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Output Control Approach for Delayed Linear Systems with Adaptive Rejection of Multiharmonic Disturbance

Anton A. Pyrkin * **Alexey A. Bobtsov** *,** **Vladimir O. Nikiforov** * **Alexey A. Vedyakov** * **Sergey A. Kolyubin** *** **Oleg I. Borisov** *

* *Department of Control Systems and Informatics, ITMO University, St. Petersburg, Russia*
** *Laboratory "Control of Complex Systems", Institute for Problems of Mechanical Engineering, St. Petersburg, Russia*
*** *Department of Engineering Cybernetics, Norwegian University of Science and Technology, Trondheim, NO-7491, Norway*

Abstract: We present a new output control approach for a linear plant with input delay and an unknown multiharmonic disturbance acting on the input and the output. To solve this problem, we combine the well-known predictor feedback approach with the state observer and adaptive scheme that identifies the frequency of the disturbance. Compared to the existing approaches, the structure of our adaptive scheme is the simplest and the results apply to plants that are unstable, non-minimum phase, and have an arbitrary relative degree.

Keywords: input delay, disturbance rejection, adaptive control, frequencies identification.

1. INTRODUCTION

In this paper we consider a unstable linear plant with an additive multisinusoidal disturbance affected the input and the output. The plant has a delay in the control channel and the disturbance has unknown frequencies, phases, amplitudes, and the common offset. We design the adaptive control law that stabilizes the system and at the same time cancels the unknown disturbance providing stabilization and exponentially decaying to zero of the state variables. First, we design the adaptive scheme to identify the frequencies of the disturbance. Then we establish the observer for the disturbance in current and predicted time. And eventually we stabilize the plant with predictor-based feedback, the concept that originated with the Smith predictor (Smith [1959]).

A lot of papers exist today that devoted to dealyed system with various problem formulation. The most important of them deal with input delay. Fundamental and elegant solutions were found and investigated for linear and nonlinear unstable systems (Krstic et al. [2008], Krstic [2009]). However the control problem for systems with uncertain input and output disturbances and the input delay is still actual because of difficulty. The stability result (Krstic et al. [2008], Krstic [2009]) for unstable linear system with the input delay was extended for the case of cancellation the unknown external disturbance (Pyrkin et al. [2010 a,b]). In this paper we would like to focus on more general case, complicated by unknown input and output disturbance, the input delay and open-loop instability of the nonminimum-phase plant.

The problem of the output disturbance can be found in the real technical plants. For example, the external disturbance exists and drives the plant somehow. Regulated variables are not significantly changing while the sensors located on the plant moves because of the disturbance. Such problem is very actual in dynamic positioning systems of the surface vessels.

With the understanding that the literature overview below may not be exhaustive, we consider some of the most well-known modern approaches to such problems as developed by researches R. Marino and P. Tomei and the co-authors (Marino et al. [2007, 2008], Marino and Tomei [2011, 2013]).

In (Aranovskii et al. [2009]) the algorithm is proposed to the design of an adaptive observer for an unknown sinusoidal disturbance that affects the output of a non-minimum phase linear control plant. In serial of works the frequency estimation approaches was significantly improved (Bobtsov and Pyrkin [2012], Pyrkin et al. [2010 a,b]). In (Pyrkin et al. [2010 a,b]) the exponential convergence of frequency estimator has been proved. In (Bobtsov and Pyrkin [2010]) the frequency estimation scheme was extended to a multisinusoidal case. Moreover, the

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frequency estimation scheme proposed in (Bobtsov and Pyrkin [2010], Pyrkin et al. [2010 a,b]) is simpler in comparison with known analogues (Hsu et al. [1999], Marino et al. [2007, 2008], Marino and Tomei [2011, 2013], Xia [2002]).

2. PROBLEM FORMULATION

In this paper we will consider the plant of the following view

$$\dot{x}(t) = Ax(t) + Bu(t-D) + B\delta(t), \tag{1}$$
$$y(t) = Cx(t) + \alpha\delta(t), \tag{2}$$

where $x \in \mathbb{R}^n$ is the unknown vector of state variables, $y(t)$ is the measurable scalar output, $u(t)$ is the scalar input with initial condition $u(t-D) = 0$ for $t < D$, $D \geq 0$ is known constant delay, A, B, C are corresponding matrices with known parameters, α is a non-zero number.

The input disturbance $\delta(t)$ has a view:

$$\delta(t) = \sigma + \sum_{i=1}^{k} \left[\mu_i \sin(\omega_i t) + \nu_i \cos(\omega_i t)\right], \tag{3}$$

and represented by sum of k sinusoids $\delta_i(t)$ with unknown amplitudes μ_i and ν_i that are not equal to zero simultaneously, frequencies ω_i, and σ is the common offset.

The objective is to find the control $u(t)$ that achieves regulation of the output

$$\lim_{t\to\infty} x(t) = 0\,, \tag{4}$$

under the following assumptions:

Assumption 1. A, B, C, and α are known.

Assumption 2. (A, B, C) is a completely controllable and observable triple.

