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Observer Design for a class of complex networks with unknown topoloy

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Abstract: In this paper a method for the design of state observers for a class of complex network systems with nonlinear dynamics is presented. Necessary conditions on the dynamics of the nodes are given, i.e. how the interconnection enters and which states have to be measured, such that the arbitrary complex interconnection can be dealt with as an unknown input. Sufficient existence conditions for an Unknown Input Observer (UIO) for this class of systems are stated in terms of an LMI representing the dissipativity properties of the system. The design method is applied to monitor an epidemic disease in a complex network.

1. INTRODUCTION

Analysis and control of dynamical systems interconnected in a network structure, also referred to as *Multi-Agent Systems* (MAS), has gained a lot of interest during the past years due to its emerging variety of applications [Mesbahi and Egerstedt, 2010, Ch. 1]. There exists a wide range of real-world applications including but not limited to sensor networks, coordination of autonomous systems, i.e. groups of robots, computer networks like the internet, energy networks like power grids, biological and chemical networks, and social networks. Besides control of the systems, monitoring of the system behavior is an important task in system analysis. Practical engineering systems are surveilled for safety and reliability, and state-feedback controllers demand the knowledge of the system state.

While linear observer design methods can be applied to linearly interconnected agents with linear dynamics [Siljak, 1991, Ch. 4], more complex networks, e.g. with nonlinear dynamics and time-varying interconnections/structure, are focus of ongoing research [Zecevic and Siljak, 2010, Ch. 6]. Recently, an observer for systems with unknown but bounded nonlinearities and linear time-varying (non-switching) interconnections was proposed in [Menon and Edwards, 2011]. Unless this study provided promising results, it is not clear how it could be extended to the case of networks with known or unknown nonlinear interconnections and possibly time-varying topology. The present paper focuses on the observer design method for this class of systems. As far as the authors know, this problem has not yet been addressed in the literature. The consideration of unknown topology rules out centralized observer design approaches, and requires to consider the problem within the framework of unknown input observers for nonlinear systems. For this purpose, a dissipativity-based observer design method as recently proposed in [Rocha-Cózatl and Moreno, 2011] is applied. Necessary conditions for the relative degree of the unknown input are derived and sufficient conditions for the existence of the observer are stated in terms of an (L)MI that can be solved using standard numerical methods [Boyd et al., 1994]. In order to illustrate the approach, the method is applied to monitor the spread of an epidemic in a complex network of agents that are susceptible, infected and/or in-quarantine (SIQ) with a certain probability, respectively, [Bernal J. et al., 2013].

1.1 Problem statement

Consider a network of N nonlinearly interconnected dynamical systems with nonlinear dynamics of the form

$$\mathcal{N}_i : \begin{cases} \dot{x}_i = A_i x_i + G_i \varphi_i(t, x_i) + \Phi_i(t, \boldsymbol{x}), & x_i(0) = x_{i,0}, \\ y_i = C_i x_i, & i = 1, .., N, \end{cases} \tag{1}$$

where $x_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^m$ are the state and the output vectors, and A_i, G_i and C_i are constant matrices of appropriate dimensions. The nonlinearity $\varphi_i(t, x_i)$ is an s-dimensional vector, locally Lipschitz in x_i and piecewise continuous in t, and $\Phi_i(t, \boldsymbol{x})$ is a smooth nonlinear function that depends on the state of the network $\boldsymbol{x} = (x_1, .., x_N)$.

The objective is to design a decentralized state observer for the system $\mathcal{N}_i$ (1) that, using the measured output $y_i(t)$, provides a state estimation $\hat{x}_i$ that asymptotically converges to the actual state $x_i(t)$, i.e. $\lim_{t\to\infty}(\hat{x}_i(t) - x_i(t)) = 0$.

1.2 Organization of the paper

The paper is organized as follows. In Section 2 we recall the results for dissipativity-based UIO design from

[Rocha-Cózatl and Moreno, 2011], where a dissipativity characterization of sufficient existence conditions for UIOs is generalized for a class of nonlinear systems. In Section 3 the dissipativity-based method is used to design an observer for a class of complex networks and the sufficient conditions are derived. The proposed observer is applied to monitor the epidemic spread in a complex network in Section 4. To conclude, in Section 5 the proposed approach is compared to previous ones in terms of the class of systems it can be applied to.

