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A Norm-Bounded robust MPC strategy with partial state measurements

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Abstract: In this paper a robust MPC scheme based on a partial-state availability is developed for uncertain discrete-time linear systems described by structured norm-bounded model uncertainties and subject to saturation and rate of variation constraints. The algorithm is based on the minimization, at each time instant, of a semi-definite convex optimization problem subject to Linear Matrix Inequalities (LMI) feasibility constraints which are derived by a judicious use of *S-Procedure* arguments. Numerical comparisons with competitor algorithms are finally reported by dealing with the control augmentation problem of an High Altitude Performance Demonstrator (HAPD) unmanned aircraft with redundant control surfaces.

Keywords: Constrained control, Norm-Bounded descriptions, aircraft control, output feedback.

1. INTRODUCTION

Model predictive control (MPC) has been widely applied, especially in the process industries, because of its ability to efficiently handle hard constraints. While output feedback is inevitably employed in practice, much of the literature is confined to the full-state case in which it is assumed that the system state is exactly known, see [1]-[4] and references therein. When output feedback MPC issues are concerned, the first step is to estimate the system state with the unavoidable consequence that a special type of uncertainty is introduced: the state estimation error. Because the optimality cannot be achieved with reasonable computational burdens, compromises have to be made.
The standard approach consists in designing an observer to reconstruct the (partially) unknown state and its estimate is exploited for regulation purposes. Along these lines, contributions on output-feedback MPC share as a common denominator the stability of the augmented system (observer and moving horizon controller). In [5], a moving horizon observer was developed for the model uncertainty free case set of past input/output data. In [6], an output feedback MPC scheme based on a dual mode approach has been proposed for nominal linear discrete-time plants, subject to input constraints and bounded disturbance/measurement noises. Other important contributions have been proposed in [7]-[11] where robust output feedback MPC schemes for Polytopic/Norm- Bounded uncertain linear discrete-time plants have been taken into consideration. In [12], the overall controller consists of two components, a stable state estimator and a tube-based, robustly stabilizing model predictive controller with the estimator allowed to be time varying. Finally in [13], a constrained output feedback MPC scheme for uncertain Norm-Bounded discrete-time linear systems is presented which extends a full-state availability strategy [2] achieved by the authors to the more interesting case of incomplete and noisy state information. In this paper we extend the output feedback receding horizon control framework presented in [14] by adding N free control moves to the action of the primal controller (*frozen* approach) which are computed by solving a constrained optimization problem whose numerical complexity grows up only linearly with the control horizon N.

2. PROBLEM FORMULATION

We consider the following discrete-time linear system with uncertainties appearing in the feedback loop:

$$\begin{aligned} x(t+1) &= \Phi x(t) + Gu(t) + B_p p(t) \\ y(t) &= Cx(t) \\ q(t) &= C_q x(t) + D_q u(t) \\ p(t) &= \Delta(t) q(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ is the state vector, $u \in \mathbb{R}^{n_u}$ is the control input vector and $p,\ q \in \mathbb{R}^{n_p}$ are additional variables which account for the uncertainty, with the operator Δ such that $||\Delta||_2 = \bar{\sigma}(\Delta) \leq 1$. In the sequel we shall assume that the system state is partially available, i.e. $y \in \mathbb{R}^{n_y}$ and $C = [I_{n_y}\ 0_{n_x - n_y}]$, and the following component-wise constraints are prescribed:

$$\begin{aligned} u(t) \in \Omega_u,\ \Omega_u \triangleq \{u \in \mathbb{R}^{n_u} \ :\ |u_i(t+k|t)| \leq \bar{u}_{i,max}, \\ \forall k \geq 0, \bar{u}_{i,max} \in \mathbb{R}^+, i = 1..n_u\} \end{aligned} \tag{2}$$

$$\begin{aligned} u(t) \in \Omega_{\delta u} \subseteq \Omega_u, \Omega_{\delta u} \triangleq \{u \in \Omega_u : |u_i(t+k+1|t) - \\ u_i(t+k|t)| \leq \bar{\delta} u_{i,max}, \forall k \geq 0, \bar{\delta} u_{i,max} \in \mathbb{R}^+, i = 1..n_u\} \end{aligned} \tag{3}$$

$$\begin{aligned} x(t) \in \Omega_x,\ \Omega_x \triangleq \{x \in \mathbb{R}^{n_x} : |x_j(t+k|t)| \leq \bar{x}_{j,max} \\ \forall k \geq 0, \bar{x}_{j,max} \in \mathbb{R}^+, j = 1..n_x\} \end{aligned} \tag{4}$$

Then, we want to solve the **Constrained Output Feedback Stabilization (COFS) Problem -** Given the plant model (1), find an output feedback control strategy

$$u(\cdot) = g(y(\cdot)) \tag{5}$$

complying with the prescribed constraints (2)-(4) such that the closed-loop system is asymptotically stable. □

First recall that a receding horizon control (RHC) approach has been exploited in [14] to achieve a solution to the **COFS** problem without considering constraints on the rate of variation of the control input signal. There, the main result can be summarized as follows:

Theorem 1. Let $x(t) = [x_a^T(t)\ x_{na}^T(t)]^T$ be the current state with $x_a(t) := [I_{n_y}\ 0]x(t)$ the available information and $x_{na}(t)$ the unknown components. Let

$$D(S) \triangleq \left\{ x \in \mathbb{R}^{n_x} \mid x^T H^T S H x \leq 1,\ H = [0\, I_{n_x - n_y}] \right\} \quad (6)$$

be the convex set characterizing the constrained $x_{na}(t)$ entries. Then, the **COFS** problem with the control law given by

$$u(t+k|t) = K x_a(t+k|t),\ k \geq 0 \quad (7)$$

can be solved by the following SDP:

