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A New Approach to H_∞ Model Reduction for Positive Systems ⋆

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Abstract: In this paper, we investigate model reduction for positive linear time-invariant systems in the H_∞ sense. First, the bounded real lemma is further developed to obtain a new characterization of the H_∞ performance, which does not include any product term between the Lyapunov matrix and the parameter of the reduced-order model. Then a new parameterization of a positive reduced-order model is proposed, and accordingly, an iterative algorithm is constructed, which makes use of the coarse reduced-order models resulting from the (generalized) balanced truncation as the initial value to search for a desired positive one. Both continuous- and discrete-time systems are considered in the same framework. Numerical examples clearly show the effectiveness and advantages of the proposed method.

Keywords: Model reduction, H_∞ performance, positive systems, iterative algorithm,

1. INTRODUCTION

Positive systems, whose state variables are always positive or nonnegative, are a class of dynamic systems of great importance, and are often encounter in various science and industrial engineering areas. These systems or processes typically include biological and chemical reaction, compartmental networks, economics systems and ecosystems [Farina and Rinaldi 2000]. Due to the practical significance and particularity different from general systems, positive systems have received considerable attention during the past decades, and many results on analysis and synthesis of positive systems have been proposed (see, for instance, Benvenuti [2013], Liu and Dang [2011], Zhao et al. [2013], Back and Astolfi [2008], Chen et al. [2013]).

Model reduction is a basic theme in control theory. Modelling by first principles often leads to high-order mathematical models, which is inconvenient for system analysis and synthesis. This stimulates the development of model reduction theory and techniques. Some classical methods, for instance, balanced truncation (BT) and Hankel norm approximation [Obinata and Anderson 2000], have been shown to be effective for model reduction of general linear systems. In recent years, many new approaches such H_∞ model reduction [Grigoriadis 1995, Wu 1996, Chow et al. 2013] have been proposed to handle more complicated systems; see, for instance, Zhang et al. [2008], Wu et al. [2009]. Model reduction for positive systems has been specifically addressed by some researchers very recently [Reis and Virnik 2009, Feng et al. 2010, Li et al. 2011, Sootla and Rantzer 2012]. On one hand, for model reduction of positive systems, it is naturally desired that the reduced-order model is also positive. Unfortunately, the aforementioned results on general linear systems do *not* have such a property, thus cannot be applied to positive systems. On the other hand, the existing results on model reduction for positive systems still have remarkable *limitations*. For instance, a generalized balanced truncation (GBT) method proposed in Reis and Virnik [2009] had to use diagonal Lyapunov matrices, which would lead to conservative or even ineffective results. Hence, model reduction for positive systems has not been well solved yet, and still is a challenging problem that is worth further investigation.

In the paper, we will revisit the problem of H_∞ model reduction for positive linear time-invariant systems. First, a new condition based on the bounded real lemma will be derived for characterizing the H_∞ norm of the error system in terms of linear matrix inequality. Then, a necessary and sufficient condition will be proposed for parameterizing a positive reduced-order model. It is shown that this new parameterization establishes a connection between a positive reduced-order model and a common reduced-order model. By using the coarse reduced-order models from the BT or GBT methods, an iterative algorithm is finally constructed to search for a positive reduce-order model with the H_∞ error optimized. Results on the continuous-time (CT) and discrete-time (DT) systems are given in a unified form. Numerical examples will be provided to show the effectiveness and merits of the proposed method.

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Notation: The superscripts "−1" and "T" stand for inverse and transpose of a matrix, respectively. $\mathbb{R}^{m\times n}$ is the set of all $m\times n$ real matrices. Especially, $\mathbb{R}_n$ represents $\mathbb{R}^{n\times n}$ for simplicity, $\mathbb{R}_+^{m\times n}$ represents $\mathbb{R}^{m\times n}$ with nonnegative elements, and $\mathbb{S}_n$ represents $\mathbb{R}_n$ with symmetric elements. A matrix $A\in\mathbb{R}_+^{m\times n}$ is said to be positive; a matrix $A\in\mathbb{R}_n$ is said to be Metzler, if all its off-diagonal elements are nonnegative. The notation $P>0$ (≥ 0) means that matrix P is positive definite (semi-definite). $\mathbf{I}$ denotes an identity matrix with appropriate dimension. $\mathrm{diag}\{A_1,\ldots,A_n\}$ stands for a block-diagonal matrix. For a matrix $A\in\mathbb{R}_n$, $\mathrm{sym}\{A\}$ indicates $A^{\mathrm{T}}+A$. Matrices, if their dimensions are not explicitly stated, are assumed to be compatible for algebraic operations.