2. DISSIPATIVITY-BASED UNKNOWN INPUT OBSERVER

Since the observer design approach followed in this paper is based on dissipativity, some basic notions, definitions and results from [Hill and Moylan, 1980, Willems, 1972a,b] are recalled in Sec. 2.1. The basic idea of dissipativity-based UIO design from [Rocha-Cózatl and Moreno, 2011] is presented in Sec. 2.2. This method allows to express the sufficient condition for the existence of an UIO in terms of an LMI as stated below.

2.1 Dissipativity

Dissipativity is a property of a dynamical system that interrelates input-output and state properties with the tool of (Lyapunov-like) storage functions.

Consider a nonlinear system

$$\Sigma_{NL}: \begin{cases} \dot{x} = f(t,x,u), \qquad x(0) = x_0, \\ y = h(t,x), \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^m$ are the state, input and output vectors respectively. Without input $u = 0$, $f(t,0,0) = 0$ and $h(t,0) = 0$ at the origin $x = 0$. The system Σ_{NL} is called *State Strictly Dissipative* (SSD) if there exist a continuously differentiable and positive-definite storage function $V : \mathbb{R}^n \to \mathbb{R}$, with $V(0) = 0$, a supply rate $\omega : \mathbb{R}^m \times \mathbb{R}^p \to \mathbb{R}$ and a constant $\epsilon > 0$, such that the *dissipativity inequality*

$$\dot{V}(x(t)) \le -\epsilon V(x(t)) + \omega(y(t), u(t)) \tag{3}$$

holds along any system trajectory. $\omega(y,u)$ has to be locally integrable for all input-output pairs of Σ_{NL}. If the supply rate is a quadratic form

$$\omega(y,u) = \begin{bmatrix} y \\ u \end{bmatrix}^T \begin{bmatrix} Q & S \\ S & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} \tag{4}$$

with Q, S and R of appropriate dimensions, and Q and R symmetric, the system is called SSD{Q,S,R}. In the special case of LTI systems

$$\Sigma_L: \begin{cases} \dot{x} = Ax + Bu, \qquad x(0) = x_0, \\ y = Cx, \end{cases} \tag{5}$$

and for quadratic supply rates (4) restricted to quadratic storage functions $V = x^T P x$, $P = P^T > 0$, dissipativity can be characterized as in the following proposition.

Proposition 1. [Hill and Moylan, 1980]
System Σ_L is SSD{Q,S,R} if and only if there exists a matrix $P = P^T > 0$ and a constant $\epsilon > 0$ such that

$$\begin{bmatrix} PA + A^T P + \epsilon I & PB \\ B^T P & 0 \end{bmatrix} - \begin{bmatrix} C^T Q C & C^T S \\ S^T C & R \end{bmatrix} \le 0. \tag{6}$$

For the case of memoryless (or static) nonlinearities dissipativity can be defined as stated next.

Definition 1. [Rocha-Cózatl and Moreno, 2011]
A time-varying memoryless nonlinearity $\psi : [0,\infty) \times \mathbb{R}^p \to \mathbb{R}^m$, $y^\star = \psi(t,u^\star)$, piecewise continuous in t and locally Lipschitz in $u^\star$, such that $\psi(t,0) = 0$, is dissipative with respect to a quadratic supply rate $\omega(y^\star,u^\star)$ (4) (D{Q,S,R}), if

$$\omega(y^\star,u^\star) = \omega(\psi(t,u^\star),u^\star) \ge 0 \qquad \forall t \ge 0, u^\star \in \mathbb{R}^r. \tag{7}$$

Remark 1: Sector conditions for square nonlinearities [Khalil, 2002], i.e. $m = p$, can be stated as dissipativity conditions in the following way: If the nonlinearity ψ lies in the sector $[K_1, K_2]$, i.e. it satisfies $(y^\star - K_1 u^\star)^T (K_2 u^\star - y^\star) \ge 0$ which can be written as a dissipativity condition $D\{Q,S,R\}$ with $(Q,S,R) = (-I, \frac{1}{2}(K_1 + K_2), -\frac{1}{2}(K_1^T K_2 + K_2^T K_1))$, where I denotes the identity matrix of appropriate dimension.