$$\min_{Q_1, Q_2, Y_1, X, \rho, \lambda, \tau > 0} \rho \quad (8)$$

subject to

$$\begin{bmatrix} \bar{Q} & Y^T R_u^{1/2} & \bar{Q} R_x^{1/2} & \bar{Q} C_q^T + Y^T D_q^T & \bar{Q}\Phi^T + Y^T G^T \\ R_u^{1/2} Y & \rho I_{n_u} & 0 & 0 & 0 \\ R_x^{1/2} \bar{Q} & 0 & \rho I_{n_x} & 0 & 0 \\ C_q \bar{Q} + D_q Y & 0 & 0 & \lambda I_{n_x} & 0 \\ \Phi \bar{Q} + G Y & 0 & 0 & 0 & \bar{Q} - \lambda B_p B_p^T \end{bmatrix} \geq 0 \quad (9)$$

$$\begin{bmatrix} X & Y \\ Y^T & \bar{Q} \end{bmatrix} \geq 0,\ X_{ii} \leq \bar{u}_{i,max}^2,\ i = 1, \ldots, n_u, \quad (10)$$

$$row_j(C)\bar{Q}row_j(C)^T \leq \bar{x}_{j,max}^2,\ j = 1, \ldots, n_y, \quad (11)$$

$$\begin{bmatrix} -Q_2^{-1} & 0 \\ 0 & 1 - x_a(t)^T Q_1^{-1} x_a(t) \end{bmatrix} - \tau \begin{bmatrix} -S & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \quad (12)$$

where

$$\bar{Q} = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, \quad Y(t) = [\,Y_1\ 0\,], \bar{P} = \rho\bar{Q}^{-1} \quad (13)$$

with $Q_1 \in \mathbb{R}^{n_y \times n_y}$ and $Q_2 \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)}$ positive definite symmetric matrices, $X \in \mathbb{R}^{n_u \times n_u}$ a symmetric matrix, $Y_1 \in \mathbb{R}^{n_u \times n_y}$, $K = Y_1 Q_1^{-1}$ and $row_j(C)$ is the *j-th* row of C □

A way to take care of the rate of variation on the control effort (3) consists in imposing $\bar{u}_{i,max} = \bar{\delta} u_{i,max}, i = 1, \ldots, n_u$, when solving the optimization (8)-(12). Though simple, this solution suffers of unavoidable conservative performance because a "small" Robust Positively Invariant (RPI) region

$$\bar{\zeta} = \{ x \in \mathbb{R}^{n_x} \mid x^T \bar{Q}^{-1} x \leq 1 \} = \{ x \in \mathbb{R}^{n_x} \mid x^T \bar{P} x \leq \rho \} \quad (14)$$

results. Then to overcome such a key drawback, the following proposition allows to enlarge the terminal RPI ellipsoidal set.

Proposition 1. Given the state feedback gain K computed as solution of the optimization (8)-(12) under the restriction $\bar{u}_{i,max} = \bar{\delta} u_{i,max}, i = 1 \ldots n_u$. Then the RPI ellipsoid

$$\zeta = \left\{ x \in \mathbb{R}^{n_x} \mid x^T P x \leq \rho,\ P = \begin{bmatrix} P_a & 0 \\ 0 & P_{na} \end{bmatrix} \right\}, \quad (15)$$

compatible with (2)-(3) and such that

$$\bar{\zeta} \subset \zeta \quad (16)$$

can be obtained by solving the following optimization problem

$$\min_{P, \lambda}\ log\ det\ P \quad (17)$$

subject to

$$\begin{bmatrix} \Phi_K^T P \Phi_K - P + \lambda C_K^T C_K & \Phi_K^T P B_p \\ * & B_p^T P B_p - P - \lambda I \end{bmatrix} \leq 0 \quad (18)$$

$$\lambda > 0, P > 0 \quad (19)$$

$$P \geq P_1^i, i = 1 \ldots n_u \quad (20)$$

$$P \geq P_2^i, i = 1 \ldots n_u \quad (21)$$

$$P \geq P_3 \quad (22)$$

$$\rho S \geq P_{na} \quad (23)$$

where $\Phi_K \triangleq \Phi + GK$, $C_K = C_q + D_q K$, $\bar{C}_i = row_i(K)C$, $\bar{\Phi}_{K,i} = \bar{C}_i \Phi_K$, $\bar{B}_{p,i} = \bar{C}_i B_p$, $P_1^i = \rho \cdot \bar{\delta} u_{i,max}^{-2}(\bar{\Phi}_{K,i}^T \bar{\Phi}_{K,i} + \bar{C}_i^T \bar{C}_i - 2\bar{\Phi}_{K,i}^T \bar{C}_i + \sigma_i C_K^T C_K + (\bar{B}_{p,i}^T \bar{C}_i - \bar{B}_{p,i}^T \bar{\Phi}_{K,i})^T - \bar{B}_{p,i}^T \bar{B}_{p,i} + (\sigma_i I)^{-1}(\bar{B}_{p,i}^T \bar{C}_i - \bar{B}_{p,i}^T \bar{\Phi}_{K,i}))$, $P_2^i = \rho \cdot \bar{u}_{i,max}^{-2} \bar{C}_i^T \bar{C}_i$, $P_3 = \rho \cdot diag\left[\bar{x}_{1,max}^{-2} \ldots \bar{x}_{n_x,max}^{-2}\right]$ and $row_i(K)$ is the *i-th* row of K.

