on the performance of LQG feedforward control can take place through the analysis of the state estimator alone.

We rely on four recent lucid papers dealing with the formulation of feedforward and preview control for: LQG systems Hazell and Limebeer [2010] and Roh and Park [1999], Model Predictive Control for linear systems with constraints Carrasco and Goodwin [2011], and $\mathscr{H}_\infty$ control Hazell and Limebeer [2008]. Each of these papers provides a survey of the literature in the field and we shall draw from particularly Hazell and Limebeer [2010] for the underlying problem formulation. Since we are dealing with linear systems, we omit the consideration of reference tracking aspects. For simplicity, the term *feedforward control* will refer to a control law which relies on both a feedforward measurement of the disturbance signal and a feedback measurement from the output of the plant.

The paper unfurls as follows. The underlying problem statement is developed in Section 2, where we adopt the construction from Roh and Park [1999] and Hazell and Limebeer [2010] where the plant model is augmented by a set of delay elements acting on the disturbance before it reaches the plant output. In section 3, feedforward data is incorporated into a standard LQG design and analysis through the provision of measurements from some of the upstream delays. In this section we also describe the informational properties of the feedforward data, notably that of multiple-horizon forecast data, through the incorporation of measurement noise processes into the LQG design. Unlike Hazell and Limebeer [2010], this covers both fedforward and unfedforward disturbance channels in the same breath. The formulation of LQ feedback gain, Kalman state estimator, and LQG performance follow directly. Section 4 constructs the explicit solution of the state feedback gain matrix of feedforward control for horizon N and specializes and improves the solution properties from Hazell and Limebeer [2010]. This is followed in Section 5 by the explicit solution of the associated state estimator with a given information structure and culminates in the demonstration that the N-step-ahead feedforward control signal can be generated by the M-step-ahead solution with any $M \geq N$ and the appropriate information structure. This is the core theoretical analysis of the paper and permits the restriction of the consideration of information structure for feedforward to the design of the state estimator alone. Section 6 provides a brief numerical example.

The contribution of this paper is to provide the analysis of informational aspects of feedforward control, exploiting forecast data, which were not evident in the solution derived in Hazell and Limebeer [2010], where the dependence of disturbance rejection control on the feedforward horizon of exact disturbance data is the prime focus and were not explored in Roh and Park [1999], where only a single noise-corrupted disturbance value is fedforward, as opposed to an entire forecast affected by additive noises of possibly non-uniform covariances. For this current paper, the effect of the data quality of a forecast measurement is of paramount interest. This has not been studied earlier and provides insight into the performance effects of improvements in feedforward data. This information-centric view of feedforward control admits new insights into the solution structure and into the role of the horizon, notably with the application of imperfect forecast data.

2. PROBLEM DESCRIPTION

The system depiction in Figure 1 below contains three subsystems: the plant G, the disturbance model G_d, and the disturbance delay line block G_Δ with discrete-time state-space realizations

$$G := \left[\begin{array}{c|cc} A & B & I \\ \hline C & 0 & 0 \end{array}\right], \quad G_d := \left[\begin{array}{c|c} A_d & B_d \\ \hline C_d & 0 \end{array}\right], \quad G_\Delta := \left[\begin{array}{c|c} A_\Delta & B_\Delta \\ \hline C_\Delta & 0 \\ I & 0 \end{array}\right],$$

where the disturbance model G_d is assumed to be stable.

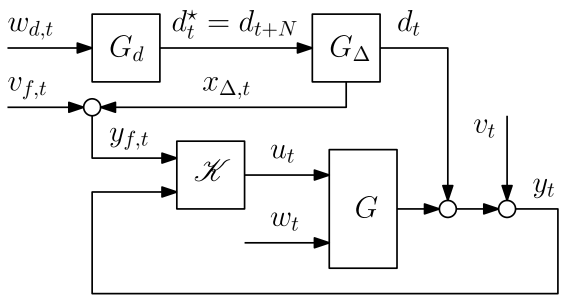

Fig. 1. A feedforward regulator problem with forecast data.