2. PROBLEM STATEMENT

Consider a stable system (Σ) in the state-space form:

$$\begin{aligned}(\Sigma):\lambda\left[x(t)\right]&=Ax(t)+Bu(t)\\ y(t)&=Cx(t)+Du(t)\end{aligned}\tag{1}$$

where $x(t)\in\mathbb{R}^{n_{\mathrm{p}}}$, $u(t)\in\mathbb{R}^{n_u}$ and $y(t)\in\mathbb{R}^{n_y}$ are the state, input and output vectors, respectively. (A,B,C,D) are real constant matrices with appropriate dimensions. Operator $\lambda\left[x(t)\right]$ denotes $\dot{x}(t)$ for CT systems (respectively, $x(t+1)$ for DT systems). Accordingly, in the frequency domain, we also use λ as Laplace operator s for the CT case (respectively, operator z for the DT case). System (Σ) is supposed to be positive. The definitions of positivity for system (Σ) and its characterization is given as follows:

Definition 1. (Farina and Rinaldi [2000]). System (Σ) in (1) is said to be positive if $x(t)\in\mathbb{R}_+^{n_{\mathrm{p}}}$ and $y(t)\in\mathbb{R}_+^{n_y}$, $t\geq 0$, for all $x(0)\in\mathbb{R}_+^{n_{\mathrm{p}}}$ and $u(t)\in\mathbb{R}_+^{n_u}$, $t\geq 0$.

Lemma 1. (Farina and Rinaldi [2000]). System (Σ) in (1) is positive if and only if A is Metzler, B, C and D are positive for the CT case (respectively, A, B, C and D are positive for the DT case).

The goal of this paper is to find a reduced-order positive stable model (Σ_{r}) to approximate (Σ). Let a reduced-order model (Σ_{r}) is described in the state-space form:

$$\begin{aligned}(\Sigma_{\mathrm{r}}):\lambda\left[x_{\mathrm{r}}(t)\right]&=A_{\mathrm{r}}x_{\mathrm{r}}(t)+B_{\mathrm{r}}u(t)\\ y_{\mathrm{r}}(t)&=C_{\mathrm{r}}x_{\mathrm{r}}(t)+D_{\mathrm{r}}u(t)\end{aligned}\tag{2}$$

where $x_{\mathrm{r}}(t)\in\mathbb{R}^{n_{\mathrm{r}}}$ with $1\leq n_{\mathrm{r}}<n_{\mathrm{p}}$ and $y_{\mathrm{r}}(t)\in\mathbb{R}^{n_y}$ are the state and output vectors of the reduced-order model. The values of matrices $(A_{\mathrm{r}},B_{\mathrm{r}},C_{\mathrm{r}},D_{\mathrm{r}})$ are to be determined later. Augmenting the state vectors as $\xi(t)\triangleq\mathrm{col}\{x(t),x_{\mathrm{r}}(t)\}$, we obtain the dynamics of the output approximation error $e(t)\triangleq y(t)-y_{\mathrm{r}}(t)$ as

$$\begin{aligned}\lambda\left[\xi(t)\right]&=A_{\mathrm{e}}\xi(t)+B_{\mathrm{e}}u(t)\\ e(t)&=C_{\mathrm{e}}\xi(t)+D_{\mathrm{e}}u(t)\end{aligned}\tag{3}$$

where

$$A_{\mathrm{e}}\triangleq\begin{bmatrix}A&0\\0&A_{\mathrm{r}}\end{bmatrix},\ B_{\mathrm{e}}\triangleq\begin{bmatrix}B\\B_{\mathrm{r}}\end{bmatrix},\ C_{\mathrm{e}}\triangleq\begin{bmatrix}C&-C_{\mathrm{r}}\end{bmatrix},\ D_{\mathrm{e}}\triangleq D-D_{\mathrm{r}}.$$

The transfer function of the error system in (3) is given by

$$G_{\mathrm{e}}(\lambda)\triangleq C_{\mathrm{e}}\left(\lambda\mathbf{I}-A_{\mathrm{e}}\right)^{-1}B_{\mathrm{e}}+D_{\mathrm{e}}.$$

As commented in Li et al. [2011], it is naturally expected that the reduced-order model (Σ_{r}) is also positive since it approximates a positive system (Σ). That is, according to Lemma 1, matrices $(A_{\mathrm{r}},B_{\mathrm{r}},C_{\mathrm{r}},D_{\mathrm{r}})$ should belong to a set $\mathbb{P}$ defined as $\mathbb{P}\triangleq\{\left[\begin{smallmatrix}A_{\mathrm{r}}&B_{\mathrm{r}};&C_{\mathrm{r}}&D_{\mathrm{r}}\end{smallmatrix}\right]:A_{\mathrm{r}}$ is Metzler, B_{r}, C_{r} and D_{r} are positive$\}$ for the CT case, and as $\mathbb{P}\triangleq\{\left[\begin{smallmatrix}A_{\mathrm{r}}&B_{\mathrm{r}};&C_{\mathrm{r}}&D_{\mathrm{r}}\end{smallmatrix}\right]:A_{\mathrm{r}}$, B_{r}, C_{r} and D_{r} are positive$\}$ for the DT case. Moreover, to reduce the approximation error, the error system in (3) is expected to satisfy the follow H_∞ norm specification:

$$\left\|G_{\mathrm{e}}(\lambda)\right\|_\infty<\gamma\tag{4}$$

where scalar $\gamma>0$ is to be minimized and

$$\left\|G_{\mathrm{e}}(\lambda)\right\|_\infty\triangleq\begin{cases}\sup\limits_{\omega\in\mathbb{R}\cup\{\infty\}}\sigma_{\max}[G_{\mathrm{e}}(\mathrm{j}\omega)], & \text{(CT)}\\ \sup\limits_{\omega\in[0,2\pi]}\sigma_{\max}[G_{\mathrm{e}}(\mathrm{e}^{\mathrm{j}\omega})], & \text{(DT)}\end{cases}.$$