Proof - Note that LMIs (18)-(19) refer to the conditions under which a linear state-feedback control law K is able to quadratically stabilize an uncertain linear system of the form (1), see [16] for details. The one-step control rate constraint (3) is recast in terms of the LMI (20) by resorting to a block-diagonal structure of the shaping matrix P. In fact by considering $\delta u = u(t+k+1|t) - u(t+k|t)$ and $\bar{x}_{k,t} = x(t+k|t)$, the constraints (3) can be rewritten as

$$\begin{bmatrix} p^T \\ 1 \end{bmatrix}^T \begin{bmatrix} -\bar{B}_{p,i}^T \bar{B}_{p,i} & (\bar{B}_{p,i}^T row_i(K) - \bar{B}_{p,i}^T \bar{\Phi}_{K,i})\bar{x}_{k,t} \\ * & \begin{matrix} \bar{\delta} u_{i,max}^2 - \bar{x}_{k,t}^T(\bar{\Phi}_{K,i}^T \bar{\Phi}_{K,i} + \\ + row_i(K)^T row_i(K) - 2\bar{\Phi}_{K,i}^T K_i)\bar{x}_{k,t} \end{matrix} \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \geq 0,$$
$$i = 1, \ldots, n_u, \quad (24)$$

that holds true for all p such that

$$[\ p^T\ \ 1\ ] \begin{bmatrix} -1 & 0 \\ * & \bar{x}_{k,t} C_K^T C_K \bar{x}_{k,t} \end{bmatrix} \begin{bmatrix} p \\ 1 \end{bmatrix} \geq 0 \quad (25)$$

Then, the implication

(24) holds for all p satisfying (25)

is valid via S-procedure arguments if and only if there exists a real $\sigma_i > 0, i = 1, \ldots, n_u$, such that the following LMI condition is satisfied

$$\begin{bmatrix} -\bar{B}_{p,i}^T \bar{B}_{p,i} + \sigma_i I & (\bar{B}_{p,i}^T K_i - \bar{B}_{p,i}^T \bar{\Phi}_{K,i})\bar{x}_{k,t} \\ * & \begin{matrix} \bar{\delta} u_{i,max}^2 - \bar{x}_{k,t}^T(\bar{\Phi}_{K,i}^T \bar{\Phi}_{K,i} + \\ + K_i^T K_i - 2\bar{\Phi}_{K,i}^T K_i - \sigma_i C_K^T C_K)\bar{x}_{k,t} \end{matrix} \end{bmatrix} \geq 0,$$
$$i = 1, \ldots, n_u. \quad (26)$$

A convenient choice of σ_i is shown in [2]. By using Schur complement, the LMI condition (26) becomes

$$\bar{x}_{k,t}^T P_1^i \bar{x}_{k,t} \leq 1$$

In order to define the invariant set ζ accounting for constraint (3), eqn.(20) must hold. Moreover, LMIs (21)-(22), accounting for input and state constraints (2) and (4) respectively, are obtained along the same lines. Finally, inequality (23) has the aim to fulfil the requirement on the *not-measurable* state component $x_{na}(\cdot)$. □

Remark 1 - All the S-procedure multipliers involved in *Proposition 1* can be computed by resorting to the ideas developed in [2].

Remark 2 - Note that *Theorem 1* and *Proposition 1* allow to define a *two-step procedure* for computing an enlarged RPI region which explicitly takes into account rate of variation constraints on the control input vector.

3. LMI CONDITIONS FOR CONSTRAINED OUTPUT-FEEDBACK MPC SCHEME

The aim of this section is to derive LMI conditions which allow to add predictive capabilities to the output receding horizon strategy developed in [14] and summarized in the previous section. To this end, let us consider the following family of virtual commands

$$u(\cdot|t) := \begin{cases} K\hat{y}_k(t) + c_k(t), & k = 0, 1, \ldots, N-1 \\ K\hat{y}_k(t), & k \geq N \end{cases} \tag{27}$$

where the vectors $c_k(t)$ provide N free perturbations to the action of a stabilizing and admissible (compatible with the prescribed constraints) controller K and $\hat{y}_k := C\hat{x}_k$ with

$$\hat{x}_k(t) := \Phi_K^k x(t) + \sum_{i=0}^{k-1} \Phi_K^{k-1-i}(Gc_i(t) + B_p p_i(t)) \tag{28}$$

are the convex set-valued state predictions such that $p_i(t) \in H_i(t)$ with

$$H_i(t) := \left\{ p \,|\, \|p\|_2^2 \leq \max_{\hat{x}_i(t)} \|C_K \hat{x}_i(t) + D_q c_i(t)\|_2^2 \right\}, \quad i = 0, 1, \ldots, k-1. \tag{29}$$

Moreover, we consider the following upper-bound to the standard LQ quadratic index [2]

$$\begin{aligned} V(x(t), P, c_k(t)) := \|x(t)\|_{R_x}^2 + \sum_{k=1}^{N-1} (\max_{\hat{x}_k(t)} \|\hat{x}_k(t)\|_{R_x}^2 \\ + \|c_{k-1}(t)\|_{R_u}^2) + \max_{\hat{x}_N(t)} \|\hat{x}_N(t)\|_P^2 + \|c_{N-1}(t)\|_{R_u}^2 \end{aligned} \tag{30}$$

Then at each time instant t, the N free control moves can be obtained by solving the following optimization w.r.t. $c_k(t)$, $k = 0, 1, \ldots, N-1$,

$$c_k^*(t) := \arg\min_{c_k(t)} V(x(t), P, c_k(t)) \tag{31}$$

subject to

$$K\hat{y}_k(t) + c_k(t) \in \Omega_u, \;\; k = 0, 1, \ldots, N-1, \tag{32}$$

$$K\hat{y}_k(t) + c_k(t) \in \Omega_{\delta u}, \;\; k = 0, 1, \ldots, N-1, \tag{33}$$

$$\hat{x}_k(t) \subset \Omega_x, \;\; k = 0, 1, \ldots, N-1, \tag{34}$$

$$\hat{x}_N(t) \subset \zeta, \tag{35}$$

where ζ is the robust invariant set under $K \triangleq Y_1 Q_1^{-1}$ computed by means of *Proposition 1*. Now by exploiting the available state measurement x_a, a computable upper-bound to the cost (30) can be determined by introducing a sequence of non-negative scalars $J_i, i = 0, 1, \ldots, N-1$, such that the inequalities (36)-(38) hold true.