All noises are assumed stationary. The plant output is corrupted by an additive disturbance $d_t \in \mathbb{R}^p$ and an unmeasured, additive measurement noise $v_t \in \mathbb{R}^p$. This new output, $y_t \in \mathbb{R}^p$, is then fedback to controller block $\mathscr{K}$, which also contains an estimator. Future disturbance $d_t^\star = d_{t+N}$ is the result of Gaussian white noise $w_{d,t} \in \mathbb{R}^{m_d}$ feeding into the known system G_d. The current disturbance d_t is generated when $d_t^\star$ is fed into the N-step delay block G_Δ, the state of which $x_{\Delta,t} \in \mathbb{R}^{Np}$ is the sequence of current and future disturbances up to horizon length N.

$$x_{\Delta,t} = \begin{bmatrix} d_t^T & d_{t+1}^T & \dots & d_{t+N-1}^T \end{bmatrix}^T.$$

At time t, a preview or forecast of the disturbance, d_{t+n}, is available for $n = 0, 1, \dots, N-1$ in the form of $y_{f,t} \in \mathbb{R}^{Np}$. The reliability of this forecast diminishes with the advancing horizon of the data, i.e. with increasing n. This is incorporated into our model through the inclusion of additive measurement noise $v_{f,t} \in \mathbb{R}^{Np}$ onto the forecast signal , which includes the whole state of the delay line instead of its input signal d_{t+N} as is typically done in preview control.

Hence, the preview signal available to the controller, $\mathscr{K}$, is

$$y_{f,t} = x_{\Delta,t} + v_{f,t},$$

where $v_{f,t}$ is assumed zero mean, white, and Gaussian with

$$\text{cov}(v_{f,t}) = \text{blockdiag}\left[V_{f,0}, V_{f,1}, \dots, V_{f,N-1}\right], \qquad (1)$$

and $V_{f,j} \in \mathbb{R}^{p \times p}$.

$$V_{f,0} \leq V_{f,1} \leq \dots \leq V_{f,N-1}. \qquad (2)$$

The delay structure is captured by taking

$$A_\Delta = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & I \\ 0 & 0 & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{Np \times Np}, \quad B_\Delta = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix} \in \mathbb{R}^{Np \times p},$$

$$C_\Delta = [I \;\, 0 \; \cdots \; 0] \in \mathbb{R}^{p \times Np},$$

with $I, 0 \in \mathbb{R}^{p \times p}$. We denote the state of the plant G and disturbance model G_d as $x_t \in \mathbb{R}^n$ and $x_{d,t} \in \mathbb{R}^{n_d}$ respectively. $w_t \in \mathbb{R}^n$ is an unmeasured, additive process noise on the plant.

Our approach, as in Hazell and Limebeer [2010], is to apply Linear Quadratic Gaussian (LQG) control to this problem and to develop the controller information architecture by studying the separation into optimal state-variable feedback for a given performance criterion and the optimal state estimation. Specifically, we demonstrate that the LQG solution for this problem possesses an underlying structure where the entire informational aspects of the control reside completely within the estimator design and the state-variable feedback remains fixed. This fixed decomposition holds even when the horizon of the forecast changes. While separation is a well understood aspect of LQG, the existence of a horizon-independent decomposition is new and extends the work of Hazell and Limebeer to provide a distinct analysis of the information aspects of performance.

2.1 Development of the Augmented System

We develop the augmented model for this system by defining the augmented state $\mathcal{X}_t$, exogenous process noise $\mathcal{W}_t$, output $\mathcal{Y}_t$, and measurement noise $\mathcal{V}_t$ as follows:

$$\mathcal{X}_t = \begin{bmatrix} x_t \\ x_{\Delta,t} \\ x_{d,t} \end{bmatrix}, \qquad \mathcal{W}_t = \begin{bmatrix} w_t \\ 0 \\ B_d w_{d,t} \end{bmatrix}, \qquad \mathcal{V}_t = \begin{bmatrix} v_t \\ v_{f,t} \end{bmatrix}, \qquad \mathcal{Y}_t = \begin{bmatrix} y_t \\ y_{f,t} \end{bmatrix}. \tag{3}$$

Using these signals, the augmented system has a state variable realization given by

$$\begin{aligned} \mathcal{X}_{t+1} &= \mathcal{A}\mathcal{X}_t + \mathcal{B}u_t + \mathcal{W}_t, \\ \mathcal{Y}_t &= \mathcal{C}\mathcal{X}_t + \mathcal{V}_t, \end{aligned} \tag{4}$$

where

$$\mathcal{A} = \begin{bmatrix} A & 0 & 0 \\ 0 & A_\Delta & B_\Delta C_d \\ 0 & 0 & A_d \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} B \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} C & C_\Delta & 0 \\ 0 & I & 0 \end{bmatrix}.$$

We will also define the set of measurements from time zero to t as $\mathcal{Y}^t := \{\mathcal{Y}_0, \ldots, \mathcal{Y}_t\}$. We note that although the augmented system is uncontrollable from u_t, it is stabilizable since the disturbance model G_d is assumed stable and is non-interacting by construction. The above augmented system is now in the appropriate form to synthesize the LQG controller and estimator pair directly.

