Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Network Reconstruction from Intrinsic Noise: Non-Minimum-Phase Systems ⋆

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Abstract: This paper considers the problem of inferring the structure and dynamics of an unknown network driven by unknown noise inputs. Equivalently we seek to identify direct causal dependencies among manifest variables only from observations of these variables. We consider linear, time-invariant systems of minimal order and with one noise source per manifest state. It is known that if the transfer matrix from the inputs to manifest states is minimum phase, then this problem has a unique solution, irrespective of the network topology. Here we consider the general case where the transfer matrix may be non-minimum phase and show that solutions are characterized by an Algebraic Riccati Equation (ARE). Each solution to the ARE corresponds to at most one spectral factor of the output spectral density that satisfies the assumptions made. Hence in general the problem may not have a unique solution, but all solutions can be computed by solving an ARE and their number may be finite.

Keywords: Linear networks, Closed-loop identification, Identifiability, Noise power spectrum, Biological networks

1. INTRODUCTION

Many phenomena are naturally described as networks of interconnected dynamical systems and the identification of the dynamics and structure of a network has recently become an important problem. Given a model class, the problem is typically underdetermined and additional assumptions on the network structure are made, such as sparsity or restriction to particular topologies. We focus here on Linear, Time-Invariant (LTI) systems, for which there are still many interesting theoretical questions outstanding, and leave the network structure unrestricted.

Gonçalves and Warnick (2008) characterized identifiability in the deterministic case where targeted, known inputs may be applied to the network. In practice however, and often in biological applications, these types of experiments are not possible or are expensive to conduct. One may simply be faced with the outputs of an existing network driven by its own intrinsic variation. This problem has been considered in various forms in the literature.

In Shahrampour and Preciado (2013) for example, networks of known, identical subsystems are treated, which can be identified using an exhaustive grounding procedure similar to that in Nabi-Abdolyousefi and Mesbahi (2012). Materassi and Salapaka (2012a) present a solution for identifying the undirected structure for a restricted class of polytree networks; and in Materassi and Salapaka (2012b) for "self-kin" networks. In Van den Hof et al. (2013) the problem is posed as a closed-loop system identification problem for a more general, but known, topology; and in Dankers et al. (2012), the authors claim that their method can also be applied to networks with unknown topology.

In all of the above-cited work, the intrinsic variation is modeled as unknown noise sources applied only to the states that are measured. Whilst being an unrealistic assumption in some applications, this input requirement was shown in Gonçalves and Warnick (2008) to be necessary for solution uniqueness even in the deterministic case. Under this assumption, in Hayden et al. (2014) we characterized all minimal solutions to this problem for minimum-phase transfer matrices with the result that both the network topology and dynamics are uniquely identifiable. Here we extend this approach to include all transfer matrices, in which case the solution is not necessarily unique.

Section 2 provides necessary background information on spectral factorization, structure in LTI systems and the network reconstruction problem. The main results are then presented in Section 3, followed by some discussion and a numerical example. Conclusions are drawn in Section 4.

Notation

Denote by $A(i,j)$, $A(i,:)$ and $A(:,j)$ element (i,j), row i and column j respectively of matrix A. For a matrix A with block (i,j) given by a_{ij}, define $a_{ij}^{(k)}$ as block (i,j) of A^k. Denote by A^T the transpose of A and by A^* the conjugate transpose. We use I and 0 to denote the identity and zero matrices with implicit dimension, where $e_i := I(:,i)$, and $\mathrm{diag}(a_1,\ldots,a_n)$ to denote the diagonal matrix with diagonal elements $a_1,\ldots,a_n$. We use standard notation to describe linear systems and omit the dependence on time t or Laplace variable s when the meaning is clear. We also define a *signed identity matrix* as any square, diagonal matrix J that satisfies $J(i,i) = \pm 1$.

⋆ This work was supported by the Engineering and Physical Sciences Research Council under Grants EP/G066477/1, EP/I03210X/1 and EP/G066477/1.

2. PRELIMINARIES

2.1 Spectral Factorization

Consider systems defined by the following Linear, Time-Invariant (LTI) representation:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{1}$$

with input $u(t) \in \mathbb{R}^m$, state $x(t) \in \mathbb{R}^n$, output $y(t) \in \mathbb{R}^p$, system matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ and transfer function:

$$G(s) = C(sI - A)^{-1}B + D$$

Make the following assumptions:

Assumption 1. The matrix A is Hurwitz.