$$\max_{p_0 \in S_0} \hat{x}_1^T R_x \hat{x}_1 + c_0^T R_u c_0 \leq J_0, \tag{36}$$

$$\max_{\substack{p_i \in S_i \\ i=0,\ldots,k \\ k=1,\ldots,N-2 \\ x(t) \in D(S)}} \hat{x}_{k+1}^T R_x \hat{x}_{k+1} + c_k^T R_u c_k \leq J_k, \tag{37}$$

$$\max_{\substack{p_i \in S_i \\ i=0,\ldots,N-1 \\ x(t) \in D(S)}} \hat{x}_N^T P \hat{x}_N + c_{N-1}^T R_u c_{N-1} \leq J_{N-1}, \tag{38}$$

By defining the following quantities $\hat{\Phi}_k \triangleq \Phi_K^{k+1} \in \mathbb{R}^{n_x \mathrm{x} n_x}$, $\hat{B}_k \triangleq \left[\Phi_K^k B_p \; \Phi_K^{k-1} B_p \; ... \; \Phi_K Bp \; B_p \right] \in \mathbb{R}^{n_x \mathrm{x}(k+1)n_p}$, $\hat{G}_k \triangleq \left[\Phi_K^k G \; \Phi_K^{k-1} G \; ... \; \Phi_K G \; G \right] \in \mathbb{R}^{n_x \mathrm{x}(k+1)n_u}$, $\underline{c}_k \triangleq [c_0^T c_1^T \cdots c_k^T]^T \in \mathbb{R}^{(k+1)n_u}$, $\underline{p}_k \triangleq [p_0^T p_1^T \cdots p_k^T]^T \in \mathbb{R}^{(k+1)n_p}$ the next two lemmas provide LMI conditions for the cost upper-bound requirements (37) and (38).

Lemma 1. Conditions (37) can be rearranged into the LMI feasibility conditions

$$\Sigma_k \triangleq \begin{bmatrix} J_k - \tau_{k+1} & -[x_a^T \underline{c}_k^T]\hat{L}_k^T \\ * & I \end{bmatrix} \geq 0, \; k = 1, \ldots, N-2, \tag{39}$$

with $\hat{L}_k^T$ the Cholesky factor of $\hat{L}_k^T \hat{L}_k = E_k + D_k^T F_k^{-1} D_k$ and

$$D_k \triangleq \begin{bmatrix} \beta^T + \sum_{i=0}^{k} \tau_i \beta_i^T & \lambda + \sum_{i=0}^{k} \tau_i \left[\lambda_i \;\; \epsilon_i \;\; 0 \right] \\ \mu^T + \sum_{i=0}^{k} \tau_i \begin{bmatrix} \omega_i^T \\ 0 \end{bmatrix} & \hat{B}_k^T R_x \hat{G}_k + \sum_{i=0}^{k} \tau_i \begin{bmatrix} \left[\hat{B}_{i-1}^T C_K^T C_K \hat{G}_{i-1} \;\; \hat{B}_{i-1}^T C_K^T D_q \;\; 0 \right] \\ 0 \end{bmatrix} \end{bmatrix}$$

$$E_k \triangleq \begin{bmatrix} \alpha + \sum_{i=0}^{k} \tau_i \alpha_i & \delta + \sum_{i=0}^{k} \tau_i \left[\delta_i \;\; \sigma_i \;\; 0 \right] \\ * & \hat{G}_k^T R_x \hat{G}_k + \begin{bmatrix} 0 & 0 \\ * & R_u \end{bmatrix} + \sum_{i=0}^{k} \tau_i \begin{bmatrix} \hat{G}_{i-1}^T C_K^T C_K \hat{G}_{i-1} & \hat{G}_{i-1} C_K^T D_q & 0 \\ * & D_q^T D_q & 0 \\ * & * & 0 \end{bmatrix} \end{bmatrix}$$

$$F_k \triangleq \begin{bmatrix} -\gamma - \sum_{i=0}^{k} \tau_i \gamma_i + \tau_{k+1} S & -\psi - \sum_{i=0}^{k} \tau_i \left[\zeta_i \;\; 0 \right] \\ * & -\hat{B}_k^T R_x \hat{B}_k - \sum_{i=0}^{k} \tau_i \begin{bmatrix} \hat{B}_{i-1}^T C_K^T C_K \hat{B}_{i-1} & 0 \\ * & -1 \end{bmatrix} \end{bmatrix}$$

Moreover $\hat{\Phi}_k^T R_x \hat{\Phi}_k = \begin{bmatrix} \alpha & \beta \\ * & \gamma \end{bmatrix}$, $\hat{\Phi}_k^T R_x \hat{G}_k = \begin{bmatrix} \delta \\ \lambda \end{bmatrix}$, $\hat{\Phi}_k^T R_x \hat{B}_k = \begin{bmatrix} \omega \\ \zeta \end{bmatrix}$, $\hat{\Phi}_{i-1}^T C_K^T C_K \hat{\Phi}_{i-1} = \begin{bmatrix} \alpha_i & \beta_i \\ * & \gamma_i \end{bmatrix}$, $\hat{\Phi}_{i-1}^T C_K^T C_K \hat{G}_{i-1} = \begin{bmatrix} \delta_i \\ \lambda_i \end{bmatrix}$, $\hat{\Phi}_{i-1}^T C_K^T C_K \hat{B}_{i-1} = \begin{bmatrix} \omega_i \\ \zeta_i \end{bmatrix}$, $\hat{\Phi}_{i-1}^T C_K^T D_q = \begin{bmatrix} \sigma_i \\ \epsilon_i \end{bmatrix}$, with $\alpha \in \mathbb{R}^{n_y \times n_y}$, $\beta \in \mathbb{R}^{n_y \times (n_x - n_y)}$, $\gamma \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)}$, $\delta \in \mathbb{R}^{n_y \times ((k+1)n_u)}$, $\lambda \in \mathbb{R}^{(n_x - n_y) \times ((k+1)n_u)}$, $\omega \in \mathbb{R}^{n_y \times ((k+1)n_p)}$, $\zeta \in \mathbb{R}^{(n_x - n_y) \times ((k+1)n_p)}$, $\alpha_i \in \mathbb{R}^{n_y \times n_y}$, $\beta_i \in \mathbb{R}^{n_y \times (n_x - n_y)}$, $\gamma_i \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)}$, $\delta_i \in \mathbb{R}^{n_y \times i n_u}$, $\lambda_i \in \mathbb{R}^{(n_x - n_y) \times i n_u}$, $\omega_i \in \mathbb{R}^{n_y \times i n_p}$, $\zeta_i \in \mathbb{R}^{(n_x - n_y) \times i n_p}$, $\sigma_i \in \mathbb{R}^{n_y \times i n_u}$, $\epsilon_i \in \mathbb{R}^{(n_x - n_y) \times i n_u}$.