3. THE LQG PROBLEM, SOLUTION, AND PERFORMANCE

The augmented system (4) describes the feedforward problem with N-step-ahead forecast disturbance measurements corrupted by noise capturing their reliability via (1-2). The complete state of the delay line, $x_{\Delta,t}$, is available to the controller but corrupted by zero-mean measurement noise $v_{f,t}$. By manipulating the covariance structure of this noise, we are able to accommodate the forecast data properties. We do this by posing an LQG optimal feedback control design problem, whose solution comprises linear state estimate feedback.

The criterion to be optimized in this disturbance rejection problem is

$$J = \lim_{T\to\infty} \frac{1}{T} E\left\{ \sum_{i=0}^{T-1} \{y_i^T Q y_i + u_i^T R u_i\} \right\}, \tag{5}$$

where Q and R are user-specified performance penalty matrices which determine the disturbance rejection problem.

To complete the infinite-horizon stochastic LQG design, we also specify

$$\mathcal{Q} = \begin{bmatrix} C^T \\ C_\Delta^T \\ 0 \end{bmatrix} Q \,[C \;\; C_\Delta \;\; 0],$$

$$\mathscr{W} = \text{cov}(\mathcal{W}_t) = \begin{bmatrix} W & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & W_d \end{bmatrix}, \quad \mathscr{V} = \text{cov}(\mathcal{V}_t) = \begin{bmatrix} V & 0 \\ 0 & V_f \end{bmatrix}.$$

We make the following assumptions to ensure the existence of a stabilizing optimal control.

(A1) The matrix A_d has all its eigenvalues strictly inside the unit circle and $[A, B]$ is stabilizable. This, together with the property that, by construction, the eigenvalues of A_Δ are all zero, ensures that $[\mathcal{A}, \mathcal{B}]$ is stabilizable.
(A2) The LQ state penalty matrix $Q \geq 0$ and the control penalty matrix $R > 0$.
(A3) The pair $[A, Q^{1/2}C]$ is detectable, which together with (A1) implies that $[\mathcal{A}, \mathcal{Q}^{1/2}\mathcal{C}]$ is also detectable. It also implies that $[A, C]$ and therefore $[\mathcal{A}, \mathcal{C}]$ are detectable.
(A4) The covariances satisfy $V > 0, V_f > 0, W \geq 0, W_d \geq 0$.
(A5) The pair $[A, W^{1/2}]$ is stabilizable, which implies that $[\mathcal{A}, \mathscr{W}]$ is stabilizable.

Under these assumptions, the LQG optimal control [1] for system (4) with criterion (5) is given by

$$\hat{\mathcal{X}}_{t+1|t} = (\mathcal{A} - \mathcal{B}\mathcal{K} - \mathcal{L}\mathcal{C})\hat{\mathcal{X}}_{t|t-1} + \mathcal{L}\mathcal{Y}_t, \tag{6}$$

$$u_t = -\mathcal{K}\hat{\mathcal{X}}_{t|t-1}. \tag{7}$$

Since $\mathcal{Y}_t = \begin{bmatrix} y_t^T & y_{f,t}^T \end{bmatrix}^T$, this implements both the feedback and feedforward control, where the feedforward control potentially uses the disturbance predictions from every horizon up to $N-1$. The state-feedback matrix, $\mathcal{K}$, and the output-injection matrix, $\mathcal{L}$, are computed by solving the two algebraic Riccati equations

$$\mathcal{P} = \mathcal{A}^T\mathcal{P}\mathcal{A} - \mathcal{A}^T\mathcal{P}\mathcal{B}(\mathcal{B}^T\mathcal{P}\mathcal{B} + R)^{-1}\mathcal{B}^T\mathcal{P}\mathcal{A} + \mathcal{Q}, \tag{8}$$

$$\Sigma = \mathcal{A}\Sigma\mathcal{A}^T - \mathcal{A}\Sigma\mathcal{C}^T(\mathcal{C}\Sigma\mathcal{C}^T + \mathscr{V})^{-1}\mathcal{C}\Sigma\mathcal{A}^T + \mathscr{W}, \tag{9}$$

for $\mathcal{P}$ and Σ and then taking

$$\begin{aligned} \mathcal{K} &= (\mathcal{B}^T\mathcal{P}\mathcal{B} + R)^{-1}\mathcal{B}^T\mathcal{P}\mathcal{A}, \\ \mathcal{L} &= \mathcal{A}\Sigma\mathcal{C}^T(\mathcal{C}\Sigma\mathcal{C}^T + \mathscr{V})^{-1}. \end{aligned} \tag{10}$$