Assumption 2. The system is driven by unknown white noise $u(t)$ with covariance $\mathbb{E}[u(t)u^T(\tau)] = I\delta(t-\tau)$.

Assumption 3. The system (A, B, C, D) is globally minimal.

The meaning of Assumption 3 is explained below. From $y(t)$, the most information about the system that can be obtained is the output spectral density:

$$\Phi(s) = \mathbb{E}[Y(s)Y^*(s)] = G(s)G^*(s)$$

The spectral factorization problem (see for example Youla (1961)) is that of obtaining spectral factors $G'(s)$ that satisfy: $G'G'^* = \Phi$. Note that the degrees of two minimal solutions may be different; hence the following definition.

Definition 1. (Global Minimality). For a given spectral density $\Phi(s)$, the globally-minimal degree is the smallest degree of all its spectral factors.

Any system of globally-minimal degree is said to be *globally minimal.* Anderson (1969) provides an algebraic characterization of all realizations of all spectral factors as follows. Given $\Phi(s)$, define the positive-real matrix $Z(s)$ to satisfy:

$$Z(s) + Z^*(s) = \Phi(s) \tag{2}$$

Minimal realizations of Z are related to globally-minimal realizations of spectral factors of Φ by the following lemma.

Lemma 1. (Anderson (1969)). Let (A, B_z, C, D_z) be a minimal realization of the positive-real matrix $Z(s)$ of (2), then the system (A, B, C, D) is a globally-minimal realization of a spectral factor of Φ if and only if the following equations hold:

$$\begin{aligned} RA^T + AR &= -BB^T \\ RC^T &= B_z - BD^T \\ 2D_z &= DD^T \end{aligned} \tag{3}$$

for some positive-definite and symmetric matrix $R \in \mathbb{R}^{n\times n}$.

This result was used by Glover and Willems (1974) to provide conditions of equivalence between any two such realizations.

Lemma 2. (Glover and Willems (1974)). If (A, B, C, D) and (A', B', C', D') are globally-minimal systems, then they have equal output spectral density if and only if:

$$A' = T^{-1}AT \tag{4a}$$
$$C' = CT \tag{4b}$$
$$SA^T + AS = -BB^T + TB'B'^T T^T \tag{4c}$$
$$SC^T = -BD^T + TB'D'^T \tag{4d}$$
$$DD^T = D'D'^T \tag{4e}$$

for some invertible $T \in \mathbb{R}^{n\times n}$ and symmetric $S \in \mathbb{R}^{n\times n}$.

For any two systems that satisfy Lemma 2 for a particular S, all additional solutions for this S may be parameterized by Corollary 1. This is adapted from Glover and Willems (1974) where it was stated for minimum-phase systems.

Corollary 1. If (A, B, C, D) and (A', B', C', D') satisfy Lemma 2 for a particular S, then all systems that also satisfy Lemma 2 with (A, B, C, D) for the same S are given by:

$$(T'A'T'^{-1}, T'B'U, C'T'^{-1}, D'U)$$

for some invertible $T' \in \mathbb{R}^{n\times n}$ and orthogonal $U \in \mathbb{R}^{m\times m}$.

2.2 Structure in LTI Systems

We now suppose that there is some unknown underlying system (A_0, B_0, C_0, D_0) with transfer function G_0 and we wish to obtain some information about this system from its spectral density Φ_0. Clearly from Lemma 2 and Corollary 1 its transfer function can only be found up to some choice of orthogonal U and symmetric S. Even given G_0, the system matrices can clearly only be found up to some change in state basis. Hence the following assumption is made:

Assumption 4. The matrices $C = [I\ 0]$ and $D = 0$.

The form of C implies a partitioning of the states into *manifest* variables which are directly observed and *latent* variables which are not. The form of D restricts the systems to be strictly proper, and hence causal. For this class of systems we seek to identify causal dependencies among manifest variables, defined in Gonçalves and Warnick (2008), as follows.