In summary, the model reduction problem to be addressed in the paper is formulated as:

Problem 1. For system (Σ), find a reduced-order model (Σ_{r}) with n_{r} $(1\leq n_{\mathrm{r}}<n)$ being prescribed, such that

1) $\left[\begin{matrix}A_{\mathrm{r}}&B_{\mathrm{r}};&C_{\mathrm{r}}&D_{\mathrm{r}}\end{matrix}\right]\in\mathbb{P}$, and

2) the error system in (3) is stable and satisfies (4).

We present the following preliminary result for later use.

Lemma 2. (Gahinet and Apkarian [1994]). Given systems (Σ) and (Σ_{r}), the error system in (3) is stable and satisfies $\left\|G_{\mathrm{e}}(\lambda)\right\|_\infty<\gamma$ if and only if there exists a matrix $P\in\mathbb{S}_{n_{\mathrm{p}}+n_{\mathrm{r}}}$ such that, for the CT case, $P>0$ and

$$\begin{bmatrix}\mathrm{sym}\{PA_{\mathrm{e}}\}&PB_{\mathrm{e}}&C_{\mathrm{e}}^{\mathrm{T}}\\B_{\mathrm{e}}^{\mathrm{T}}P&-\gamma^2\mathbf{I}&D_{\mathrm{e}}^{\mathrm{T}}\\C_{\mathrm{e}}&D_{\mathrm{e}}&-\mathbf{I}\end{bmatrix}<0\tag{5}$$

or for the DT case, $P>0$ and

$$\begin{bmatrix}A_{\mathrm{e}}^{\mathrm{T}}PA_{\mathrm{e}}-P&A_{\mathrm{e}}^{\mathrm{T}}PB_{\mathrm{e}}&C_{\mathrm{e}}^{\mathrm{T}}\\B_{\mathrm{e}}^{\mathrm{T}}PA_{\mathrm{e}}&B_{\mathrm{e}}^{\mathrm{T}}PB_{\mathrm{e}}-\gamma^2\mathbf{I}&D_{\mathrm{e}}^{\mathrm{T}}\\C_{\mathrm{e}}&D_{\mathrm{e}}&-\mathbf{I}\end{bmatrix}<0.\tag{6}$$

3. MAIN RESULTS

3.1 Performance Characterization

We present the following new result for characterizing the performance specification $\left\|G_{\mathrm{e}}(\lambda)\right\|_\infty<\gamma$ based on Lemma 2. First, define some matrices as

$$\begin{aligned}&\bar{A}\triangleq\begin{bmatrix}A&0_{n_{\mathrm{p}}\times n_{\mathrm{r}}}\\0&0_{n_{\mathrm{r}}\times n_{\mathrm{r}}}\end{bmatrix},\ \bar{B}\triangleq\begin{bmatrix}B\\0_{n_{\mathrm{r}}\times n_u}\end{bmatrix},\ \bar{F}\triangleq\begin{bmatrix}0&0_{n_{\mathrm{p}}\times n_y}\\\mathbf{I}_{n_{\mathrm{r}}}&0_{n_{\mathrm{r}}\times n_y}\end{bmatrix}\\ &\bar{M}\triangleq\begin{bmatrix}0_{n_{\mathrm{r}}\times n_{\mathrm{p}}}&\mathbf{I}_{n_{\mathrm{r}}}\\0_{n_u\times n_{\mathrm{p}}}&0\end{bmatrix},\ \bar{N}\triangleq\begin{bmatrix}0_{n_{\mathrm{r}}\times n_u}\\\mathbf{I}_{n_u}\end{bmatrix},\ K_{\mathrm{r}}\triangleq\begin{bmatrix}A_{\mathrm{r}}&B_{\mathrm{r}}\\C_{\mathrm{r}}&D_{\mathrm{r}}\end{bmatrix}\\ &\bar{C}\triangleq\begin{bmatrix}C&0_{n_y\times n_{\mathrm{r}}}\end{bmatrix},\ \bar{D}\triangleq D,\ \bar{H}\triangleq\begin{bmatrix}0_{n_y\times n_{\mathrm{r}}}&-\mathbf{I}_{n_y}\end{bmatrix}.\end{aligned}\tag{7}$$

Theorem 1. Given systems (Σ) and (Σ_{r}), the following statements are equivalent.

(i) The error system in (3) satisfies $\left\|G_{\mathrm{e}}(\lambda)\right\|_\infty<\gamma$.