Assumption 3. A lower bound on the disturbance frequencies is known.

3. FREQUENCY ESTIMATION

Our first purpose is to find and reject the disturbance. Consider the linear observer

$$\dot{\hat{x}}_1(t) = A\hat{x}_1(t) + Bu(t-D) + L\tilde{y}_1(t), \tag{5}$$
$$\hat{y}_1(t) = C\hat{x}_1(t). \tag{6}$$

Then the observer error is given by

$$\tilde{x}_1(t) = x(t) - \hat{x}_1(t), \tag{7}$$
$$\dot{\tilde{x}}_1(t) = (A - LC)\tilde{x}_1(t) + B\delta(t) - L\alpha\delta(t), \tag{8}$$
$$\tilde{y}_1(t) = y(t) - \hat{y}(t) = C\tilde{x}(t) + \alpha\delta(t), \tag{9}$$

where the matrix L makes the matrix $A - LC$ Hurwitz.

Passing to the Laplace images for (8) we can extract the state vector of the observer error

$$\tilde{x}_1(s) = (sI - (A - LC))^{-1}\left[(B - L\alpha)\delta(s) + \tilde{x}_1(0)\right], \tag{10}$$

where s is a complex variable, $\tilde{x}_1(s) = \mathcal{L}\{\tilde{x}_1(t)\}$ is a Laplace image of the function $\tilde{x}_1(t)$

Using (9) and (10) we obtain the Laplace images for $\tilde{y}_1(t)$

$$\tilde{y}_1(s) = \left[C(sI - (A - LC))^{-1}(B - L\alpha) + \alpha\right]\delta(s) + C(sI - (A - LC))^{-1}\tilde{x}_1(0). \tag{11}$$

From (3) we find $\delta(s)$

$$\delta(s) = \sigma\frac{1}{s} + \sum_{i=1}^{k} \mu_i \frac{\omega_i}{s^2 + \omega_i^2} + \nu_i \frac{s}{s^2 + \omega_i^2}. \tag{12}$$

Since the matrix $A - LC$ is Hurwitz, after the inverse Laplace transform of (11) we obtain

$$\tilde{y}_1(t) = \bar{\sigma} + \bar{\varepsilon}(t) + \sum_{i=1}^{k} \left[\bar{\mu}_i \sin(\omega_i t) + \bar{\nu}_i \cos(\omega_i t)\right], \tag{13}$$

where $\bar{\sigma}$, $\bar{\mu}_i$, $\bar{\phi}_i$ are constants and $\bar{\varepsilon}(t)$ is exponentially decaying term. Since $\varepsilon(t)$ is transients in stable system with Hurwitz state matrix these functions can be represented as a sum of decaying exponents multiplying constants, polynomials, or sinusoids. Therefore, derivatives of these functions are also exponentially decaying.

Following the idea presented in (Bobtsov and Pyrkin [2010, 2012], Pyrkin et al. [2010 a,b]) we use the signal $\tilde{y}_1$ to estimate the frequencies of the disturbance. We start by introducing the linear filter

$$\xi(s) = \frac{\lambda_0^{2k}}{\gamma(s)}\tilde{y}_1(s), \tag{14}$$

where $\lambda_0 > 0$, $\gamma(s) = s^{2k} + \gamma_{2k-1}s^{2k-1} + \cdots + \gamma_1 s + \gamma_0$ is a Hurwitz polynomial with $2k$ different eigenvalues $\lambda_j, j = 1, \ldots, 2k$. Let $\gamma_0 = \lambda_0^{2k}$ and $\lambda = \min_{j=1,\ldots,2k}\{|\mathrm{Re}\,\lambda_j|\}$.

Lemma 1. For the filter (14) and the input signal (13) the relation

$$\xi^{(2k+1)}(t) = \Omega^T(t)\bar{\Theta} + \varepsilon(t) \tag{15}$$

holds, where $\Omega^T(t) = \left[\xi^{(2k-1)}(t) \,\ldots\, \xi^{(3)}(t)\; \xi^{(1)}(t)\right]$ is a regressor of functions $\xi^{(j)}(t)$ that are derivatives of the output variable of the linear filter (14)

$$\xi^{(j)}(s) = \frac{\lambda_0^2 s^j}{\gamma(s)}\tilde{y}_1(s), \tag{16}$$

and $\bar{\Theta}^T = \left[\bar{\theta}_1 \,\ldots\, \bar{\theta}_{k-1}\; \bar{\theta}_k\right]$ is a vector of parameters depending on frequencies

$$\begin{cases} \bar{\theta}_1 = \theta_1 + \theta_2 + \cdots + \theta_k, \\ \bar{\theta}_2 = -\theta_1\theta_2 - \theta_1\theta_3 - \cdots - \theta_{k-1}\theta_k, \\ \vdots \\ \bar{\theta}_k = (-1)^{k+1}\theta_1\theta_2\cdots\theta_k. \end{cases} \tag{17}$$

where $\theta_i = -\omega_i^2$, the function $|\varepsilon(t)| \leq \rho_0 e^{-\lambda t}$ and its derivatives are bounded by an exponentially decaying function.