Partition (1) under Assumption 4:

$$\begin{bmatrix} \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u, \qquad y = [I\ 0] \begin{bmatrix} y \\ z \end{bmatrix} \tag{5}$$

where $z(t) \in \mathbb{R}^l$ are the $l = n - p$ latent states. Taking the Laplace transform of (5) and eliminating Z yields $sY = WY + VU$, for proper transfer matrices:

$$\begin{aligned} W &:= A_{11} + A_{12}\left(sI - A_{22}\right)^{-1} A_{21} \\ V &:= B_1 + A_{12}\left(sI - A_{22}\right)^{-1} B_2 \end{aligned} \tag{6}$$

Define $W_D := \operatorname{diag}(W(1,1), \dots, W(p,p))$ and subtract $W_D Y$ from both sides to give $Y = QY + PU$, where

$$\begin{aligned} Q &:= \left(sI - W_D\right)^{-1}\left(W - W_D\right) \\ P &:= \left(sI - W_D\right)^{-1} V \end{aligned} \tag{7}$$

are strictly-proper transfer matrices of dimension $p \times p$ and $p \times m$ respectively. Note that Q is constructed to have diagonal elements equal to zero (it is hollow).

Definition 2. (Dynamical Structure Function). Given any system (1) under Assumption 4, the Dynamical Structure Function (DSF) is defined as (Q, P), where Q and P are given in (7).

The DSF defines a digraph with only the manifest states and inputs as nodes. There is an edge from $Y(j)$ to $Y(i)$ if $Q(i,j) \neq 0$; and an edge from $U(j)$ to $Y(i)$ if $P(i,j) \neq 0$. In this sense, the DSF characterizes causal relations among manifest states Y and inputs U in system (1). The transfer function G is related to the DSF as follows:

$$G = (I - Q)^{-1} P \tag{8}$$

where, given G, the matrices Q and P are not unique in general, hence the following definition is made.

Definition 3. (Consistency). A DSF (Q, P) is consistent with a transfer function G if (8) is satisfied.

The relationship between state space, DSF and transfer function representations is illustrated in Figure 1, which shows that a state-space realization uniquely defines both a DSF and a transfer function. However, multiple DSFs are consistent with a given transfer function and a given DSF can be realized by multiple state-space realizations.

All realizations of a particular G are parameterized by the set of invertible matrices $T \in \mathbb{R}^{n\times n}$. A subset of these will not change the DSF as follows.

Definition 4. ((Q,P)-invariant transformation). A state transformation T of system (A_0, B_0, C_0, D_0) with DSF (Q_0, P_0) is (Q,P)*-invariant* if the transformed system:

$$(TA_0T^{-1}, TB_0, C_0T^{-1}, D_0)$$

also has DSF (Q_0, P_0).

The blue region in Figure 1(a) is the set of all (Q,P)-invariant transformations of $(A_0, B_0, [I\ 0], 0)$.

2.3 Network Reconstruction

The network reconstruction problem was cast in Gonçalves and Warnick (2008) as finding exactly (Q_0, P_0) from G_0. Since in general multiple DSFs are consistent with a given transfer function, some additional *a priori* knowledge is required for this problem to be well posed. It is common to assume the following structure of P.

Assumption 5. The matrix P is square, diagonal and full rank.

This is a standard assumption in the literature (Shahrampour and Preciado (2013); Materassi and Salapaka (2012b); Van den Hof et al. (2013)) and equates to knowing that each of the manifest states is *directly* affected only by one particular input. The following theorem is adapted form Corollary 1 of Gonçalves and Warnick (2008):

Theorem 1. (Gonçalves and Warnick (2008)). There is at most one DSF (Q,P) with P square, diagonal and full rank that is consistent with a given transfer function G.

Given a G_0 for which the generating system is known to have P_0 square, diagonal and full rank, one can therefore uniquely identify the "true" DSF (Q_0, P_0).

2.4 A Realization for Diagonal P

The presence of latent states allows some freedom in the choice of realization used to represent a particular DSF. It will be convenient to use a particular form for systems with P square, diagonal and full rank, defined here.

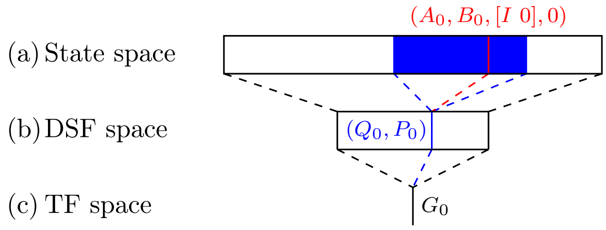


Fig. 1. Pictorial representation of relationship between state space, DSF space and transfer function space for a particular system $(A_0, B_0, [I\ 0], 0)$. In (a) is contained the set of state transformations of this system by matrices T that preserve $C_0 = [I\ 0]$; in red is the particular realization with $T = I$ and in blue the set of realizations with the same DSF (Q_0, P_0). In (b) is the set of all DSFs that have realizations in (a); in blue is the particular DSF (Q_0, P_0). In (c) is the single transfer function G_0, with which are consistent all DSFs in (b) and which can be realized by all realizations in (a).