(ii) There exist matrix $P\in\mathbb{S}_{n_{\mathrm{p}}+n_{\mathrm{r}}}$ and diagonal matrix $X\in\mathbb{R}_{n_{\mathrm{r}}+n_y}$ such that $P>0$, $X>0$ and

$$W^{\mathrm{T}}\Phi W-2U^{\mathrm{T}}XU<0\tag{8}$$

where

$$\Phi\triangleq\mathrm{diag}\{\Psi,\mathbf{I}_{n_y},-\gamma^2\mathbf{I}_{n_u}\},\ U\triangleq\begin{bmatrix}K_{\mathrm{r}}\bar{M}&K_{\mathrm{r}}\bar{N}&-\mathbf{I}\end{bmatrix}$$

$$W\triangleq\begin{bmatrix}\bar{A}&\bar{B}&\bar{F}\\\mathbf{I}&0&0\\\bar{C}&\bar{D}&\bar{H}\\0&\mathbf{I}&0\end{bmatrix},\ \Psi\triangleq\begin{cases}\begin{bmatrix}0&P\\P&0\end{bmatrix}, & \text{(CT case)}\\ \begin{bmatrix}P&0\\0&-P\end{bmatrix}, & \text{(DT case)}\end{cases}$$

and other symbols are defined in (7).

(iii) There exist matrix $\tilde{P} \in \mathbb{S}_{n_\mathrm{p}+n_\mathrm{r}}$ and diagonal matrix $\tilde{X} \in \mathbb{R}_{n_\mathrm{r}+n_u}$ such that $\tilde{P} > 0$, $\tilde{X} > 0$ and

$$\tilde{W}\tilde{\Phi}\tilde{W}^\mathrm{T} - 2\tilde{U}\tilde{X}\tilde{U}^\mathrm{T} < 0 \tag{9}$$

where $\tilde{\Phi}$ is defined as Φ but with P replaced by $\tilde{P}$ and with the values of n_y and n_u exchanged,

$$\tilde{W} \triangleq \begin{bmatrix} \bar{A} & \mathbf{I} & \bar{B} & 0 \\ \bar{C} & 0 & \bar{D} & \mathbf{I} \\ \bar{M} & 0 & \bar{N} & 0 \end{bmatrix}, \; \tilde{U} \triangleq \begin{bmatrix} \bar{F}K_\mathrm{r} \\ \bar{H}K_\mathrm{r} \\ -\mathbf{I} \end{bmatrix}$$

and other symbols are defined in (7).

Proof. First note that matrices $(A_\mathrm{e}, B_\mathrm{e}, C_\mathrm{e}, D_\mathrm{e})$ can be written in the following form:

$$\begin{bmatrix} A_\mathrm{e} & B_\mathrm{e} \\ C_\mathrm{e} & D_\mathrm{e} \end{bmatrix} = \begin{bmatrix} \bar{A} + \bar{F}K_\mathrm{r}\bar{M} & \bar{B} + \bar{F}K_\mathrm{r}\bar{N} \\ \bar{C} + \bar{H}K_\mathrm{r}\bar{M} & \bar{D} + \bar{H}K_\mathrm{r}\bar{N} \end{bmatrix}. \tag{10}$$

Moreover, the conditions in (5) and (6) can be re-arranged in a compact form as

$$M^\mathrm{T}\Phi M < 0 \tag{11}$$

where

$$M \triangleq \begin{bmatrix} A_\mathrm{e} & B_\mathrm{e} \\ \mathbf{I} & 0 \\ C_\mathrm{e} & D_\mathrm{e} \\ 0 & \mathbf{I} \end{bmatrix}.$$

((i)⇒(ii)) From Lemma 2, it follows that statement (i) is true if and only if some matrix P exists such that $P > 0$ and (11) holds. The condition in (11) implies that

$$\begin{bmatrix} M^\mathrm{T}\Phi M & M^\mathrm{T}\Phi E \\ * & E^\mathrm{T}\Phi E - 2X \end{bmatrix} < 0$$

or

$$[\, M \;\; E \,]^\mathrm{T} \Phi [\, M \;\; E \,] - 2 [\, 0 \;\; \mathbf{I} \,]^\mathrm{T} X [\, 0 \;\; \mathbf{I} \,] < 0$$

holds for some diagonal matrix $X > \alpha\mathbf{I}$ with $\alpha > 0$ being sufficiently large, where $E = \left[\, \bar{F}^\mathrm{T} \;\; 0 \;\; \bar{H}^\mathrm{T} \;\; 0 \,\right]^\mathrm{T}$. Define

$$T_1 \triangleq \begin{bmatrix} \mathbf{I} & 0 \\ [\, -K_\mathrm{r}\bar{M} \;\; -K_\mathrm{r}\bar{N} \,] & \mathbf{I} \end{bmatrix}.$$

From (10), it is easy to verify $[\, M \;\; E \,] T_1 = W$ and $[\, 0 \;\; \mathbf{I} \,] T_1 = -U$, which imply

$$W^\mathrm{T}\Phi W - U^\mathrm{T}XU = T_1^\mathrm{T} \begin{bmatrix} M^\mathrm{T}\Phi M & \Psi^\mathrm{T}\Phi E \\ * & E^\mathrm{T}\Phi E - 2X \end{bmatrix} T_1 < 0.$$

That is, statement (ii) is satisfied.