Proof - Starting from the inequality $\hat{x}_{k+1}^T R_x \hat{x}_{k+1} + c_k^T R_u c_k \leq J_k$ by considering the state decomposition $x = [x_a^T \, x_{na}^T]^T$ one has that

$$\left[x_{na}^T \; \underline{p}_k^T \; 1 \right] \begin{bmatrix} -\gamma & -\psi & -\beta^T x_a - \lambda \underline{c}_k \\ * & -\hat{B}_k^T R_x \hat{B}_k & -\mu^T x_a - \hat{B}_k^T R_x \hat{G}_k \underline{c}_k \\ * & * & \begin{matrix} J_k - x_a^T \alpha x_a - 2x_a^T \delta \underline{c}_k \\ -\underline{c}_k^T \hat{G}_k^T R_x \hat{G}_k \underline{c}_k - \underline{c}_k^T \begin{bmatrix} 0 & 0 \\ * & R_u \end{bmatrix} \underline{c}_k \end{matrix} \end{bmatrix} \begin{bmatrix} x_{na} \\ \underline{p}_k \\ 1 \end{bmatrix} \geq 0 \tag{40}$$

Moreover, requirements $p_i \in H_i$ that is

$$p_i^T p_i \leq (C_K \hat{x}_i + D_q c_i)^T (C_K \hat{x}_i + D_q c_i) \tag{41}$$

can be recast in the following form

$$\left[\, x_{na}^T \; \underline{p}_k^T \; 1 \,\right] \begin{bmatrix} \gamma_i & [\,\zeta_i \;\; 0\,] & -\beta_i^T x_a + [\,\lambda_i \;\; \epsilon_i \;\; 0\,]\, \underline{c}_k \\ * & \begin{bmatrix} \hat{B}_{i-1}^T C_K^T C_K \hat{B}_{i-1} & 0 \\ * & -1 \end{bmatrix} & \begin{bmatrix} \omega_i^T x_a + [\,\hat{B}_{i-1}^T C_K^T C_K \hat{G}_{i-1} \;\; \hat{B}_{i-1}^T C_K^T D_q \;\; 0\,]\, \underline{c}_k \\ 0 \end{bmatrix} \\ * & * & \begin{matrix} x_a^T \alpha_i x_a + 2 x_a^T [\,\delta_i \;\; \sigma_i \;\; 0\,]\, \underline{c}_k \\ + \underline{c}_k^T \begin{bmatrix} \hat{G}_{i-1}^T C_K^T C_K \hat{G}_{i-1} & \hat{B}_{i-1}^T C_K^T D_q & 0 \\ * & D_q^T D_q & 0 \\ * & * & 0 \end{bmatrix} \underline{c}_k \end{matrix} \end{bmatrix} \begin{bmatrix} x_{na} \\ \underline{p}_k \\ 1 \end{bmatrix} \geq 0 \qquad (42)$$

Finally, requirements (6) assumes the following quadratic form

$$\left[\, x_{na}^T \; \underline{p}_k^T \; 1 \,\right] \begin{bmatrix} -S & 0 & 0 \\ * & 0 & 0 \\ * & * & 1 \end{bmatrix} \begin{bmatrix} x_{na} \\ \underline{p}_k \\ 1 \end{bmatrix} \geq 0 \qquad (43)$$

Then by resorting to *S-Procedure* arguments

condition (37) subject to (40), (42) and (43)

is satisfied if there exist $k+2$ scalars $\tau_0 \geq 0, \tau_1 \geq 0, \ldots, \tau_{k+1} \geq 0$ such that the inequality (44) holds true.

$$\begin{bmatrix} F_k & -D_k \begin{bmatrix} x_a \\ \underline{c}_k \end{bmatrix} \\ * & J_k - \tau_{k+1} - \left[\, x_a^T \;\; \underline{c}_k^T \,\right] E_k \begin{bmatrix} x_a \\ \underline{c}_k \end{bmatrix} \end{bmatrix} \geq 0 \qquad (44)$$

By applying Schur complement arguments to (44), it is straightforward to complete the proof. □

Remark 3 Note that the *S-Procedure* multipliers $\tau_0, ..., \tau_{k+1}$ can be computed by following the lines indicated in [2], i.e. by solving Generalized Eigenvalue Problems (GEVP):

$$[\tau_0, ..., \tau_{k+1}] = \underset{\tau_0 \geq 0, \ldots, \tau_{k+1} \geq 0}{\arg\min} \; \bar{\lambda} \hat{L}_k^T \hat{L}_k \qquad (45)$$

$$\textit{subject to } F_k > 0, \; J_k - \tau_{k+1} > 0$$

where $\bar{\lambda}$ denotes the largest eigenvalue operator. □

Lemma 2. The terminal constraint (38) translates into

$$\Sigma_N \triangleq \begin{bmatrix} \rho - \hat{\tau}_N & -[x_a^T \underline{c}_{N-1}^T] \hat{L}_N^T \\ * & I \end{bmatrix} \geq 0 \qquad (46)$$