Lemma 3. The matrix V (and hence P) is diagonal if and only if the matrices:

$$B_1 \qquad \text{and} \qquad A_{12}A_{22}^k B_2$$

for $k = 0, 1, \ldots, l-1$ are diagonal, where $l = \dim(A_{22})$.

The proof is omitted but follows from expressing V in (6) as a Neumann series and making use of the Cayley-Hamilton Theorem. It is clear that P is diagonal if and only if V is. Then without loss of generality, we may order the manifest states of any system (A_0, B_0, C_0, D_0) with P_0 square, diagonal and full rank such that it can be transformed using (Q,P)-invariant transformations into one in the following form (see Hayden et al. (2013)).

Lemma 4. Any DSF (Q,P) with P square, diagonal and full rank has a realization with A_{12}, A_{22}, B_1 and B_2 as follows:

$$\left[\begin{array}{c|c} A_{12} & B_1 \\ \hline A_{22} & B_2 \end{array}\right] = \left[\begin{array}{cccc|ccc} 0 & 0 & I & 0 & 0 & 0 & 0 \\ 0 & \gamma_{22} & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \gamma_{34} & 0 & 0 & B_{22} \\ \hline \times & \times & \times & \alpha_{14} & I & 0 & 0 \\ \times & \times & \times & \alpha_{24} & 0 & I & 0 \\ \alpha_{31} & 0 & \times & \alpha_{34} & 0 & 0 & 0 \\ \times & \times & \times & \alpha_{44} & 0 & 0 & 0 \end{array}\right] \tag{9}$$

where $B_{22} \in \mathbb{R}^{p_3\times p_3}$ and $\gamma_{22} \in \mathbb{R}^{p_2\times p_2}$ are square, diagonal and full rank and $\times$ denotes an unspecified element. The dimension of B_1 is $p = p_1 + p_2 + p_3$ and the matrix $\alpha_{31} \in \mathbb{R}^{p_1\times p_1}$ is square and diagonal but not necessarily full rank. The matrix A_{22} satisfies the following property for $i = 1, \ldots, p_1$:

$$\begin{aligned} &\alpha_{31}(i,:) = \alpha_{31}^{(2)}(i,:) = \cdots = \alpha_{31}^{(k_i-1)}(i,:) = 0^T \\ &\alpha_{31}^{(k_i)}(i,:) = \alpha_{31}^{(k_i)}(i,i)e_i^T \neq 0^T \\ &\alpha_{34}^{(k_i)}(i,:) = 0^T \end{aligned}$$

for some k_i such that $0 < k_i < l$, where $l = \dim(A_{22})$ and $\alpha_{ij}^{(k)}$ denotes block (i,j) of A_{22}^k.

The proof is omitted but follows by applying state transformations to the original system and showing that the transformed system still satisfies Lemma 3. From Theorem 1 any state transformation that preserves P diagonal is (Q,P)-invariant.

3. NETWORK RECONSTRUCTION BY SPECTRAL FACTORIZATION

Given an underlying system (A_0, B_0, C_0, D_0) with DSF (Q_0, P_0), transfer function G_0 and output spectral density Φ_0 under Assumptions 1 - 5, we seek to identify (Q_0, P_0) from Φ_0. From Theorem 2 of Hayden et al. (2014) we know that $G_0 J$, and hence $(Q_0, P_0 J)$, can be found up to a choice of signed identity matrix J from Φ_0 if G_0 is minimum phase. Using Corollary 1 we can extend this as follows.

Corollary 2. If two systems (A, B, C, D) and (A', B', C', D') under Assumptions 1 - 5 with DSFs (Q, P) and (Q', P') satisfy Lemma 2 for a particular S, then all additional systems that also satisfy Lemma 2 with (A, B, C, D) for the same S have DSFs:

$$(Q', P'J)$$

for some signed identity matrix J. Each solution S to Lemma 2 therefore corresponds to at most one solution for the DSF $(Q', P'J)$ for some choice of J.

In this section we prove that for a given system (A, B, C, D), solutions S to Lemma 2 can be partitioned into two parts: the first must be zero and the second must solve an Algebraic Riccati Equation (ARE) whose parameters are determined only by the original system.