2.2 Unknown Input Observer design

In [Rocha-Cózatl and Moreno, 2011], a dissipativity-based method to design an observer for nonlinear systems with unknown input of the form

$$\Sigma: \begin{cases} \dot{x} = Ax + G\psi(t,\sigma) + \varphi(t,y,u) - Bw, \quad x(0) = x_0, \\ y = Cx, \\ \sigma = Hx, \end{cases} \tag{8}$$

is described, where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $w \in \mathbb{R}^q$, $y \in \mathbb{R}^m$, are the state, the known and unknown input, and the measured output vectors, respectively. The unknown input w is assumed to be a piecewise continuous time signal and can be considered as arbitrary unbounded disturbance. The unmeasured or not measurable part $\sigma \in \mathbb{R}^r$ is a linear function of the state, and A, G, B, C and H are constant matrices of appropriate dimensions. The nonlinearity $\varphi(t,y,u)$ is locally Lipschitz in y, continuous in u and piecewise continuous in t. The s-dimensional vector $\psi(t,\sigma)$ is assumed to be locally Lipschitz in σ and piecewise continuous in t. It is assumed that there exists a solution for Σ (8).

For this system an observer is proposed of the form

$$\Omega: \begin{cases} \dot{\zeta} = A\zeta + G\psi(t,\hat{\sigma} + N(\Delta y)) + L(\Delta y) + \varphi(t,y,u), \\ \hat{y} = C\zeta, \\ \hat{\sigma} = H\zeta, \\ \hat{x} = D\zeta + Ey, \end{cases} \tag{9}$$

with $\Delta y = \hat{y} - y$ and $\zeta(0) = \zeta_0$, where the output injection matrices $L \in \mathbb{R}^{n \times m}$ and $N \in \mathbb{R}^{r \times m}$, and matrices $D \in \mathbb{R}^{n \times n}$ and $E \in \mathbb{R}^{n \times m}$ are design variables. Ω (9) is a copy of the the plant Σ (8) including two output injections terms to ensure observer convergence.

Plant Σ (8) and observer Ω (9) together form a cascade system with the dynamics of the state error $e := \zeta - x$ in *Lure form* [Khalil, 2002]

$$\Sigma_e^{obs}: \begin{cases} \dot{e} = A_L e + Gv - Bw, & e(0) = e_0, \\ z = H_N e, \\ \tilde{y} = Ce, \\ v = -\underbrace{[\varphi(t,\sigma) - \varphi(t,\sigma+z)]}_{\phi(t,z,\sigma)}, \end{cases} \tag{10}$$

with $A_L = A + LC$ and $H_N = H + NC$, where $\tilde{y} := \hat{y} - y$ is the output error, $\tilde{\sigma} := \hat{\sigma} - \sigma$ and $z := H_N e = \tilde{\sigma} + N\tilde{y}$. Note that the one-sided coupling of the error dynamics Σ_e^{obs} to the plant through an incremental version of the system's nonlinearity $\phi(t,z,\sigma) = \varphi(t,\sigma) - \varphi(t,\sigma+z)$ is effected by the unmeasured states σ.

For nonlinear systems of this form (with a linear forward path and only a nonlinearity in the feedback path) dissipativity can ensure stability as in the following lemma.

Lemma 2. [Hill and Moylan, 1980]
Consider the negative feedback interconnection between a linear system and nonlinearity ψ of the form

$$\begin{aligned} \dot{e} &= A_L e + Gv, \quad e(0) = e_0, \\ z &= H_N e, \\ v &= -\phi(t,z). \end{aligned} \tag{11}$$

If the linear system is $SSD\{-R, S^T, -Q\}$, then the origin $e = 0$ of system (11) is globally exponentially stable for every $D\{Q,S,R\}$ nonlinearity ϕ.

Based on dissipativity, [Rocha-Cózatl and Moreno, 2011] generalizes the sufficient existence conditions for an UIO for LTI systems [Moreno, 2001, Rocha-Cózatl and Moreno, 2004], that can be expressed as a matrix inequality (MI) (12).