Proof 1. It is well known (Bobtsov and Pyrkin [2010, 2012]) that signal (13) is the solution of the system

$$p^{2k+1}\tilde{y}_1(t) = \bar{\theta}_1 p^{2k-1}\tilde{y}_1(t) + \ldots + \bar{\theta}_k p\tilde{y}_1(t) + \varepsilon_1(t), \tag{18}$$

where $p = d/dt$ is the differentiation operator, $\varepsilon_1(t)$ is an exponentially decaying function of time accosiated with the term $\bar{\varepsilon}(t)$ in (13).

Taking the Laplace transformation in (18) we obtain

$$s^{2k+1}\tilde{y}_1(s) = \bar{\theta}_1 s^{2k-1}\tilde{y}_1(s) + \ldots + \bar{\theta}_k s\tilde{y}_1(s) + d(s), \tag{19}$$

where the polynomial $d(s)$ denotes initial conditions and terms caused by $\varepsilon_1(t)$.

Multiplying (19) on $\frac{\lambda_0^{2k}}{\gamma(s)}$ with respect to (14) yields

$$s^{2k+1}\xi(s) = \bar{\theta}_1 s^{2k-1}\xi(s) + \ldots + \bar{\theta}_k s\xi(s) + \frac{\lambda_0^{2k}}{\gamma(s)}d(s). \tag{20}$$

After inverse Laplace transformation in (20) we have necessary equation (15), where $\varepsilon(t) = \mathcal{L}^{-1}\{\frac{\lambda_0^{2k} d(s)}{\gamma(s)}\}$. By force of polynomial $\gamma(s)$ structure the function $\varepsilon(t)$ can be represented as a sum of decaying exponents. Thus, derivatives of these functions are also exponentially decaying.

Remark 1. Since θ_i is the root of polynomial $q^{2k} + \bar{\theta}_1 q^{2k-2} + \cdots + \bar{\theta}_{k-1} q^2 + \bar{\theta}_k$, where q is an algebraic variable, it is possible to calculate values θ_i using $\bar{\theta}_i$.

The adaptive scheme for frequencies estimation is presented in the following theorem.

Theorem 2. The update law

$$\hat{\omega}_i = \sqrt{\left|\hat{\theta}_i\right|}, \tag{21}$$

where estimates θ_i calculated using $\hat{\bar{\theta}}_i$ that are elements of a vector $\hat{\bar{\Theta}}$:

$$\hat{\bar{\Theta}} = \Upsilon(t) + K\Omega(t)\xi^{(2k)}(t), \tag{22}$$

$$\dot{\Upsilon}(t) = -K\Omega(t)\Omega^T(t)\hat{\bar{\Theta}}(t) - K\dot{\Omega}(t)\xi^{(2k)}(t). \tag{23}$$

where $K = \mathrm{diag}\{k_i > 0, i = \overline{1,k}\}$, guarantees that the estimation error $\tilde{\omega}_i = \omega_i - \hat{\omega}_i$ exponentially converges to zero:

$$|\tilde{\omega}_i(t)| \leq \rho_1 e^{-\beta_1 t}, \quad \rho_1, \beta_1 > 0, \quad \forall t \geq 0. \tag{24}$$

Proof 2. Using Lemma 1, we compute the derivative of the estimation error $\tilde{\bar{\Theta}} = \bar{\Theta} - \hat{\bar{\Theta}}$:

$$\begin{aligned}
\dot{\tilde{\bar{\Theta}}}(t) &= \dot{\bar{\Theta}} - \dot{\hat{\bar{\Theta}}}(t) \\
&= -\dot{\Upsilon}(t) - K\dot{\Omega}(t)\xi^{(2k)} - K\Omega(t)\xi^{(2k+1)} \\
&= K\Omega(t)\Omega^T(t)\hat{\bar{\Theta}}(t) + K\dot{\Omega}(t)\xi^{(2k)}(t) \\
&\quad - K\dot{\Omega}(t)\xi^{(2k)} - K\Omega(t)\xi^{(2k+1)} \\
&= K\Omega(t)\Omega^T(t)\hat{\bar{\Theta}}(t) - K\Omega(t)\left(\Omega^T(t)\bar{\Theta} + \varepsilon(t)\right) \\
&= -K\Omega(t)\Omega^T(t)\tilde{\bar{\Theta}}(t) - K\Omega(t)\varepsilon(t).
\end{aligned} \tag{25}$$

For the Lyapunov function $V(t) = \tilde{\bar{\Theta}}^T K^{-1}\tilde{\bar{\Theta}}/2$ one can see from (25) that the derivative $\dot{V}(t)$ is non-positive. This fact provides only the convergence of $\tilde{\bar{\Theta}}$ to some constants.