3.1 Main Results

The main result is obtained by evaluating solutions to Lemma 2 for any two realizations that satisfy Assumptions 1 - 5 and are in the form of Lemma 4. Immediately (4) yields:

$$A' = T^{-1} A T \tag{10a}$$
$$B_1' B_1'^T = B_1 B_1^T \tag{10b}$$
$$A_{12} S_2 = B_1 B_1^T T_1^T \tag{10c}$$
$$S_2 A_{22}^T + A_{22} S_2 = -B_2 B_2^T + T_1 B_1 B_1^T T_1^T + T_2 B_2 B_2^T T_2^T \tag{10d}$$

where

$$S = \begin{bmatrix} 0 & 0 \\ 0 & S_2 \end{bmatrix} \quad \text{and} \quad T = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{11}$$

with $S_2 \in \mathbb{R}^{l \times l}$, $T_1 \in \mathbb{R}^{l \times p}$ and $T_2 \in \mathbb{R}^{l \times l}$. We can further partition S_2 as follows.

Lemma 5. For any two systems (A, B, C, D) and (A', B', C', D') in the form of Lemma 4 that satisfy Assumptions 1-5 and Lemma 2, the matrix S_2 in (11) satisfies:

$$S_2 = \begin{bmatrix} 0 & 0 \\ 0 & s_{22} \end{bmatrix}$$

where $s_{22} \in \mathbb{R}^{(l-p) \times (l-p)}$.

Proof. The proof is given for the case where $p_3 = p_2 = 0$, for which the notation is considerably simpler. The proof of the general case follows in exactly the same manner. In this case, (9) simplifies to:

$$\left[\begin{array}{c|c} A_{12} & B_1 \\ \hline A_{22} & B_2 \end{array}\right] = \left[\begin{array}{ccc|c} 0 & I & 0 & 0 \\ \hline \times & \times & \alpha_{14} & I \\ \alpha_{31} & \times & \alpha_{34} & 0 \\ \times & \times & \alpha_{44} & 0 \end{array}\right] \tag{12}$$

Equations (10c) and (10d) now simplify to:

$$A_{12} S_2 = 0 \tag{13a}$$
$$S_2 A_{22}^T + A_{22} S_2 + B_2 B_2^T - T_2 B_2 B_2^T T_2^T = 0 \tag{13b}$$

from which S_2 and $T_2 B_2$ are required to be in the following forms, partitioned as A_{22} and B_2:

$$S_2 = \begin{bmatrix} s_{11} & 0 & s_{12} \\ 0 & 0 & 0 \\ s_{12}^T & 0 & s_{22} \end{bmatrix} \quad \text{and} \quad T_2 B_2 = \begin{bmatrix} t_1 \\ 0 \\ t_3 \end{bmatrix} \tag{14}$$

We will now prove by induction that we must have $s_{11} = 0$ and $s_{12} = 0$ to satisfy (13) for any valid choice of T_2.

Hypothesis

Recall that for $i = 1, \ldots, p_1$, the number k_i is the smallest value of j in the range $0 < j < l$ such that $\alpha_{31}^{(j)}(i, i) \neq 0$. Hypothesize that the following statement holds for $i = 1, \ldots, p_1$ and for all $j = 1, \ldots, k_i - 1$:

$$\boxed{\begin{array}{c} \alpha_{34}^{(j)}(i,:) s_{12}^T = 0^T \\ \alpha_{34}^{(j)}(i,:) s_{22} = 0^T \end{array}} \tag{15}$$

Base case: $\mathbf{j = 1}$

Multiply (13b) by $A_{12}(i,:)$ on the left for some i in the range $1 \leq i \leq p_1$ with $k_i > 1$:

$$A_{12}(i,:) S_2 A_{22}^T + A_{12}(i,:) A_{22} S_2 + A_{12}(i,:) B_2 B_2^T - A_{12}(i,:) T_2 B_2 B_2^T T_2^T = 0^T$$

and note that directly from (12) and (14) terms one, three and four are zero. Hence we have:

$$A_{12}(i,:) A_{22} S_2 = 0^T \tag{16}$$

Since $\alpha_{31}(i,:) = 0^T$ (because $k_i > 1$), (16) gives:

$$\alpha_{34}(i,:) s_{12}^T = 0^T \quad \text{and} \quad \alpha_{34}(i,:) s_{22} = 0^T$$

The hypothesis (15) therefore holds for $j = 1$ for all $i = 1, \ldots, p_1$ with $k_i > 1$.