Theorem 1. [Rocha-Cózatl and Moreno, 2011]
If there exist constant matrices $P = P^T > 0$, L, N, $\mathbb{S}$ (of full row rank) and a constant $\epsilon > 0$, such that the MI

$$\begin{bmatrix} P^\star & PG & PB \\ G^T P & 0 & 0 \\ B^T P & 0 & 0 \end{bmatrix} - \begin{bmatrix} -H_N^T R H_N & H_N^T S^T & C^T \mathbb{S} \\ S H_N & -Q & 0 \\ \mathbb{S} C & 0 & 0 \end{bmatrix} \leq 0, \tag{12}$$

with $P^\star = P(A+LC) + (A+LC)^T P + \epsilon I$ and $H_N = H + NC$, is satisfied, then

(1) There exist constant matrices L, N, $\mathbb{S}$ (of full row rank) such that the observer error dynamics Σ_e^{obs} (10) is $SSD\{0, \mathbb{S}^T, 0\}$.

(2) Both conditions for the observer error dynamics Σ_e^{obs} (10) without injection term (i.e. $L = 0$, $N = 0$) are satisfied:
   (a) It is minimum phase and
   (b) It fulfills the relative degree condition
   $$\text{rank}\{CB\} = \text{rank}\{B\} = q = \dim(w). \tag{13}$$

(3) There exists an UIO for Σ (8).

The proof was presented in [Rocha-Cózatl and Moreno, 2011].

Remark 2: The MI (12) is nonlinear in the design parameters but if one of the additional degrees of freedom (DOFs) is fixed, e.g. $N = N_0 = 0$, it becomes an LMI in P, PL, ϵ and $\mathbb{S}$ that can be solved efficiently [Boyd et al., 1994]. The existence of a solution for the inequality is only a sufficient condition for the existence of an UIO, and there is no guaranteed method to find a value for N if it exists.

3. OBSERVER DESIGN FOR NETWORKS

In this section the method of dissipativity-based observer design for complex networks is presented. The fundamental idea is to bring the network dynamics to the form Σ (8) by dealing with the interconnections as unknown inputs, such that the dissipativity-based UIO presented in Sec. 2.2 can be applied. The observer error dynamics form a Lure system with only one nonlinearity in the feedback path which is coupled to the unmeasured states of the plant. By characterizing the nonlinearity in terms of dissipativity, a sufficient existence condition for an UIO can be stated in form of a Matrix Inequality (MI) (Sec. 2.2). If one of the additional DOFs is fixed, an LMI is obtained that ca be solved efficiently using standard numerical methods [Boyd et al., 1994]. In the following, the procedure is presented in detail and the necessary condition for the transformation to the UIO form and sufficient conditions for the existence of an UIO are stated.

Recalling the complex network dynamics $\mathcal{N}_i$ (1)

$$\begin{aligned} \dot{x}_i &= A_i x_i + G_i \varphi_i(t, x_i) + \Phi_i(t, \boldsymbol{x}), \quad x_i(0) = x_{i,0}, \\ y_i &= C_i x_i, \qquad\qquad\qquad\qquad i = 1,..,N, \\ \sigma_i &= H_i x_i, \end{aligned} \tag{14}$$

the unknown interconnection $\Phi_i(t,\boldsymbol{x})$ can be replaced by $B_i w_i$ with unknown input w_i and B_i containing the information how, i.e. in which channels, the unknown input enters the dynamics, and is selected such that

$$\mathcal{R}(B_i) \supseteq \text{span}\{\Phi_i(t,\boldsymbol{x}), \forall x, t\}, \tag{15}$$

where $\mathcal{R}(B_i)$ denotes the range of the matrix B_i. There exist solutions for the system if w_i is smooth. Since $\mathcal{R}(B_i) \supseteq \text{span}\{\Phi_i\}$, the solutions of system (16) contain the solutions of $\mathcal{N}_i$ (1).