Temporary assume that $\varepsilon = 0$. It is very well-known that if the regressor $\Omega(t)$ is satisfied the persistence excitation condition then $\tilde{\bar{\Theta}}$ as a solution of

$$\dot{\tilde{\bar{\Theta}}}(t) = -K\Omega(t)\Omega^T(t)\tilde{\bar{\Theta}}(t)$$

tends to zero exponentially fast (see Theorem 4.3.2 in Ioanou and Sun [1996]).

Since $|\varepsilon(t)| \leq \rho_0 e^{-\lambda t}$ is an exponentialy decayng function of time then it is straightforward to show with respect to comparison principle (Khalil [2002]) that for an arbitrary exponential term ε estimate error $\tilde{\bar{\Theta}}$ converges to zero exponentially fast in the system (25)

$$\left\|\tilde{\bar{\Theta}}(t)\right\| \leq \rho_2 e^{-\beta_2 t}. \tag{26}$$

Regard to remark 1 we will hold that vector of estimates θ_i is accessible based on vector of estimates $\bar{\theta}_i$. Since the calculation θ_i basing on $\bar{\theta}_i$ can be considered as an algebraic task, that the estimate errors $\tilde{\theta}_i = \theta_i - \hat{\theta}_i$ are also tends to zero and bounded by a exponentialy decaing function of time.

Frequencies of the multiharmonic disturbance can be calculated using (21). Let us show that the frequency estimation error $\tilde{\omega}_i = \omega_i - \hat{\omega}_i$ for (21) has an exponentially decaying behavior (24)

$$\tilde{\omega}_i(t) = \sqrt{\left|\tilde{\theta}_i(t) + \hat{\theta}_i(t)\right|} - \sqrt{\left|\hat{\theta}_i(t)\right|} \leq \sqrt{\left|\tilde{\theta}_i(t)\right|}. \tag{27}$$

$$\tilde{\omega}_i(t) = \sqrt{|\theta_i|} - \sqrt{\left|\theta_i(t) - \tilde{\theta}_i(t)\right|} \geq -\sqrt{\left|\tilde{\theta}_i(t)\right|}. \tag{28}$$

$$|\tilde{\omega}_i(t)| \leq \sqrt{\left|\tilde{\theta}_i(t)\right|} \leq \rho_1 e^{-\beta_1 t}, \tag{29}$$

where ρ_1 and β_1 depend on ρ_2 and β_2.

4. DISTURBANCE OBSERVER

On the next step we need to design the observers for the disturbance $\delta(t)$ and predicted estimate $\delta(t+D)$ that is necessary in the compensation task.

The main problem is that parameters of the disturbance are unknown. Firstly, we will solve the problem of identifying the unknowns σ, μ_i, and ν_i. Using estimates $\hat{\sigma}$, $\hat{\mu}_i$, and $\hat{\nu}_i$ one can get the observer for the disturbance $\delta(t)$ in the form

$$\hat{\delta}(t) = \hat{\sigma} + \sum_{i=1}^{k} \left[\hat{\mu}_i \sin(\omega_i t) + \hat{\nu}_i \cos(\omega_i t)\right] \tag{30}$$

The second observer for predicted estimate of the disturbance we can find directly from (30)

$$\begin{aligned}
\hat{\delta}(t+D) &= \hat{\sigma} + \sum_{i=1}^{k} \left[\hat{\mu}_i \sin(\omega_i(t+D)) + \hat{\nu}_i \cos(\omega_i(t+D))\right] \\
&= \hat{\sigma} + \sum_{i=1}^{k} \left[\kappa_i \sin(\omega_i t) + \zeta_i \cos(\omega_i t)\right],
\end{aligned} \tag{31}$$

where

$$\kappa_i = \hat{\mu}_i \cos(\omega_i D) - \hat{\nu}_i \sin(\omega_i D), \tag{32}$$

$$\zeta_i = \hat{\mu}_i \sin(\omega_i D) + \hat{\nu}_i \cos(\omega_i D). \tag{33}$$

The next step and the main part of the section is to design the update laws for the estimates $\hat{\sigma}$, $\hat{\mu}_i$, and $\hat{\nu}_i$.

From (11) we have

$$\tilde{y}_1(t) = \left[\frac{b(p)}{a(p)}\right]\delta(t) + \bar{\varepsilon}(t), \tag{34}$$

where $p = d/dt$ is the differential operator and the transfer function

$$\frac{b(p)}{a(p)} = C(pI - (A - LC))^{-1}(B - L\alpha) + \alpha. \tag{35}$$

Assumption 4. Let us assume that the polynomial $b(p)$ does not have roots on the imaginary axis.

4.1 Extracting harmonics

From (14) we get

$$\xi(t) = \left[\frac{\lambda_0}{\gamma(p)}\right]\left(\left[\frac{b(p)}{a(p)}\right]\delta(t) + \bar{\varepsilon}(t)\right). \tag{36}$$

From (36) we can represent ξ as follows:

$$\xi(t) = \xi_0(t) + \varepsilon_2(t) + \sum_{i=1}^{k} \xi_i(t), \tag{37}$$

where ξ_0 is the constant, ξ_i is the sinusoidal function of time with frequency ω_i, and ε_ξ exponentially decays to zero.