with $\hat{L}_N^T$ the Cholesky factor of

$$\hat{L}_N^T \hat{L}_N = E_N + D_N^T F_N^{-1} D_N$$

and

$$D_N \triangleq \begin{bmatrix} \hat{\beta}^T + \sum_{i=0}^{N-1} \hat{\tau}_i \beta_i^T & \hat{\lambda} + \sum_{i=0}^{N-1} \hat{\tau}_i [\,\lambda_i \;\; \epsilon_i \;\; 0\,] \\ \hat{\mu}^T + \sum_{i=0}^{N-1} \hat{\tau}_i \begin{bmatrix} \omega_i^T \\ 0 \end{bmatrix} & \hat{B}_{N-1}^T P \hat{G}_{N-1} + \sum_{i=0}^{N-1} \hat{\tau}_i \begin{bmatrix} \hat{B}_{i-1}^T C_K^T C_K \hat{G}_{i-1} & \hat{B}_{i-1}^T C_K^T D_q & 0 \\ & 0 & \end{bmatrix} \end{bmatrix}$$

$$E_N \triangleq \begin{bmatrix} \hat{\alpha} + \sum_{i=0}^{N-1} \hat{\tau}_i \alpha_i & \hat{\delta} + \sum_{i=0}^{N-1} \hat{\tau}_i [\,\delta_i \;\; \sigma_i \;\; 0\,] \\ * & \hat{G}_{N-1}^T P \hat{G}_{N-1} + \sum_{i=0}^{N-1} \hat{\tau}_i \begin{bmatrix} \hat{G}_{i-1}^T C_K^T C_K \hat{G}_{i-1} & \hat{G}_{i-1} C_K^T D_q & 0 \\ * & D_q^T D_q & 0 \\ * & * & 0 \end{bmatrix} \end{bmatrix}$$

$$F_N \triangleq \begin{bmatrix} -\hat{\gamma} - \sum_{i=0}^{N-1} \hat{\tau}_i \gamma_i + \hat{\tau}_N S & -\hat{\psi} - \sum_{i=0}^{N-1} \hat{\tau}_i [\,\zeta_i \;\; 0\,] \\ * & -\hat{B}_{N-1}^T P \hat{B}_{N-1} - \sum_{i=0}^{N-1} \hat{\tau}_i \begin{bmatrix} \hat{B}_{i-1}^T C_K^T C_K \hat{B}_{i-1} & 0 \\ * & -1 \end{bmatrix} \end{bmatrix}$$

Moreover $\hat{\Phi}_{N-1}^T P \hat{\Phi}_{N-1} = \begin{bmatrix} \hat{\alpha} & \hat{\beta} \\ * & \hat{\gamma} \end{bmatrix}$, $\hat{\Phi}_{N-1}^T P \hat{G}_{N-1} = \begin{bmatrix} \hat{\delta} \\ \hat{\lambda} \end{bmatrix}$, $\hat{\Phi}_{N-1}^T P \hat{B}_{N-1} = \begin{bmatrix} \hat{\omega} \\ \hat{\zeta} \end{bmatrix}$, $\hat{\Phi}_{i-1}^T C_K^T C_K \hat{\Phi}_{i-1} = \begin{bmatrix} \hat{\alpha}_i & \hat{\beta}_i \\ * & \hat{\gamma}_i \end{bmatrix}$, $\hat{\Phi}_{i-1}^T C_K^T C_K \hat{G}_{i-1} = \begin{bmatrix} \hat{\delta}_i \\ \hat{\lambda}_i \end{bmatrix}$, with $\hat{\alpha} \in \mathbb{R}^{n_y \times n_y}$, $\hat{\beta} \in \mathbb{R}^{n_y \times (n_x - n_y)}$, $\hat{\gamma} \in \mathbb{R}^{(n_x - n_y) \times (n_x - n_y)}$, $\hat{\delta} \in \mathbb{R}^{n_y \times (N n_u)}$, $\hat{\lambda} \in \mathbb{R}^{(n_x - n_y) \times (N n_u)}$, $\hat{\omega} \in \mathbb{R}^{n_y \times (N n_p)}$, $\hat{\zeta} \in \mathbb{R}^{(n_x - n_y) \times (N n_p)}$.

Proof - Follows the same lines of *Lemma 1.* □

Remark 4 Multipliers $\hat{\tau}_i, i = 0, \ldots, N-1$, are obtained by solving the GEVP

$$[\hat{\tau}_0 \ldots \hat{\tau}_N] = \underset{\hat{\tau}_0 \geq 0, \ldots, \hat{\tau}_N \geq 0}{\arg\min} \; \bar{\lambda} \hat{L}_N^T \hat{L}_N \qquad (47)$$

$$\textit{subject to } F_N > 0, \rho - \hat{\tau}_N > 0$$

Finally, constraints (32) , (33) and (34) can be recast in terms of LMI conditions. In the sequel for the sake of brevity we shall describe the simplest case $N = 1$. The extension to arbitrary horizon length is straightforward.

Input constraint - Conditions (32) can be rewritten as

$$\Gamma_0 \triangleq \begin{bmatrix} \bar{u}_{i,max}^2 & -(\bar{C}_i x + J_i^T c_0)^T \\ * & I \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u, \qquad (48)$$

where

$$J_i = \Big[\, 0 \ldots 0 \; \overbrace{1}^{i-th} \; 0 \ldots 0 \,\Big]^T \in \mathbb{R}^{n_u}$$

One-step control rate - Given $u(t-1) = Ky(t-1)$, constraints (33) can be rewritten as

$$\Psi_0 \triangleq \begin{bmatrix} \delta u_{i,max}^2 & (\bar{C}_i x + J_i^T c_0 - J_i^T u(t-1))^T \\ * & I \end{bmatrix} \geq 0, \quad i = 1, \ldots, n_u. \qquad (49)$$