((i)⇐(ii)) Suppose the condition in (8) holds for some matrix $P > 0$ and diagonal matrix $X > 0$. Define matrix

$$T_2 \triangleq \begin{bmatrix} \mathbf{I} \\ [\, K_\mathrm{r}\bar{M} \;\; K_\mathrm{r}\bar{N} \,] \end{bmatrix}. \tag{12}$$

Note that $WT_2 = M$ and $UT_2 = 0$, which imply

$$M^\mathrm{T}\Phi M = T_2^\mathrm{T}\left(W^\mathrm{T}\Phi W - 2U^\mathrm{T}XU\right) T_2 < 0.$$

Then, by Lemma 2, one can obtain statement (i).

Statement (iii) is the dual version of statement (ii), and the equivalence between statements (i) and (iii) can be proven similarly, and thus is omitted for brevity. ∎

Remark 1. Condition (ii) of Theorem 1 is an equivalent characterization of the index $\|G_\mathrm{e}(\lambda)\|_\infty < \gamma$. In this new characterization, the parameter of the reduced-order model, i.e., K_r, is only multiplied by X, rather than by Lyapunov matrix P, for the condition in Lemma 2, which is more appealing for the reduced-order model synthesis. Such a similar matrix-separation idea has been widely utilized in robust control [de Oliveira and Skelton 2001, Li and Gao 2014, Gao and Li 2011]. Moreover, the simply diagonal structure of X is very beneficial for parameterizing the structurized $K_\mathrm{r} \in \mathbb{P}$.

3.2 Parameterization of Reduced-order Model

Based on Theorem 1, we further develop a necessary and sufficient condition for parameterizing a reduced-order positive model satisfying a prescribed H_∞ error level.

Theorem 2. Given system (Σ), the following statements are equivalent.

(i) Problem 1 is solvable.

(ii) There exist matrix $P \in \mathbb{S}_{n_\mathrm{p}+n_\mathrm{r}}$, diagonal matrix $X \in \mathbb{R}_{n_\mathrm{r}+n_y}$, matrices $\mathcal{K}_\mathrm{r} \in \mathbb{R}^{(n_\mathrm{r}+n_y)\times(n_\mathrm{r}+n_u)}$ and $L_\mathrm{r} \in \mathbb{P}$ such that $P > 0$, $X > 0$ and

$$\Xi(\mathcal{K}_\mathrm{r}) \triangleq W^\mathrm{T}\Phi W - \mathrm{sym}\{\mathcal{U}^\mathrm{T}V\} < 0 \tag{13}$$

where Φ, W are defined in (8),

$$\mathcal{U} \triangleq \left[\, \mathcal{K}_\mathrm{r}\bar{M} \;\; \mathcal{K}_\mathrm{r}\bar{N} \;\; -\mathbf{I} \,\right], \; V \triangleq \left[\, L_\mathrm{r}\bar{M} \;\; L_\mathrm{r}\bar{N} \;\; -X \,\right]$$

and other symbols are defined in (7). The reduced-order model (Σ_r) can be obtained as $K_\mathrm{r} = X^{-1}L_\mathrm{r}$.

(iii) There exist matrix $\tilde{P} \in \mathbb{S}_{n_\mathrm{p}+n_\mathrm{r}}$, diagonal matrix $\tilde{X} \in \mathbb{R}_{n_\mathrm{r}+n_u}$, matrices $\mathcal{K}_\mathrm{r} \in \mathbb{R}^{(n_\mathrm{r}+n_y)\times(n_\mathrm{r}+n_u)}$ and $L_\mathrm{r} \in \mathbb{P}$ such that $\tilde{P} > 0$, $\tilde{X} > 0$ and

$$\tilde{\Xi}(\mathcal{K}_\mathrm{r}) \triangleq \tilde{W}\tilde{\Phi}\tilde{W}^\mathrm{T} - \mathrm{sym}\{\tilde{V}\tilde{\mathcal{U}}^\mathrm{T}\} < 0 \tag{14}$$

where $\tilde{\Phi}$, $\tilde{W}$ are defined in (9),

$$\tilde{\mathcal{U}} \triangleq \begin{bmatrix} \bar{F}\mathcal{K}_\mathrm{r} \\ \bar{H}\mathcal{K}_\mathrm{r} \\ -\mathbf{I} \end{bmatrix}, \; \tilde{V} \triangleq \begin{bmatrix} \bar{F}L_\mathrm{r} \\ \bar{H}L_\mathrm{r} \\ -\tilde{X} \end{bmatrix}$$

and other symbols are defined in (7). The reduced-order model (Σ_r) can be obtained from $K_\mathrm{r} = L_\mathrm{r}\tilde{X}^{-1}$.

Sketch of the Proof. The proof can be completed by invoking Theorem 1, applying a change of variable $L_\mathrm{r} = \tilde{X}K_\mathrm{r}$ for statement (ii) and $L_\mathrm{r} = K_\mathrm{r}\tilde{X}$ for statement (iii), and finally seeing $\mathcal{U} = U$ and $\tilde{\mathcal{U}} = \tilde{U}$. It can be shown that the non-singularity of X is guaranteed by (13) while that of $\tilde{X}$ by (14). Consequently, if (13) or (14) is valid, one can recover the positive reduced-order model through $K_\mathrm{r} = X^{-1}L_\mathrm{r}$ or $K_\mathrm{r} = L_\mathrm{r}\tilde{X}^{-1}$. For the sake of space limitation, details are omitted here. ∎

Compared with Theorem 1, the condition in (13) includes an additional matrix $\mathcal{K}_\mathrm{r}$. It is shown that $\mathcal{K}_\mathrm{r}$ satisfies the following property.