Closed-Loop LQG Performance

Denote the augmented state-estimate error as

$$\mathcal{E}_t = \mathcal{X}_t - \hat{\mathcal{X}}_{t|t-1}.$$

Substituting the control input signal u_t from (7) into the augmented system dynamics in (4) and into the augmented state estimate dynamics from (6), the closed-loop system can be written as

$$\begin{bmatrix} \mathcal{X}_{t+1} \\ \mathcal{E}_{t+1} \end{bmatrix} = \begin{bmatrix} (\mathcal{A} - \mathcal{B}\mathcal{K}) & \mathcal{B}\mathcal{K} \\ 0 & (\mathcal{A} - \mathcal{L}\mathcal{C}) \end{bmatrix} \begin{bmatrix} \mathcal{X}_t \\ \mathcal{E}_t \end{bmatrix} + \begin{bmatrix} \mathcal{W}_t \\ \mathcal{W}_t - \mathcal{L}\mathcal{V}_t \end{bmatrix}.$$

The closed-loop covariance matrix can be found by solving the following Lyapunov equation:

$$\begin{aligned} &\begin{bmatrix} E\{\mathcal{X}_t\mathcal{X}_t^T\} & E\{\mathcal{X}_t\mathcal{E}_t^T\} \\ E\{\mathcal{E}_t\mathcal{X}_t^T\} & E\{\mathcal{E}_t\mathcal{E}_t^T\} \end{bmatrix} \\ &= \begin{bmatrix} (\mathcal{A} - \mathcal{B}\mathcal{K}) & \mathcal{B}\mathcal{K} \\ 0 & (\mathcal{A} - \mathcal{L}\mathcal{C}) \end{bmatrix} \begin{bmatrix} E\{\mathcal{X}_t\mathcal{X}_t^T\} & E\{\mathcal{X}_t\mathcal{E}_t^T\} \\ E\{\mathcal{E}_t\mathcal{X}_t^T\} & E\{\mathcal{E}_t\mathcal{E}_t^T\} \end{bmatrix} \\ &\times \begin{bmatrix} (\mathcal{A} - \mathcal{B}\mathcal{K}) & \mathcal{B}\mathcal{K} \\ 0 & (\mathcal{A} - \mathcal{L}\mathcal{C}) \end{bmatrix}^T + \text{cov}\left(\begin{bmatrix} \mathcal{W}_t \\ \mathcal{W}_t - \mathcal{L}\mathcal{V}_t \end{bmatrix} \right). \end{aligned}$$

Using the properties of the trace of a matrix and noting that the covariance of the estimation error is simply the Kalman filter covariance, $E\{\mathcal{E}_t\mathcal{E}_t^T\} = \Sigma$, it can be verified that the closed-loop LQG performance criterion can be computed as

$$\begin{aligned} J = \text{trace}\,\big((&\mathcal{Q} + \mathcal{K}^T R\mathcal{K})E\{\mathcal{X}_t\mathcal{X}_t^T\} \\ &+ \mathcal{K}^T R\mathcal{K}(\Sigma - E\{\mathcal{X}_t\mathcal{E}_t^T\} - E\{\mathcal{E}_t\mathcal{X}_t^T\})\big) + \text{trace}\,(QV). \end{aligned} \tag{11}$$

For the remainder of this paper, we shall refer to the above problem as the *forecast feedforward control problem*. In the subsequent sections, we refine the structures of $\mathcal{K}$ and $\mathcal{L}$,

[1] We have chosen, for notational clarity only, to use the LQG optimal control with the prediction estimator even though the plant itself is strictly proper and the filtered estimator is truly optimal. See Ishihara and Takeda [1986]

extending the observations of Hazell and Limebeer [2010] concerned with "efficient computation," which in turn extended those of Bitmead et al. [1990] for the control only. Then the detailed structure of V_f, the feedforward measurement noise covariance, will be introduced, since this is the embodiment of the problem's *Information Structure*.