State constraint - Constraints (34) can be rewritten as

$$\chi_0 \triangleq \begin{bmatrix} \bar{x}_{j,max}^2 - \underline{\tau}_{1,j} & -[x_a^T c_0^T] \underline{L}_0^T \\ * & I \end{bmatrix} \geq 0, j = 1, \ldots, n_x \qquad (50)$$

with $\underline{L}_0^T$ the Cholesky factor of $\underline{L}_0^T \underline{L}_0 = \underline{E}_j + \underline{D}_j^T \underline{F}_j^{-1} \underline{D}_j$ where

$$\underline{D}_j \triangleq \begin{bmatrix} -\underline{\beta}_j^T - \underline{\tau}_{0,j} \bar{\beta}^T & -\underline{\zeta}_j - \underline{\tau}_{0,j} \bar{\zeta} \\ -\underline{\sigma}_j^T & -\underline{\Delta}_j^T \end{bmatrix}$$

$$\underline{E}_j \triangleq \begin{bmatrix} \underline{\alpha}_j + \underline{\tau}_{0,j} \bar{\alpha} & \underline{\omega}_j + \underline{\tau}_{0,j} \bar{\omega} \\ * & \underline{\chi}_j + \underline{\tau}_{0,j} D_q^T D_q \end{bmatrix}$$

$$\underline{F}_j \triangleq \begin{bmatrix} -\underline{\gamma}_j - \underline{\tau}_{0,j}\bar{\gamma} + \underline{\tau}_{1,j} S & -\underline{\epsilon}_j \\ * & -\underline{\Lambda}_j + \underline{\tau}_{0,j} \end{bmatrix}$$

with $\Phi_k^T J_j J_j^T \Phi_k = \begin{bmatrix} \underline{\alpha}_j & \underline{\beta}_j \\ * & \underline{\gamma}_j \end{bmatrix}$, $\Phi_k^T J_j J_j^T G = \begin{bmatrix} \underline{\omega}_j \\ \underline{\zeta}_j \end{bmatrix}$, $\Phi_k^T J_j J_j^T B_p = \begin{bmatrix} \underline{\sigma}_j \\ \underline{\epsilon}_j \end{bmatrix}$, $\underline{\chi}_j = G^T J_j J_j^T G$, $\Delta_j = G^T J_j J_j^T B_p$, $\Lambda_j = B_p^T J_j J_j^T B_p$, $\underline{\alpha}_j \in \mathbb{R}^{n_y \times n_y}$, $\underline{\beta}_j \in \mathbb{R}^{n_y \times (n_x - n_y)}$, $\underline{\gamma}_j \in \mathbb{R}^{(n_x-n_y)\times(n_x-n_y)}$, $\underline{\omega}_j \in \mathbb{R}^{n_y \times n_u}$, $\underline{\zeta}_j \in \mathbb{R}^{(n_x-n_y)\times n_u}$, $\underline{\sigma}_j \in \mathbb{R}^{n_y \times n_p}$, $\underline{\epsilon}_j \in \mathbb{R}^{(n_x-n_y)\times n_p}$.

Multipliers $\underline{\tau}_{0,j}, \underline{\tau}_{1,j}, j = 1,\ldots,n_x$, are obtained via the GEVP:

$$[\underline{\tau}_{0,j}, \underline{\tau}_{1,j}] = arg \min_{\underline{\tau}_{0,j} \geq 0, \underline{\tau}_{1,j} \geq 0} \bar{\lambda} \underline{L}_0^T \underline{L}_0 \qquad (51)$$

$$subject\ to\ \bar{F}_j > 0,\ \bar{x}_{j,max}^2 - \underline{\tau}_{1,j} > 0$$

The above developments allow to write down a computable MPC scheme, hereafter denoted as **OUT-MPC**:

OUT-MPC-Algorithm- Off-line

0.1 Solve the optimization (8) subject to (9)-(12) under the restriction $\bar{u}_{i,max} = \bar{\delta} u_{i,max}$, $i = 1 \ldots n_u$. Compute $K = Y_1 Q_1^{-1}$;
0.2 Solve GEVPs (45), (47) and (51);

OUT-MPC-Algorithm- On-line

1.1 Solve $[J_k^*(t), c_k^*(t)] \triangleq \arg\min_{J_k, c_k} \sum_{k=0}^{N-1} J_k$ subject to (39), (46), (48), (49) and (50);
1.2 Apply $u(t) = Ky(t) + c_0^*(t)$, $t \rightarrow t+1$ and goto 1.1.

Finally the following theorem shows that the proposed strategy enjoys feasibility retention and closed-loop stability properties.

Theorem 2. Let the **Output-MPC-Algorithm** have solution at time $t = 0$. Then, it has solution at each future time instant t, satisfies input and state constraints and yields a quadratically stable closed-loop system.

Proof - It follows similar lines as in [1] and [2]. □

4. SIMULATION EXAMPLE

In this section, we present a numerical example that illustrates the features of the proposed output feedback norm-bounded MPC strategy. Moreover, numerical comparison are carried out by considering the RHC counterpart algorithm proposed in [14] and hereafter named as **OUT-RHC-Algorithm**. To this end, the High Altitude Performance Demonstrator (HAPD) aircraft nonlinear model (see [15] for details) is considered for simulation purposes. The HAPD is an over-actuated aircraft designed by C.I.R.A (Italian Aerospace Research Centre) having three pairs of elevators, two pairs of ailerons, two rudders and eight propellers. Main HAPD geometrical parameters and constraints on the actuators capability are : Wing Area S= $13.5\ m^2$; Wing Span $S_b = 16.55\ m$; Mean Chord $S_c = 0.557\ m$; Mass M= $184.4\ kg$; Maximum allowable control surfaces deflections ± 25 deg; Maximum allowable control surfaces rates-of-variation $\pm 200\ deg/s$. In order to comply with the proposed framework, the nonlinear model has been first recast into a Polytopic Linear Differential Inclusion (PLDI) obtained by using some linearized models characterizing different flight conditions within the following operating envelope: the true air speed V_0 belongs to $[17, 23]$ *m/s*; the altitude varies between 300 *m* and 700 *m*. Then, the PLDI has been outer-approximated as the Norm-bound Linear Differential Inclusion (NDLI) (1) by using the optimization procedure described in [16]. The following statements hold for linear model: i) actuator and sensor dynamics have been considered negligible; ii) only the two slowest aeroelastic modes accounting for symmetrical and asymmetrical deformation of HAPD aircraft, have been considered; iii) faster aeroelastic dynamics have been assumed to be instantaneous; iv) thrust motors have been considered fixed. Numerical simulation with the full nonlinear model of the HAPD aircraft is performed starting from forward flight equilibrium conditions (*Altitude*= $500m$ and $V_0 = 20m/s$). The aircraft is driven by a doublet variation demand of $7deg/s$ on roll-rate p , see dashed line in Fig. 1. All the numerical results summarized in Figs. 1-3 are obtained by considering the proposed MPC strategy with a control horizon $N = 1$. In particular, Fig. 1 depicts the tracking capabilities of the two competitors. As it clearly results, the proposed improved Output MPC version shows a significant improvement in terms of control performance (*quasi*-perfectly tracking of the prescribed set-points) at an expanse of an affordable computational load increase as discussed in [2]. Figs. 2-3 show the behaviours pertaining to deflections and deflection rates of variation, respectively. Moreover, all the prescribed constraints on the control surfaces (dashed line) are always satisfied and, finally, it is interesting to observe that the **OUT-MPC-Algorithm** (solid line) shows a superior capability w.r.t. its RHC counterpart (dash-dot line) to take advantage from the redundancy of the control surface deflections.