Induction

Suppose for some i in the range $1 \leq i \leq p_1$ with $k_i > k$, the hypothesis holds for $j = k - 1$. This implies that:

$$\alpha_{34}^{(k-1)}(i,:) s_{12}^T = 0^T \quad \text{and} \quad \alpha_{34}^{(k-1)}(i,:) s_{22} = 0^T \tag{17}$$

Now show that the hypothesis is satisfied for $j = k$ as follows. First multiply (13b) on the left by $A_{12}(i,:) A_{22}^{k-1}$ to give:

$$\begin{aligned} &A_{12}(i,:) A_{22}^{k-1} S_2 A_{22}^T + A_{12}(i,:) A_{22}^{k} S_2 \\ &+ A_{12}(i,:) A_{22}^{k-1} B_2 B_2^T \\ &- A_{12}(i,:) A_{22}^{k-1} T_2 B_2 B_2^T T_2^T = 0^T \end{aligned} \tag{18}$$

Since $\alpha_{31}^{(k-1)}(i,:) = 0^T$, the expression $A_{12}(i,:) A_{22}^{k-1} S_2$ is equal to zero from (17) and hence the first term in (18) is equal to zero. The third term is also zero due to $\alpha_{31}^{(k-1)}(i,:) = 0^T$. The remaining two terms give:

$$\alpha_{34}^{(k)}(i,:) s_{12}^T - \alpha_{34}^{(k-1)}(i,:) t_3 t_1^T = 0^T \tag{19a}$$
$$\alpha_{34}^{(k)}(i,:) s_{22}^T - \alpha_{34}^{(k-1)}(i,:) t_3 t_3^T = 0^T \tag{19b}$$

where we have used the fact that $\alpha_{31}^{(k)}(i,:) = 0^T$ since $k_i > k$. Now multiply (19b) on the right by $\alpha_{34}^{(k-1)}(i,:)^T$, which, from (17), gives:

$$\alpha_{34}^{(k-1)}(i,:)t_3 t_3^T \alpha_{34}^{(k-1)}(i,:)^T = 0$$

which implies $\alpha_{34}^{(k-1)}(i,:)t_3 = 0^T$. This eliminates all T_2 terms from (19), giving the desired result:

$$\alpha_{34}^{(k)}(i,:)s_{12}^T = 0^T \qquad \text{and} \qquad \alpha_{34}^{(k)}(i,:)s_{22}^T = 0^T$$

By induction the hypothesis (15) therefore holds for all $i = 1, \ldots, p_1$ for $j = 1, \ldots, k_i - 1$.

Termination

To show that s_{11} and s_{12} must be equal to zero, multiply (13b) on the left by $A_{12}(i,:)A_{22}^{k_i-1}$ for any i such that $1 \le i \le p_1$. Recall that:

$$\alpha_{31}^{(k_i)}(i,:) = \alpha_{31}^{(k_i)}(i,i)e_i^T \neq 0$$
$$\alpha_{34}^{(k_i)}(i,:) = 0^T$$

and hence the equivalent of (19) is:

$$\alpha_{31}^{(k_i)}(i,i)s_{11}(i,:) - \alpha_{34}^{(k_i-1)}(i,:)t_3 t_1^T = 0^T \tag{20a}$$
$$\alpha_{31}^{(k_i)}(i,i)s_{12}(i,:) - \alpha_{34}^{(k_i-1)}(i,:)t_3 t_3^T = 0^T \tag{20b}$$

Since the hypothesis (15) holds for $j = k_i - 1$, we know that $\alpha_{34}^{(k_i-1)}(i,:)s_{12}^T = 0^T$. Multiply (20b) on the right by $\alpha_{34}^{(k_i-1)}(i,:)^T$ to give $\alpha_{34}^{(k_i-1)}(i,:)t_3 = 0^T$ and (20) then simplifies to:

$$\alpha_{31}^{(k_i)}(i,i)s_{11}(i,:) = 0^T \qquad \Rightarrow \qquad s_{11}(i,:) = 0^T$$
$$\alpha_{31}^{(k_i)}(i,i)s_{12}(i,:) = 0^T \qquad \Rightarrow \qquad s_{12}(i,:) = 0^T$$

Since the above holds for every $i = 1, \ldots, p_1$, we therefore have $s_{11} = 0$ and $s_{12} = 0$.