4. STRUCTURE OF THE STATE-ESTIMATE FEEDBACK GAIN AND FILTER GAIN

Before we can analyze the information structure of the forecast feedforward control problem, we must first apply an important result from Hazell and Limebeer [2010] which gives explicit formulæ for the augmented control gain $\mathcal{K}$ and its constituent components. Prior to this, we develop an appropriate partition. After establishing the efficient computation of $\mathcal{K}$ in the context of this paper, we summarize why the analogous results, from Hazell and Limebeer [2010], for the efficient computation of $\mathcal{L}$ and Σ do not apply to the case of an uncertain forecast as the feedforward measurement.

Consider the conformable partitioning of the augmented state-estimate feedback gain $\mathcal{K}$ for multiplication by the augmented state estimate $\hat{\mathcal{X}}_{t|t-1}$ (which inherits its partition from (3)) as

$$\mathcal{K} = [K \;\; K_{(0)} \;\; K_{(1)} \;\ldots\; K_{(N-1)} \;\; K_{dN}], \tag{12}$$

where, to form controller u_t, $-K$ multiplies the estimate of plant state x_t, $-K_n$, with $0 \leq n \leq N-1$, multiplies the estimate of the delay line state $x_{\Delta,t}$, and $-K_{dN}$ multiplies the estimated disturbance model state $x_{d,t}$.

We partition the associated algebraic Riccati equation solution $\mathcal{P}$ from (8) in similar fashion as

$$\mathcal{P} = \begin{bmatrix} P & P_\Delta & P_d \\ P_\Delta^T & P_{\Delta\Delta} & P_{\Delta d} \\ P_d^T & P_{\Delta d}^T & P_{dd} \end{bmatrix}. \tag{13}$$

4.1 Efficient Computation for $\mathcal{P}$ and $\mathcal{K}$ of Hazell and Limebeer

The following theorem is the direct application of the results of Hazell and Limebeer [2010] in the context of the forecast feedforward control problem, and is therefore stated without proof.

Theorem 1. (Efficient Formulae for $\mathcal{K}$ and $\mathcal{P}$). Define the matrix A_c as

$$A_c = A - BK.$$

The components of the augmented, unique, stabilizing control gain $\mathcal{K}$ in (10), with $\mathcal{P}$ partitioned as in (13), are given by

$$K = (B^T P B + R)^{-1} B^T P A, \tag{14}$$

$$K_\Delta = (B^T P B + R)^{-1} B^T \times \left[0 \;\; (C^T Q) \;\; (A_c^T C^T Q) \;\ldots\; (A_c^{T^{N-2}} C^T Q)\right], \tag{15}$$

$$K_{dN} = \left(B^T P B + R\right)^{-1} B^T \left(A_c^{T^{N-1}} C^T Q C_d + A_c^{T^N} \sum_{j=0}^{\infty} \left\{A_c^{T^j} \left(C^T Q C_d\right) A_d^j\right\} A_d\right). \tag{16}$$

where P, the solution to the discrete-time algebraic Riccati equation (DARE) associated with pure feedback to the plant state, is found by solving

$$P = A^T P A - A^T P B (B^T P B + R)^{-1} B^T P A + C^T Q C.$$

We offer the following comments.

- The plant state-estimate feedback control gain, K, and its associated algebraic Riccati equation solution, P, are unaffected by the presence of feedforward.
- The first block element of K_Δ is zero because, in our predictor feedback case, it represents a past disturbance, which is subsumed by current and future information.
- As noted by Hazell and Limebeer, as the feedforward horizon $N \to \infty$, the tail elements ofK_Δ and K_{dN} tend to zero exponentially fast. Likewise, the block elements of $\mathcal{K}$ apart from the tail elements K_{dN} are fixed. Thus, as $N \to \infty$ the feedback control gain tends to a semi-infinite block vector, which is simply precomputed.

In Hazell and Limebeer [2010], the feedforward signal is provided exactly, i.e. is noise free, and hence $V_f = 0$ above. In this case the combined state estimate covariance Σ in (9) is zero except for the upper left corner corresponding to the plant state estimate. Accordingly, Hazell et al. are able to derive a structure of $\mathcal{L}$ which mirrors with that of $\mathcal{K}$, the feedback gain matrix. This yields their "computationally efficient" approach to calculating these feedforward estimator gain values.