After this inclusion, the dynamics of the single agents can be considered decoupled including a disturbance by the interconnected environment. In the dynamics of the decoupled agents, the nonlinearity $G_i\varphi_i(t,x_i)$ depends in general on the whole state x_i. It can be split up in a part $\widetilde{\varphi}_i(t,y_i)$ that depends on the output of the measured states y_i and another one $\psi_i(t,\sigma_i)$ that depends on the unmeasured states σ_i. The nonlinear term $\widetilde{\varphi}_i(t,y)$ can be completely compensated (in the error dynamics) by adding its copy to the observer dynamics. The nonlinear terms in the measured channels can be neglected by considering them as part of the unknown input. By this transformation we finally obtain the form Σ (8) (without input $u_i = 0$)

$$\begin{aligned} \dot{x}_i &= A_i x_i + G_i \psi_i(t,\sigma_i) + \widetilde{\varphi}_i(t,y_i) - B_i w_i, \quad x_i(0) = x_{i,0}, \\ y_i &= C_i x_i, \qquad\qquad\qquad\qquad i = 1,..,N, \\ \sigma_i &= H_i x_i. \end{aligned} \tag{16}$$

It is critical how the unknown inputs enter the dynamics and which states are measured for the existence of an UIO. B_i and C_i have to satisfy the matchability condition (13)

$$\text{rank}\{C_iB_i\} = \text{rank}\{B_i\} = q_i = \dim(w_i),$$

which is a necessary assumption on the relative degree of the unknown input. It is obvious that this condition can only be satisfied if the number of outputs is greater or equal the number of unknown input.

Consider the observer error dynamics Σ_e^{obs} (10)

$$\Sigma_{e,i}^{obs} : \begin{cases} \dot{e}_i = A_{L,i}e_i + G_iv_i - B_iw_i, \quad e_i(0) = e_{i,0}, \\ z_i = H_{N,i}e_i, \\ \tilde{y}_i = C_ie_i, \\ v_i = -\phi_i(t, z_i, \sigma_i), \end{cases}$$

to characterize the nonlinearity $\phi_i(t, z_i, \sigma_i)$ in terms of dissipativity. If the nonlinearity can be characterized dissipative to a quadratic supply rate $D\{Q, S, R\}$, e.g. by applying and transforming classical sector conditions $[K_1, K_2]$, the existence of an UIO can be checked by means of MI (12). The result can be summarized as in the following Theorem 2 which is a direct consequence of Theorem 1.

Theorem 2.
If B_i is chosen such that $\mathcal{R}(B_i) \supseteq \text{span}\{\Phi_i(t, \boldsymbol{x}), \forall x, t\}$ and satisfies $\text{rank}\{C_iB_i\} = \text{rank}\{B_i\} = q_i = \dim(w_i)$, and there exist constant matrices $P_i = P_i^T > 0$, L_i, N_i, $\mathbb{S}_i$ (of full row rank) and a constant $\epsilon_i > 0$, such that the MI (12)

$$\begin{bmatrix} P_i^\star & P_iG_i & P_iB_i \\ G_i^TP_i & 0 & 0 \\ B_i^TP_i & 0 & 0 \end{bmatrix} - \begin{bmatrix} -H_{N_i}^TR_iH_{N,i} & H_{N,i}^TS_i^T & C_i^T\mathbb{S}_i \\ S_iH_{N,i} & -Q_i & 0 \\ \mathbb{S}_iC_i & 0 & 0 \end{bmatrix} \leq 0,$$

with $P_i^\star = P_i(A_i + L_iC_i) + (A_i + L_iC_i)^TP_i + \epsilon_iI$, is satisfied $\forall i = 1, .., N$, then the observer Ω (9) asymptotically converges to the actual state of network $\mathcal{N}$ (1).

MI (12) contains the arguments P_i, L_i, ϵ_i, N_i and $\mathbb{S}_i$, besides the system parameters in matrices A_i, B_i, C_i, G_i and H_i, and information about the nonlinearity Q_i, S_i, R_i. The inequality is nonlinear in the entries $P_iA_{L,i} + A_{L,i}^TP_i + \epsilon_iI = P_iA_i + \mathbf{P_i}\mathbf{L_i}C_i + A_i^TP_i + C_i^T\mathbf{L_i^T}\mathbf{P_i} + \epsilon_iI$ and $-H_{N,i}^TR_iH_{N,i} = -(H_i + \mathbf{N_i}C_i)^TR_i(H_i + \mathbf{N_i}C_i)$. By fixing one of the additional DOFs, i.e. $N_i = N_{i,0} = 0$ (and replacing the product P_i times L_i by a new variable PL_i), an LMI is obtained that can be solved efficiently [Boyd et al., 1994], for example in *MATLAB* with the the optimization toolbox *YALMIP* [Löfberg, 2004].