Then we find the relation between ξ_1, ξ_2 and the disturbance δ from (36) by replacing $p = j\omega$, where j is the complex unit.

$$\xi_0(t) = \sigma \frac{b_0}{a_0}, \tag{38}$$

$$\begin{aligned}\xi_i(t) &= M_i\mu_i \sin(\omega_i t + \varphi_i) + M_i\nu_i \cos(\omega_i t + \varphi_i) \\ &= [\, \mu_i \quad \nu_i \,] \begin{bmatrix} M_i \sin(\omega_i t + \varphi_i) \\ M_i \cos(\omega_i t + \varphi_i) \end{bmatrix},\end{aligned} \tag{39}$$

where

$$M_i = \left| \frac{\lambda_0}{\gamma(j\omega_i)} \frac{b(j\omega_i)}{a(j\omega_i)} \right|, \tag{40}$$

$$\varphi_i = arg\left(\frac{\lambda_0}{\gamma(j\omega_i)} \frac{b(j\omega_i)}{a(j\omega_i)} \right). \tag{41}$$

For the variable $\xi(t)$ we have:

$$\xi(t) = \xi_0 + \xi_1(t) + \xi_2(t) + \cdots + \xi_k(t). \tag{42}$$

After differentiation (42) $2k$ times, we obtain two systems of k linear equations:

$$\begin{cases} \xi^{(1)}(t) = \dot{\xi}_1(t) + \dot{\xi}_2(t) + \cdots + \dot{\xi}_k(t), \\ \xi^{(3)}(t) = \theta_1\dot{\xi}_1(t) + \theta_2\dot{\xi}_2(t) + \cdots + \theta_k\dot{\xi}_k(t), \\ \vdots \\ \xi^{(2k-1)}(t) = \theta_1^{k-1}\dot{\xi}_1(t) + \cdots + \theta_k^{k-1}\dot{\xi}_k(t), \end{cases} \tag{43}$$

and

$$\begin{cases} \xi^{(2)}(t) = \theta_1\xi_1(t) + \theta_2\xi_2(t) + \cdots + \theta_k\xi_k(t), \\ \xi^{(4)}(t) = \theta_1^2\xi_1(t) + \theta_2^2\xi_2(t) + \cdots + \theta_k^2\xi_k(t), \\ \vdots \\ \xi^{(2k)}(t) = \theta_1^k\xi_1(t) + \theta_2^k\xi_2(t) + \cdots + \theta_k^k\xi_k(t). \end{cases} \tag{44}$$

From (42) and (44) we get the realizable estimation scheme for variables ξ_0 and $\xi_i(t)$:

$$\begin{bmatrix} \hat{\xi}_1(t) \\ \hat{\xi}_2(t) \\ \vdots \\ \hat{\xi}_k(t) \end{bmatrix} = \begin{bmatrix} \hat{\theta}_1 & \cdots & \hat{\theta}_k \\ \hat{\theta}_1^2 & \cdots & \hat{\theta}_k^2 \\ \vdots & \ddots & \vdots \\ \hat{\theta}_1^k & \cdots & \hat{\theta}_k^k \end{bmatrix}^{-1} \begin{bmatrix} \xi^{(2)}(t) \\ \xi^{(4)}(t) \\ \vdots \\ \xi^{(2k)}(t) \end{bmatrix}, \tag{45}$$

and

$$\hat{\xi}_0 = \xi(t) - \sum_{i=1}^{k} \hat{\xi}_i(t). \tag{46}$$

4.2 Amplitudes identification

The bias can be found in the easiest way from (38) and (46)

$$\hat{\sigma} = \frac{a_0}{b_0}\hat{\xi}_0(t). \tag{47}$$

To identify amplitudes μ_i and ν_i we consider the equation (39). This equation is a trivial regressor model where $\xi_i(t)$ is a measurable function, μ_i and ν_i are the unknown parameters and $\varsigma_i(t) = \begin{bmatrix} M_i \sin(\omega_i t + \varphi_i) \\ M_i \cos(\omega_i t + \varphi_i) \end{bmatrix}$ is the regressor.

The problem is that the regressor ς_i is unmeasurable directly due to uncertain parameters M_i and φ_i. On the next step we show how to get the regressor using available variables without complex calculations of M_i and φ_i.

Introduce the auxiliary function of time

$$\Delta(t) = \sum_{i=0}^{k} \sin(\omega_i t). \tag{48}$$

Then we consider the auxiliary filter

$$\vartheta(t) = \left[\frac{\lambda_0}{\gamma(p)} \frac{b(p)}{a(p)} \right] \Delta(t). \tag{49}$$

Remark 2. We have to note that since the transfer function $\frac{\lambda_0}{\gamma(p)}\frac{b(p)}{a(p)}$ has a relative degree more than $2k$, then from the filter (49) it is possible to get all derivatives of $\vartheta(t)$ till $\vartheta^{(2k)}(t)$.