When the feedforward noise, V_f, is not zero, this efficiency disappears and the elements of the state estimate, the plant state estimate and the disturbance/delay state estimate, are coupled through the output measurement y_t. That being said, the case of $V_f > 0$ is precisely that dealing with forecast data and at this point our treatment deviates significantly from that of Hazell and Limebeer [2010]. Indeed, the effect of feedforward reliability is our prime focus, since it encapsulates the information structure of the problem. We next develop analytical tools for this.

5. NESTING OF FEEDFORWARD CONTROLLERS

Consider two forecast feedforward problems with respective forecast horizon lengths N and M, where we take $M > N$. Our aim in this section is to demonstrate that the N-horizon feedforward controller is identical to the M-horizon feedforward controller with the information structure with

$$V_{f,N} = V_{f,N+1} = \cdots = V_{f,M-1} = \xi I,$$

for $\xi > 0$, so that as $\xi \to \infty$, these noise variances become infinite.

First, we need a lemma concerned with jointly Gaussian processes with possibly infinite covariances.

Lemma 1. Consider the jointly Gaussian distributed random vectors

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} m_X \\ m_Y \\ m_Z \end{bmatrix}, \begin{bmatrix} \Sigma_X & \Sigma_{XY^T} & \Sigma_{XZ^T} \\ \Sigma_{YX^T} & \Sigma_Y & \Sigma_{YZ^T} \\ \Sigma_{ZX^T} & \Sigma_{ZY^T} & \Sigma_Z \end{bmatrix}\right),$$

and let $\Sigma_Z = \xi I$, with $\xi > 0$. Then

$$\lim_{\xi\to\infty} \mathrm{E}\{X|Y=y, Z=z\} = E\{X|Y=y\},$$
$$\lim_{\xi\to\infty} \mathrm{cov}(X|Y=y, Z=z) = \mathrm{cov}(X|Y=y).$$

Proof: For jointly Gaussian random variables

$$\begin{bmatrix} A \\ B \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} m_A \\ m_B \end{bmatrix}, \begin{bmatrix} \Sigma_A & \Sigma_{AB^T} \\ \Sigma_{BA^T} & \Sigma_B \end{bmatrix}\right),$$

$$\mathrm{E}\{A|B=b\} = m_A + \Sigma_{AB^T}\Sigma_B^{-1}(b - m_B),$$
$$\mathrm{cov}(A|B=b) = \Sigma_A - \Sigma_{AB^T}\Sigma_B^{-1}\Sigma_{BA^T}.$$

We apply this result with $A = X$ and $B = \left[Y^T \;\; Z^T\right]^T$.

Using $\Delta = A - BD^{-1}C$,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} \Delta^{-1} & -\Delta^{-1}BD^{-1} \\ -D^{-1}C\Delta^{-1} & D^{-1}(D + C\Delta^{-1}B)D^{-1} \end{bmatrix}.$$

Whence, as $\xi \to \infty$, $\Sigma_Z^{-1} \to 0$ yielding

$$\begin{bmatrix} \Sigma_Y & \Sigma_{YZ^T} \\ \Sigma_{ZY^T} & \Sigma_Z \end{bmatrix}^{-1} = \begin{bmatrix} \Sigma_Y^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Applying these results to the conditional expectation $\mathrm{E}\{X|Y = y, Z = z\}$ as $\xi \to \infty$ produces the result. $\square$

We now proceed to the main result of this paper, which demonstrates the information structure of the forecast feedforward control problem.

Theorem 2. (Data Nesting Property). Consider two forecast feedforward control problems with the same plant model, disturbance model and driving noise covariance, LQ criterion, and plant process and measurement noise covariances,

$$[A, B, C, A_d, B_d, C_d, W_d, Q, R, W, V],$$

but with differing forecast horizon lengths N and M with $N \leq M$. Let the feedforward information structure of the N-horizon problem be described by the forecast noise covariance

$$\mathrm{cov}(v_{f,t}^N) = \mathrm{blockdiag}[V_{f,0}, \dots, V_{f,N-1}] \tag{17}$$

and result in control signal $\{u_{N,t} : t \geq 0\}$. Further, let the feedforward information structure of the M-horizon problem be described by forecast noise covariance

$$\mathrm{cov}(v_{f,t}^M) = \mathrm{blockdiag}[V_{f,0}, \dots, V_{f,N-1}, \underbrace{\xi I, \dots, \xi I}_{M-N \text{ terms}}], \tag{18}$$

with $\xi > 0$ and corresponding control signal $\{u_{M,t} : t \geq 0\}$. Then,

$$u_{N,t} = \lim_{\xi\to\infty} u_{M,t}, \quad \text{for all } t \geq 0.$$

Proof: The case where $N = M$ is trivially satisfied: assume $M > N$. From (12) the N-horizon control gain is given by

$$\mathcal{K}^N = [K \;\; K_{(0)} \;\; \dots \;\; K_{(N-1)} \;\; K_{dN}],$$

and the M-horizon control gain is given by

$$\mathcal{K}^M = [K \;\; K_{(0)} \;\; \dots \;\; K_{(N-1)} \;\; K_{(N)} \;\; \dots \;\; K_{(M-1)} \;\; K_{dM}],$$

From (15-16), the first $N + 1$ block elements of these control gains are identical (i.e. up to $K_{(N-1)}$).