Notes on implementation: If an objective function is provided to the solver, it tries to optimize its value subject to the provided constraints. Since the UIO design problem is just a feasibility problem $L(x) < R(x) = 0$ (and no objective function is provided), the solver tries to minimize t and maximize the gap of inequality $L(x) < t * I + R(x)$ to obtain a value. Since the matrix has to be negative *semi*-definite, the value t only converges numerically to zero for a feasible solution (and the solver gives a warning about the possibility of marginal infeasibility). Furthermore, it can be possible to fix more parameters (sometimes by guessing), e.g. fix the Lyapunov matrix P_i to a diagonal structure to reduce the number of parameters and speed up the computation significantly, or to use the identity matrix. The parameter ϵ_i (in combination with P_i) contains information about the rate of convergence. It is also advisable to avoid very large observer gains L_i containing the inverse of P_i. The DOFs can be used by the designer for optimization, but it is difficult to give necessary conditions for inequality (12) to be solvable (besides the conditions required of other methods that are generalized) and needs further studies.

Remark 3: Compared to the very restrictive conditions for exact error linearization that can only be applied to a very small class of systems, the approximate error linearization using dissipativity used in this approach can deal with a larger class of systems [Moreno, 2005]. As mentioned above it is difficult and needs further studies to give conditions under which inequality (12) is solvable. But since the proposal is a generalization of other known methods like the High-Gain Observer (HGO), the conditions required by these methods provide a characterization [Rocha-Cózatl and Moreno, 2011]. The existence of an convergent observer depends on two properties: (i) the input-output (relative degree) structure between unknown inputs and measured outputs, and (ii) the shape of the nonlinearities determining a sector condition.

Remark 4: Compared to the approach of [Menon and Edwards, 2011], that applies a Sliding Mode Observer (SMO) to deal with the nonlinearities as unknown inputs, the method proposed in this paper can deal with a wider class of systems because the nonlinearities can explicitly be taken into account. Furthermore, by considering the interconnections as unknown input, the approach can deal with arbitrary complex interconnections. They could be nonlinear and time-varying, and thereby also include the case of switching topologies. The parameters or the whole function of the interconnection or the structure of the network can be unknown as it can reasonably be the case for large and complex real-world networks like biological networks, social networks, or the internet.

This approach raises the question of applicability in the case that some information about the interconnections, i.e. the form of the interconnection with uncertain edge weights, or only measurements in some of the nodes are available. While the rate of convergence of the UIO is restricted by the zero-dynamics of the observer error, in the other extreme of full information about the interconnections the network dynamics could be fully observable, and thereby arbitrary fast state estimation by an observer like the Luenberger is possible. The possibility to do this depends on the specific network dynamics in consideration and has to be analyzed for the specific example.

4. EPIDEMICS MONITORING

To illustrate the proposed network observer, we want to monitor the spread of a virus in a network of interconnected systems with the states of susceptibility (S), infection (I) and quarantine (Q) considering the SIQ-model from [Bernal J. et al., 2013]

$$\begin{aligned} \dot{p}_i &= -\tau_ip_i + (1 - p_i - q_i)[1 - \zeta_i(\boldsymbol{p})], \\ \dot{q}_i &= \tau_ip_i - [1 - \mu_i(q_i)]q_i, \end{aligned} \tag{17}$$

where p_i and q_i denote the probabilities of agent i to be infected or in quarantine, respectively. The transition probability from state (I) to (Q) is a constant parameter τ_i. $\zeta_i(\boldsymbol{p}) = \prod_{j=1}^{N}[1 - r_{ij}(t)\beta p_j(t)]$ is the probability of *not* becoming infected in one time unit that depends on the connection to the neighbors $r_{ij}(t)$, their infection states $p_j(t)$ and the probability of infection during a single contact β. In this set-up the transition probability from (Q) to (S) $\mu_i(q_i) = \frac{k_{0,i}}{q_i^2 - 2k_{1,i}q_i + k_{2,i}}, k_{0,i}, k_{1,i}, k_{2,i} > 0, k_{2,i} > k_{1,i}^2$ is represented by a nonlinearity.