It is easy to see that the signal $\vartheta(t)$ is a sum of sinusoids with the same frequencies ω_i and exponential decaying term that describes the transient process.

$$\vartheta(t) = \sum_{i=1}^{k} \vartheta_i(t) + \varepsilon_3(t), \tag{50}$$

Making the same calculations as in (36), (39), (40), (41) one can check that the signals $\vartheta_i(t)$ equal

$$\vartheta_i(t) = M_i \sin(\omega_i t + \varphi_i). \tag{51}$$

Derivatives of $\vartheta_i(t)$ correspondingly equal

$$\dot{\vartheta}_i(t) = \omega_i\, M_i \cos(\omega_i t + \varphi_i). \tag{52}$$

Using the algorithm of harmonic extraction (43)-(45) it is straightforward to show how to design the observer for all components of regressors ς_i

$$\begin{bmatrix} \hat{\vartheta}_1(t) \\ \hat{\vartheta}_2(t) \\ \vdots \\ \hat{\vartheta}_k(t) \end{bmatrix} = \begin{bmatrix} \hat{\theta}_1 & \cdots & \hat{\theta}_k \\ \hat{\theta}_1^2 & \cdots & \hat{\theta}_k^2 \\ \vdots & \ddots & \vdots \\ \hat{\theta}_1^k & \cdots & \hat{\theta}_k^k \end{bmatrix}^{-1} \begin{bmatrix} \vartheta^{(2)}(t) \\ \vartheta^{(4)}(t) \\ \vdots \\ \vartheta^{(2k)}(t) \end{bmatrix}, \tag{53}$$

$$\begin{bmatrix} \hat{\dot{\vartheta}}_1(t) \\ \hat{\dot{\vartheta}}_2(t) \\ \vdots \\ \hat{\dot{\vartheta}}_k(t) \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 \\ \hat{\theta}_1 & \cdots & \hat{\theta}_k \\ \vdots & \ddots & \vdots \\ \hat{\theta}_1^{k-1} & \cdots & \hat{\theta}_k^{k-1} \end{bmatrix}^{-1} \begin{bmatrix} \vartheta^{(1)}(t) \\ \vartheta^{(3)}(t) \\ \vdots \\ \vartheta^{(2k-1)}(t) \end{bmatrix}, \tag{54}$$

Remark 3. Inverse matrices in (45), (53), and (54) exist if the disturbance $\delta(t)$ has no less then k harmonics with various each from other frequencies. Since the functions $\hat{\theta}_i$ exponentially converges to θ_i it is straightforward though lengthy to show that errors $\tilde{\xi}_0 = \xi_0 - \hat{\xi}_0$, $\tilde{\xi}_i = \xi_i(t) - \hat{\xi}_i(t)$, and $\tilde{\dot{\xi}}_i(t) = \dot{\xi}_i(t) - \hat{\dot{\xi}}_i(t)$ also tend to zero exponentially fast.

So we get the algorithm to calculate the regressor functions $\varsigma_i(t)$

$$\hat{\varsigma}_i^T(t) = \begin{bmatrix} \hat{\vartheta}_i(t) & \dfrac{\dot{\hat{\vartheta}}_i(t)}{\eta_i(t)} \end{bmatrix}, \tag{55}$$

$$\eta_i(t) = \begin{cases} \hat{\omega}_i(t), \text{ if } \hat{\omega}_i \geq \omega_0, \\ \omega_0, \text{ otherwise}, \end{cases} \tag{56}$$

Eventually, we design the update laws for th amplitudes μ_i and ν_i by the standard gradient approach (Ioanou and Sun [1996]) basing on the regressor model (39)

$$\dot{\hat{\mu}}_i(t) = k_\mu \hat{\vartheta}_i(t) \left(\hat{\xi}_i(t) - \hat{\mu}_i(t)\hat{\vartheta}_i(t) - \hat{\nu}_i(t)\frac{\dot{\hat{\vartheta}}_i(t)}{\eta_i(t)} \right), \tag{57}$$

$$\dot{\hat{\nu}}_i(t) = k_\nu \frac{\dot{\hat{\vartheta}}_i(t)}{\eta_i(t)} \left(\hat{\xi}_i(t) - \hat{\mu}_i(t)\hat{\vartheta}_i(t) - \hat{\nu}_i(t)\frac{\dot{\hat{\vartheta}}_i(t)}{\eta_i(t)} \right), \tag{58}$$

where $k_\mu, k_\nu > 0$ and $\eta_i(t)$ is defined above in (56) .

Analysis of the estimate errors for $\tilde{\mu}_i = \mu_i - \hat{\mu}_i$ and $\tilde{\nu}_i = \nu_i - \hat{\nu}_i$ gives the similar calculations as in the theorem 2 in formula (25) with further conclusions. Let us skip reduplication of that fragment to reduce the paper size. We note only that it is straightforward though lengthy to show that $\tilde{\mu}_i$ and $\tilde{\nu}_i$ are exponentially decays to zero (Ioanou and Sun [1996]). The same fact about exponential convergence is fulfilled correspondingly for observer errors $\tilde{\delta}_1(t) = \delta(t) - \hat{\delta}(t)$ and $\tilde{\delta}_2(t) = \delta(t+D) - \hat{\delta}(t+D)$.