These control gains multiply the following augmented state estimates. For the N-horizon problem,

$$\hat{\mathcal{X}}_{t|t-1}^N = \begin{bmatrix} \mathrm{E}\{x_t|\mathcal{Y}_N^{t-1}\} \\ \mathrm{E}\{d_t|\mathcal{Y}_N^{t-1}\} \\ \vdots \\ \mathrm{E}\{d_{t+N-1}|\mathcal{Y}_N^{t-1}\} \\ \mathrm{E}\{x_{d,t}^N|\mathcal{Y}_N^{t-1}\} \end{bmatrix},$$

and for the M-horizon problem, as $\xi \to \infty$,

$$\hat{\mathcal{X}}_{t|t-1}^M = \begin{bmatrix} \mathrm{E}\{x_t|\mathcal{Y}_M^{t-1}\} \\ \mathrm{E}\{d_t|\mathcal{Y}_M^{t-1}\} \\ \vdots \\ \mathrm{E}\{d_{t+N-1}|\mathcal{Y}_M^{t-1}\} \\ \mathrm{E}\{d_{t+N}|\mathcal{Y}_M^{t-1}\} \\ \vdots \\ \mathrm{E}\{d_{t+M-1}|\mathcal{Y}_M^{t-1}\} \\ \mathrm{E}\{x_{d,t}^M|\mathcal{Y}_M^{t-1}\} \end{bmatrix} \to \begin{bmatrix} \mathrm{E}\{x_t|\mathcal{Y}_N^{t-1}\} \\ \mathrm{E}\{d_t|\mathcal{Y}_N^{t-1}\} \\ \vdots \\ \mathrm{E}\{d_{t+N-1}|\mathcal{Y}_N^{t-1}\} \\ \mathrm{E}\{d_{t+N}|\mathcal{Y}_N^{t-1}\} \\ \vdots \\ \mathrm{E}\{d_{t+M-1}|\mathcal{Y}_N^{t-1}\} \\ \mathrm{E}\{x_{d,t}^M|\mathcal{Y}_N^{t-1}\} \end{bmatrix},$$

where we have invoked Lemma 1 and (17) to alter the conditioning and a notational convenience has been used to indicate that the disturbance state $x_{d,t}$ is further in the future for the M problem versus the N problem.

Now note that the latter $M - N$ elements in $\hat{\mathcal{X}}_{t|t-1}^M$ are predictions of the disturbance process without the aid of further feedforward measurements. Thus, as $\xi \to \infty$,

$$\mathrm{E}\{d_{t+N+j}|\mathcal{Y}_N^{t-1}\} = C_d A_d^j \mathrm{E}\{x_{d,t}^N|\mathcal{Y}_N^{t-1}\},$$

for $j \geq 0$. Further,

$$\mathrm{E}\{x_{d,t}^M|\mathcal{Y}_N^{t-1}\} = A_d^{M-N} \mathrm{E}\{x_{d,t}^N|\mathcal{Y}_N^{t-1}\}.$$

With these observations, plus the formulæ (15-16), it is apparent that

$$u_{N,t} = -\mathcal{K}^N \hat{\mathcal{X}}_{t|t-1}^N = -\mathcal{K}^M \lim_{\xi\to\infty} \hat{\mathcal{X}}_{t|t-1}^M = \lim_{\xi\to\infty} u_{M,t}. \quad \square$$

6. NUMERICAL EXAMPLE

We consider three variants of the feedforward control problem which illustrate the effect of information structure on the closed-loop performance and permit comparison to Hazell and Limebeer [2010]. The plant and disturbance models are characterized by the following.

$$A = \begin{bmatrix} 1 & 0.0.7869 \\ 0 & 0.6065 \end{bmatrix}, \; B = \begin{bmatrix} 0.4261 \\ 0.7869 \end{bmatrix}, \; C = [1 \;\; 0],$$
$$A_d = 0.999, \quad B_d = 0.5, \quad C_d = 0.4,$$
$$W = 0.001\, I_2, \quad V = 0.001, \quad W_d = 1,$$
$$Q = 10, \quad R = 0.001.$$

We note that this plant-disturbance pair exhibits low observability in that an eigenvalue of the plant dynamics A is relatively close to the eigenvalue of the disturbance dynamics A_d, reflecting the amplified benefits of disturbance feedforward. We take feedforward horizon $N = 10$ and consider the following three cases, plotted in Figure 2.