If it is assumed that the connections $r_{ij}(t)$ are time-varying and cannot be known for all time instances t, or if the structure of the network is unknown, the above model (17) can be written alternatively as

$$\begin{aligned} \dot{p}_i &= -(1+\tau_i)p_i - q_i + w_i, \\ \dot{q}_i &= \tau_i p_i - \varphi_i(q_i), \end{aligned} \tag{18}$$

with unknown input $w_i = 1 - (1 - p_i - q_i)\zeta_i(\boldsymbol{p}) \in [0; 1]$ and $\varphi_i(q_i) = [1 - \mu_i(q_i)]q_i$.

Since the unknown input enters in the state p_i, measuring all values p_i, i.e. $y_i = p_i, i = 1, .., N$, satisfies the necessary relative degree condition (13). Applying the proposed dissipativity-based UIO Ω (9) with $N_i = 0, \forall i$, where $\widehat{p}_i$ and $\widehat{q}_i$ denote the observer states, yields dynamics of the errors $e_{p,i} = \widehat{p}_i - p_i$ and $e_{q,i} = \widehat{q}_i - q_i$

$$\begin{bmatrix} \dot{e}_{p,i} \\ \dot{e}_{q,i} \end{bmatrix} = \begin{bmatrix} -(1+\tau_i) + L_{1,i} & -1 \\ \tau_i + L_{2,i} & 0 \end{bmatrix} \begin{bmatrix} e_{p,i} \\ e_{q,i} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v_i + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_i,$$
$$v_i = -[\varphi(q_i) - \varphi(q_i + e_{q,i})]. \tag{19}$$

Since the states p_i are measured, i.e. $\widehat{p}_i \equiv y_i \equiv p_i$ and $e_{p,i} \equiv 0$, the system (19) is minimum-phase and the so-called *reduced-order* observer converges asymptotically (Fig. 3) if the origin of the error dynamics

$$\dot{e}_{q,i} = -[\varphi(q_i + e_{q,i}) - \varphi(q_i)] \tag{20}$$

is asymptotically stable. This is the case if φ is strictly increasing, e.g.

$$k_{2,i} > k_{1,i}^2 + k_{0,i}. \tag{21}$$

If the measured states p_i shall also be estimated by the so-called *full-order* observer, e.g. to reconstruct the unknown input w_i, the observer gain has to satisfy $L_{1,i} < 1 + \tau_i$ in order to converge. The zero-dynamics of the error (20) can be decoupled from the unknown input by choosing $L_{2,i} = -\tau_i$ in (19) and than equal the dynamics of the reduced-order observer (20). For a non-vanishing unknown input or persistently acting disturbance, i.e. in the endemic case, the state estimate only converges to a tube around the real state of the plant (Fig. 2) as can be seen in the steady state of the error dynamics:

$$\begin{aligned} \dot{e}_{p,i} &= (-1 - \tau_i + L_{1,i})e_{p,i} - e_{q,i} + w_i = 0 \\ \Rightarrow e_{p,i} &= \frac{1}{1 + \tau_i - L_{1,i}}(w_i - \cancelto{0}{e_{q,i}}). \end{aligned} \tag{22}$$

This area can be made arbitrary small by increasing the absolute value of the observer gain $L_{1,i} << 0$. An extension of the proportional observer by an integral component of the observer error as proposed in [Moreno, 2008] can compensate the remaining offset and the error converges asymptotically to zero.

The observer is simulated for a scale-free network of $N = 500$ nodes (Fig. 1). In Fig. 2 the norm of the estimates and the true values of the states $\boldsymbol{p} = (p_1, .., p_N)$ are depicted for the endemic case with uniformly distributed random $\tau_i \in [0.1; 0.3]$, $\beta = 0.5$ and for two proportional observers with the gains $L_{1,i} = -1$ and $L_{1,i} = -10$, $\forall i$, respectively. The remaining offset against the true value becomes smaller with a larger absolute value of the observer gain. The proportional-integral observer with $L_{I,i} = -1$ (and $L_{1,i} = 0$), $\forall i$, can compensate this offset. The estimates of the states $\boldsymbol{q} = (q_1, .., q_N)$ converge asymptotically to the real value with the same behavior for all observers (Fig. 2) because this part of the error dynamics is decoupled by choosing $L_{2,i} = -\tau_i$ and asymptotically stable for all $k_{0,i}$, $k_{1,i}$, $k_{2,i}$ chosen from a uniform distribution that satisfies Eq. (21).