Therefore, we obtain two observers for $\delta(t)$ and $\delta(t+D)$ that are required the estimates $\hat{\sigma}$, $\hat{\mu}_i$, and $\hat{\nu}_i$. All necessary calculations to get these estimates are presented above.

5. MAIN RESULT

In this section we present the control law that provides the stability of the closed loop and convergence of the state to zero.

The feedback will be chosen in the form

$$u(t) = -\hat{\delta}(t+D) + \psi(t), \tag{59}$$

where $\hat{\delta}(t)$ is given by (30) and $\psi(t)$ is the new control is to be designed to stabilize the closed-loop system.

Substituting (59) into (1) yields

$$\dot{x}(t) = Ax(t) + B\psi(t-D) + B\tilde{\delta}_1(t). \tag{60}$$

Basing on (60) we introduce the second observer for state variables.

$$\dot{\hat{x}}_2(t) = A\hat{x}_2 + B\psi(t-D) + L\tilde{y}_2(t), \tag{61}$$

$$\hat{y}_2(t) = C\hat{x}_2(t) + \alpha\hat{\delta}(t), \tag{62}$$

Consider a model of the observer error $\tilde{x}_2(t) = x(t) - \hat{x}(t)$ with respect to (1), (2), (61), (62)

$$\dot{\tilde{x}}_2(t) = (A - LC)\tilde{x}_2(t) + B\tilde{\delta}_1(t) - L\alpha\tilde{\delta}_2(t), \tag{63}$$

$$\tilde{y}_2(t) = C\tilde{x}_2(t) + \alpha\tilde{\delta}_2(t). \tag{64}$$

Since the system (63) and (64) is linear, the matrix $A - LC$ is Hurwitz, and the functions of time $\tilde{\delta}_1(t)$ and $\tilde{\delta}_2(t)$ exponentially decay one can find that the observer errors $\tilde{x}_2(t)$ and $\tilde{y}_2(t)$ converge to zero

$$\|\tilde{x}_2(t)\| \leq \rho_4 e^{-\beta_4 t}, \qquad |\tilde{y}_2(t)| \leq \rho_5 e^{-\beta_5 t}, \tag{65}$$

where $\rho_4, \beta_4, \rho_5, \beta_5 > 0$.

Thus, we obtain the state observer for the plant (1), (2) such that all state estimates exponentially converges to true state variables. So we will use the second observer (61), (62) to design stabilizing controller.

Following the approach in (Krstic [2009], Pyrkin et al. [2010 a,b]), we model the delay by the transport PDE

$$\Psi_t(z,t) = \Psi_z(z,t), \quad 0 < z < D \tag{66}$$

$$\Psi(D,t) = \psi(t) \tag{67}$$

with the initial condition $\Psi(z,0) = \psi(z-D)$. The solution of this PDE is $\Psi(z,t) = \psi(t+z-D)$, and therefore $\Psi(0,t) = \psi(t-D)$ gives the delayed input. We can now rewrite (60) in the form

$$\dot{x}(t) = Ax(t) + B\Psi(0,t) + B\tilde{\delta}_1(t). \tag{68}$$

Following the idea of the backstepping algorithm proposed in (Krstic [2009]) and investigated in (Pyrkin et al. [2010 a,b]) we consider the transformation

$$W(z,t) = \Psi(z,t) - Ke^{Az}\hat{x}_2(t) - K\int_0^z e^{A(z-\tau)}B\Psi(\tau,t)d\tau + K\int_z^D e^{A(z+D-\tau)}L\tilde{Y}(\tau,t)d\tau, \tag{69}$$

$$\tilde{Y}(z,t) = \tilde{y}_2(t+z-D), \tag{70}$$

$$\tilde{Y}_t(z,t) = \tilde{Y}_z(z,t), \tag{71}$$

$$\tilde{Y}(D,t) = \tilde{y}_2(t). \tag{72}$$

and the control law

$$\psi(t) = Ke^{AD}\hat{x}_2(t) + K\int_0^D e^{A(D-\tau)}B\Psi(\tau,t)d\tau\,. \tag{73}$$

Lemma 3. The transformation (69) and the control law (73) maps the plant (68) into an internally stable system

$$\dot{x}(t) = (A + BK)x(t) + BW(0,t) + B\varepsilon_2(t), \tag{74}$$

$$W_t(z,t) = W_z(z,t), \tag{75}$$

$$W(D,t) = 0, \tag{76}$$

where the term $w(0,t)$ goes to zero for finite time, the matrix K makes the matrix $A + BK$ Hurwitz, and exponentially decaying term $\varepsilon_2(t)$.