Theorem 3. Suppose that a matrix $\mathcal{K}_\mathrm{r}$ satisfying (13) or (14) exists, then the state-space model given by

$$\begin{aligned} (\Sigma'_\mathrm{r}) : \lambda\,[x_\mathrm{r}(t)] &= \mathcal{A}_\mathrm{r}x_\mathrm{r}(t) + \mathcal{B}_\mathrm{r}u(t) \\ y_\mathrm{r}(t) &= \mathcal{C}_\mathrm{r}x_\mathrm{r}(t) + \mathcal{D}_\mathrm{r}u(t) \\ \mathcal{K}_\mathrm{r} &= [\, \mathcal{A}_\mathrm{r} \;\; \mathcal{B}_\mathrm{r};\; \mathcal{C}_\mathrm{r} \;\; \mathcal{D}_\mathrm{r} \,] \end{aligned} \tag{15}$$

is a stable n_rth-order model for the system (Σ) such that the resulting error system (Σ_e) satisfies $\|G_\mathrm{e}(\lambda)\|_\infty < \gamma$.

Sketch of the Proof. The proof can be completed by following the part (i)⇐(ii) of the proof of Theorem 1 with K_r replaced by $\mathcal{K}_\mathrm{r}$. Details are omitted for saving space. ∎

Theorem 3 shows that the matrix $\mathcal{K}_\mathrm{r}$ also gives rise to a reduced-order model (Σ'_r) for the same system (Σ). The difference between (Σ_r) and (Σ'_r) is that system (Σ'_r) is not necessary to be positive. In other words, according to Theorem 3, statement (ii) or (iii) in Theorem 2 denotes a connection between a positive reduced-order model and a common reduced-order one, which will be made use of to search for a positive reduced-order model.

3.3 Iterative Algorithm for Positive Model Reduction

The inequalities in (13) and (14) are not convex constraints. However, if $\mathcal{K}_\mathrm{r}$ is fixed, the two conditions become convex with respect to the remaining variables. Hence, an algorithm is naturally proposed for computing the reduced-order model: First fix $\mathcal{K}_\mathrm{r}$ corresponding to some reduced-order model known *a priori*, and then solve the conditions in (13) and (14) to obtain $K_\mathrm{r} \in \mathbb{P}$ corresponding to a positive reduced-order model. Note that $\mathcal{K}_\mathrm{r}$ is not required to be a positive reduced-order model, thus it is easy to obtain some $\mathcal{K}_\mathrm{r}$ via the existing methods. Moreover, this process can be repeated for further reducing the approximation error.

Consequently, the following iterative algorithm is proposed for searching for a positive reduced-order model.

Iterative Algorithm for Positive Model Reduction

Step 1 Find the parameter $\mathcal{K}_\mathrm{r}$ of an n_rth-order model (Σ'_r) in (15) via the existing model reduction methods. Set $i = 1$ and $\mathcal{K}_\mathrm{r}^{(1)} = \mathcal{K}_\mathrm{r}$.

Step 2 (Primal) Solve the optimization problem:

$$\min\ \gamma : \text{s.t.} \begin{cases} \Xi(\mathcal{K}_\mathrm{r}^{(i)}) < 0,\ P > 0 \\ \text{diagonal } X > 0,\ L_\mathrm{r} \in \mathbb{P} \end{cases} \quad (16)$$
$$\text{for } P,\ X,\ L_\mathrm{r} \text{ and } \gamma.$$

Set $\mathcal{K}_\mathrm{r}^{(i)} = X^{-1} L_\mathrm{r}$.

Step 3 (Dual) Solve the following optimization problem:

$$\min\ \gamma : \text{s.t.} \begin{cases} \tilde{\Xi}(\mathcal{K}_\mathrm{r}^{(i)}) < 0,\ \tilde{P} > 0 \\ \text{diagonal } \tilde{X} > 0,\ L_\mathrm{r} \in \mathbb{P} \end{cases} \quad (17)$$
$$\text{for } \tilde{P},\ \tilde{X},\ L_\mathrm{r} \text{ and } \gamma.$$

Denote the optimum of γ as $\gamma^{(i)}$ and set $K_\mathrm{r}^{(i)} = L_\mathrm{r}\tilde{X}^{-1}$.

Step 4 If $\left|\gamma^{(i)} - \gamma^{(i-1)}\right| / \gamma^{(i)} < \delta$ with δ being a prescribed tolerance or if $i = k$ with k being the prescribed maximum allowable number of iterations, then output $K_\mathrm{r} = K_\mathrm{r}^{(i)}$ and $\gamma = \gamma^{(i)}$ as the optimized reduced-order model, and EXIT; else, set $i \leftarrow i+1$, $\mathcal{K}_\mathrm{r}^{(i)} = K_\mathrm{r}^{(i-1)}$ and go back to Step 2.