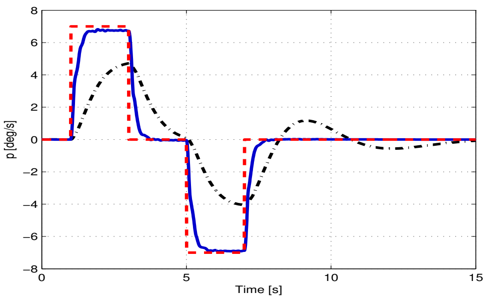

Fig. 1. Time responses for p during test manouvres: **OUT-MPC-Algorithm** (solid line) and **OUT-RHC-Algorithm** strategy (dash-dot line).

5. CONCLUSIONS

In this paper, we have presented a novel robust predictive control strategy for constrained uncertain norm-bounded model plants when only a partial state measurement is available. The key idea to avoid the design of the standard state-feedback/observer pair was pursued by extensively using *S-Procedure* arguments. This gives rise to an output feedback MPC scheme which is capable to reduce the computational intractability typically arising in complex systems and, at the same time, to significantly improve the overall control performance. Finally, in order to show the benefits of the proposed strategy some numerical simulation involving a full nonlinear model of HAPD aircraft has carried out.

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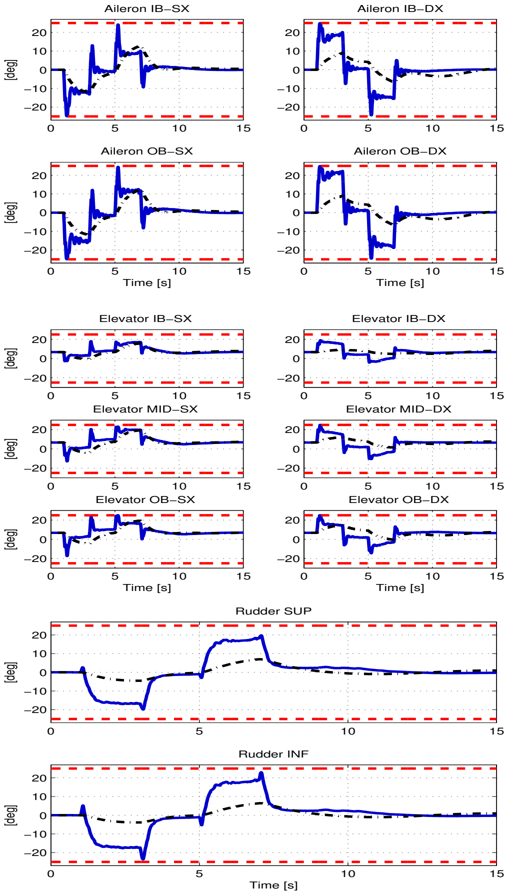

Fig. 2. Control Surfaces deflections: **OUT-MPC-Algorithm** (solid line) and **OUT-RHC-Algorithm** strategy (dash-dot line).

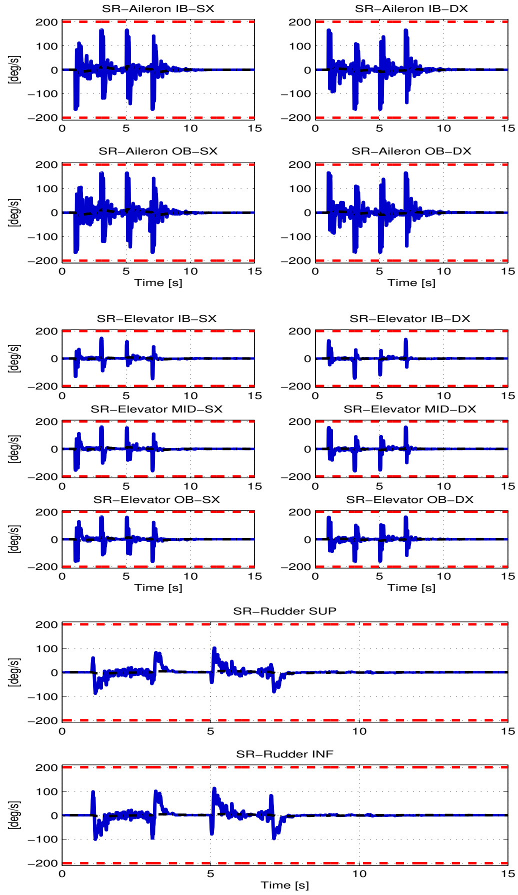

Fig. 3. Control surfaces rates-of-variation: **OUT-MPC-Algorithm** (solid line) and **OUT-RHC-Algorithm** strategy (dash-dot line).

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