Proof 3. Let us to calculate the time and spatial derivatives of the transformation (69)

$$W_z(z,t) = \Psi_z(z,t) - KAe^{Az}\hat{x}_2(t) - KB\Psi(z,t) - KA\int_0^x e^{A(z-\tau)}B\Psi(\tau,t)d\tau - Ke^{AD}L\tilde{Y}(z,t) + KA\int_z^D e^{A(z+D-\tau)}L\tilde{Y}(\tau,t)d\tau, \tag{77}$$

$$W_t(z,t) = \Psi_t(z,t) - Ke^{Az}\dot{\hat{x}}_2(t) - K\int_0^z e^{A(z-\tau)}B\Psi_t(\tau,t)d\tau + K\int_z^D e^{A(z+D-\tau)}L\tilde{Y}_t(\tau,t)d\tau$$

$$
\begin{aligned}
&= \Psi_z(z,t) - KAe^{Az}\hat{x}_2(t) - KB\Psi(z,t) \\
&\quad - Ke^{Az}B\Psi(0,t) - Ke^{Az}L\tilde{Y}(D,t) \\
&\quad + Ke^{Az}B\Psi(0,t) + Ke^{Az}L\tilde{Y}(D,t) \\
&\quad - KA\int_0^z e^{A(z-\tau)}B\Psi(\tau,t)d\tau - Ke^{AD}L\tilde{Y}(z,t) \\
&\quad + KA\int_z^D e^{A(z+D-\tau)}L\tilde{Y}(\tau,t)d\tau \\
&= W_z(z,t). \qquad (78)
\end{aligned}
$$

Setting $z = D$ in (69) gives (76). Setting $z = 0$ in (69) and substituting the resulting $\Psi(0,t)$ in (60), we have

$$
\begin{aligned}
\dot{x}(t) &= (A + BK)x(t) + BW(0,t) + B\tilde{\delta}_1(t) \\
&\quad - BK\tilde{x}_2(t) - BK\int_0^D e^{A(D-\tau)}L\tilde{Y}(\tau,t)d\tau. \qquad (79)
\end{aligned}
$$

From (65) we have that terms $\tilde{x}_2(t)$ and $\tilde{y}_2(t)$ converge to zero and bounded by exponential decaying functions. Therefore, it is straightforward to show the same for the term $K\int_0^D e^{A(D-\tau)}L\tilde{Y}(\tau,t)d\tau$. Finally, we denote

$$
\varepsilon_2(t) = \tilde{\delta}_1(t) - K\tilde{x}_2(t) - K\int_0^D e^{A(D-\tau)}L\tilde{Y}(\tau,t)d\tau, \qquad (80)
$$

which completes the proof.

Following (Krstic et al. [2008], Pyrkin et al. [2010 a,b]) consider the Lyapunov function

$$
V(t) = x^T(t)Px(t) + \frac{\gamma}{2}\int_0^D (1+z)W(z,t)^2 dz, \qquad (81)
$$

where $P = P^T > 0$ is the solution to the Lyapunov equation

$$
P(A+BK) + (A+BK)^T P = -Q \qquad (82)
$$

for some $Q + Q^T > 0$ and some positive number γ.

After differentiating (81) we obtain

$$
\begin{aligned}
\dot{V}(t) &= x^T(t)(P(A+BK) + (A+BK)^T P)x(t) \\
&\quad + 2x^T PBW(0,t) + 2x^T PB\varepsilon_2(t) \\
&\quad - \frac{\gamma}{2}W(0,t)^2 - \frac{\gamma}{2}\int_0^D W(z,t)^2 dz \\
&\le -x^T(t)Qx(t) + \frac{4}{\gamma}x^T(t)PBB^T Px(t) \\
&\quad + \frac{\gamma}{2}\varepsilon_2^2(t) - \frac{\gamma}{2}\int_0^D W(z,t)^2 dz. \qquad (83)
\end{aligned}
$$

Taking $\gamma = 8\lambda_{max}(PBB^T P)/\lambda_{min}(Q)$ we get

$$
\begin{aligned}
\dot{V}(t) &\le -\frac{\lambda_{min}(Q)}{2}x^T(t)x(t) + \frac{\gamma}{2}\varepsilon_2^2(t) \\
&\quad - \frac{\gamma}{2(1+D)}\int_0^D (1+z)W(z,t)^2 dz. \qquad (84)
\end{aligned}
$$

and eventually

$$
\dot{V}(t) \le -C_0 V(t) + \frac{\gamma}{2}\varepsilon_2^2(t), \qquad (85)
$$

where $C_0 = \min\left\{\frac{\lambda_{min}(Q)}{2\lambda_{max}(P)}, \frac{1}{1+D}\right\}$ and $\varepsilon_2^2(t)$ is exponentially decaying term.

To complete analysis of the Lyapunov function (81) for the closed-loop system we have to use the comparison principle (Khalil [2002]). Thus one can find that $|V(t)| \le \rho_6 e^{-\beta_6 t}$ with some $\rho_6, \beta_6 > 0$. Hence the main goal (4) is achieved.

6. CONCLUSIONS

We design the adaptive controller for the linear plant with the input delay that cancels the unknown disturbance affected the input and the output providing exponentially decaying to zero for the state variables.

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