Remark 2. The optimization of the H_∞ level of the error system is directly realized by running the algorithm. It can be shown that $\gamma^{(i)}$ is monotonically non-increasing with iteration proceeding. On the other hand, if the goal of the model reduction problem is to find a reduced-order model with a prescribed H_∞ level γ^*, one can add an extra criterion $\gamma \le \gamma^*$ to decide whether to stop the iteration.

Remark 3. The choice of the initial $\mathcal{K}_\mathrm{r}$ will inevitable affect the converging performance of the algorithm, but there seems no tractable method to guarantee that one initial $\mathcal{K}_\mathrm{r}$ must be better than another. However, numerical experiment shows that the proposed algorithm could produce satisfactory results, even when initialized by the coarse reduced-order models generated from, for instance, the BT or GBT method [Moore 1981, Reis and Virnik 2009]. Especially, due to their simplicity, this paper only focuses on the BT and GBT as the initialization methods.

4. NUMERICAL EXAMPLES

To initialize the developed algorithm, the standard BT method and the GBT method as aforementioned will be applied to generate the coarse initial reduced-order models (the two cases are labelled as **IA-BT** and **IA-GBT**, respectively). Numerical solver SeDuMi [Sturm 1999], invoked through the interface Yalmip [Löfberg 2004], will be used to solve the optimization problems in (16) and (17).

Example 1. Consider an example of the CT positive system (Σ) with state-space parameters given by

$$A = \begin{bmatrix} -1.5 & 0.6 & 1.0 & 0 & 0 & 0 \\ 0.3 & -1.9 & 0.2 & 0 & 0 & 0 \\ 0.2 & 0.5 & -2.7 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -3 & 0.6 & 0.5 \\ 0 & 0 & 0 & 0.4 & -1.6 & 0.3 \\ 0 & 0 & 0 & 0.6 & 0.5 & -1.6 \end{bmatrix},\ B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$
$$C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix},\ D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This numerical example was composed in Li et al. [2011] to describe a compartmental network with two subsystems. The model reduction algorithm developed in Section 3.3 and the one in Li et al. [2011] will be utilized to explore positive 2nd-order models. For the algorithm in Li et al. [2011], tolerances are set as $\delta_1 = \delta_2 = \delta_3 = 10^{-2}$, which is also the setting of the reference; Algorithms 1 and 3 in the reference will be invoked alternately and repeat ten times at most if it does not give a solution in the last iteration.

Table 1. Number of primal and dual iterations for different algorithms (Example 1)

Method	$\gamma^* = 0.1$	$\gamma^* = 0.05$
Li et al. [2011]	$15 + 27$	$55 + 17$
IA-GBT	$1 + 1$	$2 + 2$
IA-BT	$3 + 3$	$5 + 5$

For prescribed H_∞ error level γ^* (see Remark 2), Table 1 lists the results on the number of primal and dual iterations for different algorithms, where, e.g., "$15 + 27$" means that 15 primal iterations and 27 dual iterations are needed for the algorithm in Li et al. [2011]. It is observed that, for the considered situations, the proposed algorithm, when initialized by the reduced-order models from the GBT or BT method, has lighter computational burden than the one in Li et al. [2011]. Especially, for $\gamma^* = 0.1$, IA-GBT actually needs no iteration and directly generates a desired reduced-order model.

Example 2. In this example, a random experiment will be conducted to compare the developed model reduction algorithm when initialized by the BT and GBT methods. The random systems are obtained as follows. First, all the elements of A, B, C and D are randomly generated from a normal distribution with zero mean and unitary variance. Then replace A, B, C and D by their absolute value. To guarantee stability, A is further updated by $A - 1.1\eta\mathbf{I}$ for the CT case and by $\frac{5}{6\rho}A$ for the DT case, respectively, where η and ρ are respectively the maximum real part of the eigenvalues and the spectral radius of A. These randomly generated systems with order $n_{\rm p}$ from 3 to 15 are all 2-input-2-output. We compute all the reduced-order models with order $n_{\rm r}$ from 2 to $n_{\rm p} - 1$. For comparison, define an index for a specific reduced-order model: $E \triangleq \|G_{\rm e}(\lambda)\|_\infty / \|G(\lambda)\|_\infty$. Apparently, the smaller an E, the better the reduced-order model in the H_∞ sense.

Experimental results are depicted in Figure 1, where the subscript of E indicates the used method. We also present the result that are directly obtained from the GBT method (see the first row). It is observed that

1) IA-BT outperforms GBT and IA-GBT in this random experiment. This is because, via the GBT method, it is difficult to obtain a good reduced-order model. Particularly, $E_{\rm GBT}$ for some systems is (much) larger than 100%, implying the ineffectiveness of the corresponding reduced-order model.