The result of the above lemma significantly simplifies (10). Whilst there is some freedom in the choice of T_1 and T_2, the number of solutions for (Q, P) is always bounded by the number of solutions for s_{22}, as follows.

Theorem 2. Two systems (A, B, C, D) and (A', B', C', D') in the form of Lemma 4 that satisfy Assumptions 1-5 have equal output spectral density if and only if the following equations are satisfied:

$$s_{22}\bar{a} + \bar{a}^T s_{22} - s_{22}\bar{b}\bar{b}^T s_{22} = 0 \tag{21}$$
$$\alpha_{34}s_{22} = 0 \tag{22}$$

$$T_1 B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & s_{22}\gamma_{34}^T B_{22} \end{bmatrix} \tag{23a}$$

$$T_2 B_2 = \begin{bmatrix} t_1 & 0 \\ 0 & 0 \\ s_{22}\left[\alpha_{14}^T \; \alpha_{24}^T\right] t_1 & 0 \end{bmatrix} \tag{23b}$$

$$A' = T^{-1}AT \quad \text{for} \quad T = \begin{bmatrix} I & 0 \\ T_1 & T_2 \end{bmatrix} \tag{23c}$$

$$B'_{22} = B_{22}J \tag{23d}$$

for some symmetric $s_{22} \in \mathbb{R}^{(l-p)\times(l-p)}$, where $\bar{a} = \alpha_{44}^T$, $\bar{b} = \left[\gamma_{34}^T B_{22}^{-T} \; \alpha_{14}^T \; \alpha_{24}^T \; \alpha_{34}^T\right]$, $t_1 \in \mathbb{R}^{(p_1+p_2)\times(p_1+p_2)}$ is orthogonal and $J \in \mathbb{R}^{p_3 \times p_3}$ is a signed identity matrix.

Proof. The proof follows directly from Lemma 5 by substituting $S_2 = \begin{bmatrix} 0 & 0 \\ 0 & s_{22} \end{bmatrix}$ into (10).

Remark 1. From Corollary 2, the number of DSFs that have equal spectral density to that of any given (Q_0, P_0) (up to a choice of signed identity matrix) is therefore at most equal to the number of solutions to the ARE (21). This significantly restricts the solution set.

Remark 2. It is straightforward to see that $(\bar{a}, \bar{b})$ is controllable due to the minimality of (A, B, C, D). The number of solutions to (21) is therefore a function of the Jordan Form of α_{44} and in particular is finite if and only if every eigenvalue of α_{44} has geometric multiplicity of one (see Lancaster and Rodman (1995)). In general the solution will not be unique.

Remark 3. Any solution to the ARE must also satisfy (22) in order to satisfy Theorem 2. In general this condition will not necessarily be satisfied, reducing the size of the DSF solution set. Given any system (A_0, B_0, C_0, D_0) with DSF (Q_0, P_0), the solution set of (21) can be calculated; for any solution that also satisfies (22) it is straightforward to choose T and J to satisfy (23) and hence construct the set of DSFs with equal spectral density to (Q_0, P_0).

Remark 4. Given a particular spectral density Φ_0, one can partition (3) in Lemma 1 to obtain equations similar to (21)-(23). This provides a constructive procedure to compute all DSFs with spectral density Φ_0.

3.2 Example

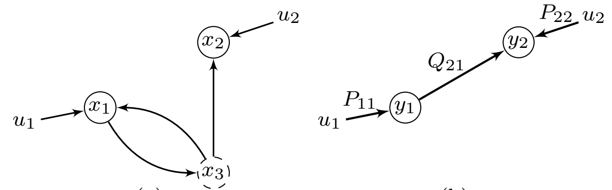
A simple example is given in which two globally-minimal systems with P diagonal have the same output spectral density. These systems are obtained from the spectral density by solving the equations in Lemma 1 to illustrate that they are the only such systems with this particular spectral density. Start with the following stable, minimal system with two manifest states and one latent state:

$$A_0 = \begin{bmatrix} -1 & 0 & 4 \\ 0 & -2 & 5 \\ -6 & 0 & -3 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

with $C_0 = [I \; 0]$ and $D_0 = 0$. The transfer matrix $G_0 = C_0 \left(sI - A_0\right)^{-1} B_0$ can be realized by an infinite variety of A and B matrices, whereas the DSF:

$$Q_0(s) = \begin{bmatrix} 0 & 0 \\ \frac{-30}{(s+2)(s+3)} & 0 \end{bmatrix}, \quad P_0(s) = \begin{bmatrix} \frac{s+3}{s^2+4s+27} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix}$$

is the only valid Q and diagonal P that is consistent with G_0. This system is represented graphically for state space (a) and DSF (b) like so:



(a) (b)

The DSF describes causal interactions among manifest states that may occur via latent states in the underlying system. Compute the output spectral density $\Phi(s)$ for this system and from it construct the positive real matrix $Z(s)$, such that $Z(s) + Z^*(s) = \Phi(s)$. Construct any minimal realization of Z by standard methods, such as:

$$A = \begin{bmatrix} -3.9 & -0.97 & 1.9 \\ -3.6 & -3.2 & 2.4 \\ -15.5 & -1.5 & 1.1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.17 & 0.032 \\ 0.032 & 0.57 \\ 0.092 & 0.60 \end{bmatrix},$$

with $C = [I\ 0]$ and $D = 0$. Partition B_z as B, then all solutions to (3) are given by:

$$B_z = \begin{bmatrix} B_{z1} \\ B_{z2} \end{bmatrix}, \qquad R = \begin{bmatrix} B_{z1} & B_{z2}^T \\ B_{z2} & R_2 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}$$

where R_2 must solve a scalar ARE. This ARE has exactly two solutions: $R_2 = 1.02$ or 1.65, and correspondingly

$$B_2 = [1.49\ 0.5] \quad \text{or} \quad [0.28\ -1.01]$$

with $D = D_z = 0$. Choose the signs of B_1 to be positive for simplicity, then there are exactly two solutions (A, B, C, D), one for each of the possible choices of B_2. Transform both of these into the form of Lemma 4, then the transformed systems are both realizations of DSFs with diagonal P. The first corresponds to the original system (A_0, B_0, C_0, D_0) and the second to the following stable, minimal system:

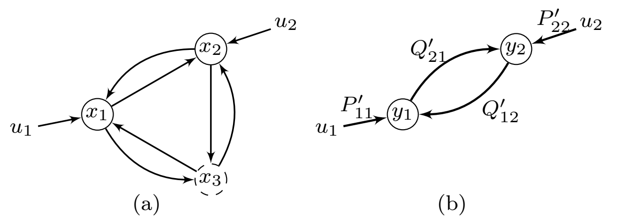
$$A' = \begin{bmatrix} -3.3 & -2.9 & 4 \\ -2.9 & -5.7 & 5 \\ -8.3 & -3.7 & 3 \end{bmatrix}, \quad B' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

with $C' = [I\ 0]$, $D' = 0$ and DSF:

$$Q'(s) = \begin{bmatrix} 0 & \frac{-2.9(s+2.0)}{s^2+0.34s+23.3} \\ \frac{-2.9(s+11.1)}{s^2+2.7s+1.3} & 0 \end{bmatrix},$$

$$P'(s) = \begin{bmatrix} \frac{s-3}{s^2+0.34s+23.3} & 0 \\ 0 & \frac{s-3}{s^2+2.659s+1.3} \end{bmatrix}$$

Note that this system has a different network structure to the original system for both state space (a) and DSF (b):



(a) (b)

The reader may verify that these systems do indeed have the same output spectral density $\Phi(s)$. It may also be verified that (10) is satisfied for the following S and T:

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -0.15 \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -0.59 & -0.73 & 1 \end{bmatrix}$$

These are the only two DSFs (up to a choice of signed identity matrix) with this particular spectral density.

4. CONCLUSIONS

This paper characterizes solutions for the structure and dynamics of an unknown network driven by unknown intrinsic noise. For stable LTI systems with standard assumptions on the noise, we provide necessary and sufficient conditions that must be satisfied by any solution. One of these conditions is an Algebraic Riccati Equation (ARE) and we show that the number of solutions for the network is at most equal to the number of solutions to this ARE. From the output spectral density we can therefore construct the network topology and dynamics of every solution, regardless of the topology. To our knowledge this is the first such result that places no restrictions on the structure of the network. The fact that the solution may be non unique highlights a fundamental limitation, whilst the fact that the number of solutions may be finite is a very positive result. This paper addresses the *identifiability* of the network; how to efficiently find solutions in practice remains the subject of substantial future work.

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