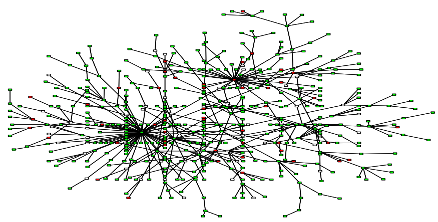

Fig. 1. Force-directed layout of scale-free network with $N = 500$ nodes.

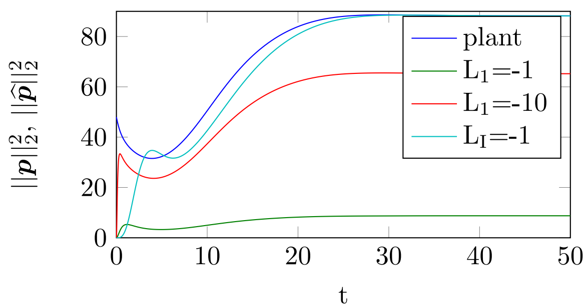

Fig. 2. Full-order observer estimates in state p

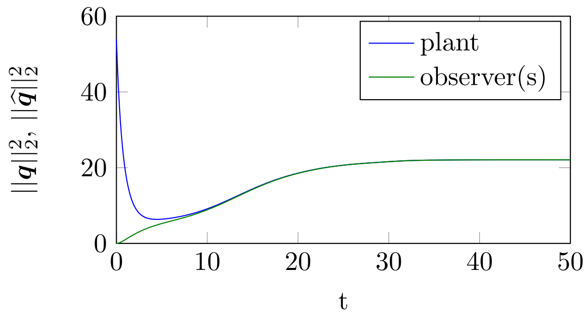

Fig. 3. Reduced-order observer estimates in state q

Considering an unknown input the output has to be measured in every node and the rate of convergence of the observer is restricted by the internal nonlinear system dynamics and there is no possibility to increase it. Furthermore, the stability depends on the (nominal) nonlinearity and there is no way to increase the robustness (by the observer correction term).

5. CONCLUSION AND OUTLOOK

In this paper a state observer design method for complex networks dynamics based on dissipativity was developed and presented. The proposed approach was applied to monitor an epidemic in a complex network based on an SIQ-model. The proposed full-order observer form was extended to converge under persistently acting disturbances.

The proposed network state observer can be applied to a wide class of systems. The network agents can have nonlinear dynamics and the form of the interconnection can be arbitrary complex, i.e. nonlinear, time-varying, time-switching or even unknown if the topology of the network is unknown. As long as the interconnections satisfy the necessary rank condition on the relation in which states they enter the dynamics of the agents and which states are measured, the network dynamics can be transformed to the UIO form. Compared to a recent approach of [Menon and Edwards, 2011] that applies an SMO to a network with linear interconnections to treat the nonlinearities as disturbances, the proposed dissipativity-based approach can deal with a wider class of systems since the nonlinearity can be explicitly taken into account. Compared to the SMO the proposed observer can deal with unbounded unknown inputs and the design simplifies to solving an LMI while it can be difficult to find a sufficiently large observer gain for the SMO. Furthermore, in contrast to the SMO the proposed UIO can deal with unbounded unknown inputs. A full characterization of the class of systems requires further studies although it is characterized as the generalization of other methods like the High-Gain Observer (HGO). For the transformed system a sufficient existence condition for an UIO is presented in terms of an LMI that can be solved efficiently and yields the observer parameters.

The approach is illustrated for the example of monitoring an epidemic in a complex network based on a nonlinear SIQ-model. A reduced-order observer is obtained that converges asymptotically if the dynamics are detectable. The full-order observer is extended by an integral component to converge under a non-vanishing unknown input in the endemic case.

Aspects which should be analyzed in future studies are:

- the detectability/observability-analysis w.r.t.
    - · the sensor locations and
    - · the level of information about the interconnections, i.e. parametric uncertainties;
- the characterization of the class of systems
- and the optimization of the additional DOFs.

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