1. (Almost) perfect feedforward: $V_{f,i} = 10^{-6}, \; i = 1, \dots, 10$. This corresponds to a disturbance feedforward noise level which is negligible in terms of its effect on closed-loop performance. This is the case studied in Hazell and Limebeer [2010].

2. Moderate reliability: $V_{f,i} = 0.01 \times 1.85^i$.

3. Diminished reliability: $V_{f,i} = 0.01 \times 2.3^i$.

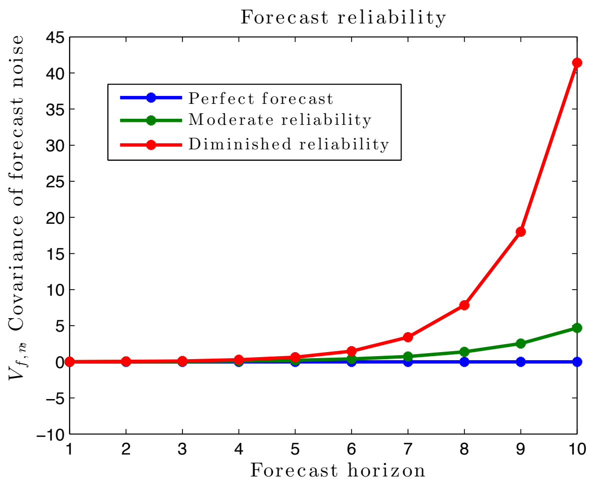

Fig. 2. Forecast noise covariances for the three cases: no noise, moderate reliability and diminished reliability.

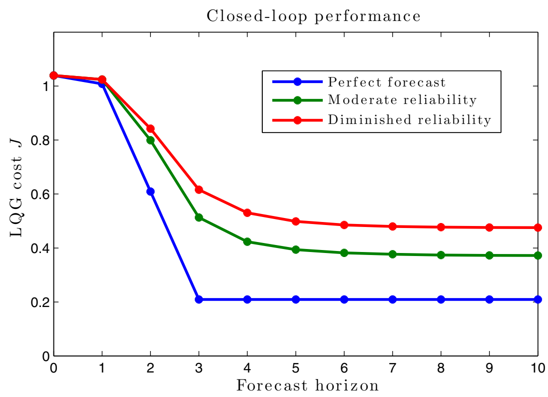

Fig. 3. Closed-loop performances.

Figure 3 depicts the closed-loop performance achieved with feedforward control for the three information structures.

All three curves demonstrate the performance benefit from feedforward and are: monotonically non-increasing in horizon n, and convergent to a limiting value as the horizon becomes large. The closed-loop performance without feedforward for each is identical. Further, they demonstrate the role played by forecast data reliability. Improved reliability is linked to improved performance at every horizon and therefore in the limit.

7. CONCLUSION

We have developed the analysis of a feedforward control design problem which incorporates the information structure associated with longer term imperfect forecast data having diminished reliability. This involved the inclusion of measurement variances into the standard $\mathscr{H}_2$ feedforward design of Hazell and Limebeer [2010] and the use of multiple-horizon forecasts for the disturbance acting on the system, extending the single-valued stochastic feedforward measurement of Roh and Park [1999]. We found that the LQ control gain $\mathcal{K}$ is completely unaltered by, and thus independent of, the inclusion of forecast data quality into the feedforward problem, and thus could still be calculated by means of the efficient computation derived in Hazell and Limebeer [2010]. This observation revealed that the improvements in performance due to the reliability of feedforward are more specifically a consequence of an improved state estimate. Although the inherently coupled structure of the augmented Kalman estimator gain and covariance prevented their efficient computation, we instead showed that a horizon-N forecast feedforward problem can always be equivalently considered to be a horizon $N + k$ problem by assigning the final k forecast covariances to be infinite, thereby elucidating the information structure inherent in the forecast feedforward control problem.

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