2) All the $E_{\text{IA-BT}}$ and $E_{\text{IA-GBT}}$ are less than 100%, showing the effectiveness of the proposed model reduction method. Although the reduced-order models obtained from the GBT are not satisfactory, they can be improved by the proposed optimization method. Especially, the sub-figures in the third row show that all the reduced-order models have the index $E < 10\%$ or even less, that is, the optimized reduced-order models are satisfactory. It is worth mentioning that all the results on IA-BT are obtained by running the optimization algorithm *once only*, showing the potential of the proposed model reduction method.

5. CONCLUSION

H_∞ model reduction for positive linear time-invariant systems has been addressed in this paper. The proposed results unify the CT and DT systems in the same framework. To preserve positivity of the reduced-order model, a novel necessary and sufficient condition for the existence of a positive reduced-order model has been proposed, which parameterizes a positive reduced-order model through another common reduced-order one. By virtue of this property, an iterative algorithm has been accordingly developed to search for and optimize the positive reduced-order model. Numerical examples show that, the proposed algorithm, when initialized by the simple BT or GBT method, can produce satisfactory reduced-order models.

REFERENCES

Back, J. and Astolfi, A. (2008). Design of positive linear observers for positive linear systems via coordinate transformations and positive realizations. *SIAM Journal on Control and Optimization*, 47(1), 345–373.

Benvenuti, L. (2013). Minimal positive realizations of transfer functions with real poles. *IEEE Trans. Autom. Control*, 58(4), 1013–1017.

Chen, X., Lam, J., Li, P., and Shu, Z. (2013). ℓ_1-induced norm and controller synthesis of positive systems. *Automatica*, 49(5), 1377–1385.

Chow, Y.L., Hu, Y.B., Li, X., Kominek, A., and Lam, J. (2013). Mixed additive/multiplicative H_∞ model reduction. *J. Dyn. Syst., Measurement Control—Trans. ASME*, 135(5), 051005 (10 pages).

de Oliveira, M.C. and Skelton, R.E. (2001). Stability tests for constrained linear systems. In *Perspectives in Robust Control, Lecture Notes in Control and Information Sciences*, 241–257. Springer, Berlin.

Farina, L. and Rinaldi, S. (2000). *Positive Linear Systems: Theory and Applications*. Pure and Applied Mathematics. John Wiley & Sons, New York.

Feng, J., Lam, J., Shu, Z., and Wang, Q. (2010). Internal positivity preserved model reduction. *Int. J. Control*, 83(3), 575–584.

Gahinet, P. and Apkarian, P. (1994). A linear matrix inequality approach to H_∞ control. *Int. J. Robust Nonlin. Control*, 4(4), 421–448.

Gao, H. and Li, X. (2011). H_∞ filtering for discrete-time state-delayed systems with finite frequency specifications. *IEEE Trans. Autom. Control*, 56, 2935–2941.

Grigoriadis, K.M. (1995). Optimal H_∞ model reduction via linear matrix inequalities: continuous- and discrete-time cases. *Syst. Control Lett.*, 26, 321–333.

Li, P., Lam, J., Wang, Z., and Date, P. (2011). Positivity-preserving H_∞ model reduction for positive systems. *Automatica*, 47, 1504–1511.

Li, X. and Gao, H. (2014). A heuristic approach to static output-feedback controller synthesis with restricted frequency-domain specifications. *IEEE Trans. Autom. Control*, 59(4), 1008–1014.

Liu, X. and Dang, C. (2011). Stability analysis of positive switched linear systems with delays. *IEEE Trans. Autom. Control*, 56(7), 1684–1690.

Löfberg, J. (2004). YALMIP: a toolbox for modeling and optimization in MATLAB. In *Proceedings of the 2004 IEEE International Symposium on Computer Aided Control System Design*, 284–289. Taipei, Taiwan.

Moore, B. (1981). Principal component analysis in linear systems: controllability, observability, and model reduction. *IEEE Trans. Autom. Control*, 26(5), 17–31.

Obinata, G. and Anderson, B.D.O. (2000). *Model Reduction for Control System Design*. Springer-Verlag, London.

Reis, T. and Virnik, E. (2009). Positivity preserving balanced truncation for descriptor systems. *SIAM Journal on Control and Optimization*, 48(4), 2600–2619.

Sootla, A. and Rantzer, A. (2012). Scalable positivity preserving model reduction using linear energy functions. In *2012 IEEE 51 Annual Conf. Decision Control*, 4285–4290. Maui, Hawaii, USA.

Sturm, J.F. (1999). Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones. *Optimization Methods and Software*, 11(1), 625–653.

Wu, F. (1996). Induced L_2 norm model reduction of polytopic uncertain linear systems. *Automatica*, 32(10), 1417–1426.

Wu, L., Ho, D.W.C., and Lam, J. (2009). H_∞ model reduction for continuous-time switched stochastic hybrid systems. *Int. J. Syst. Sci.*, 40(12), 1241–1251.

Zhang, L., Shi, P., Boukas, E.K., and Wang, C. (2008). H_∞ model reduction for uncertain switched linear discrete-time systems. *Automatica*, 44, 2944–2949.

Zhao, X., Zhang, L., and Shi, P. (2013). Stability of a class of switched positive linear time-delay systems. *Int. J. Robust Nonlin. Control*, 23(5), 578–589.

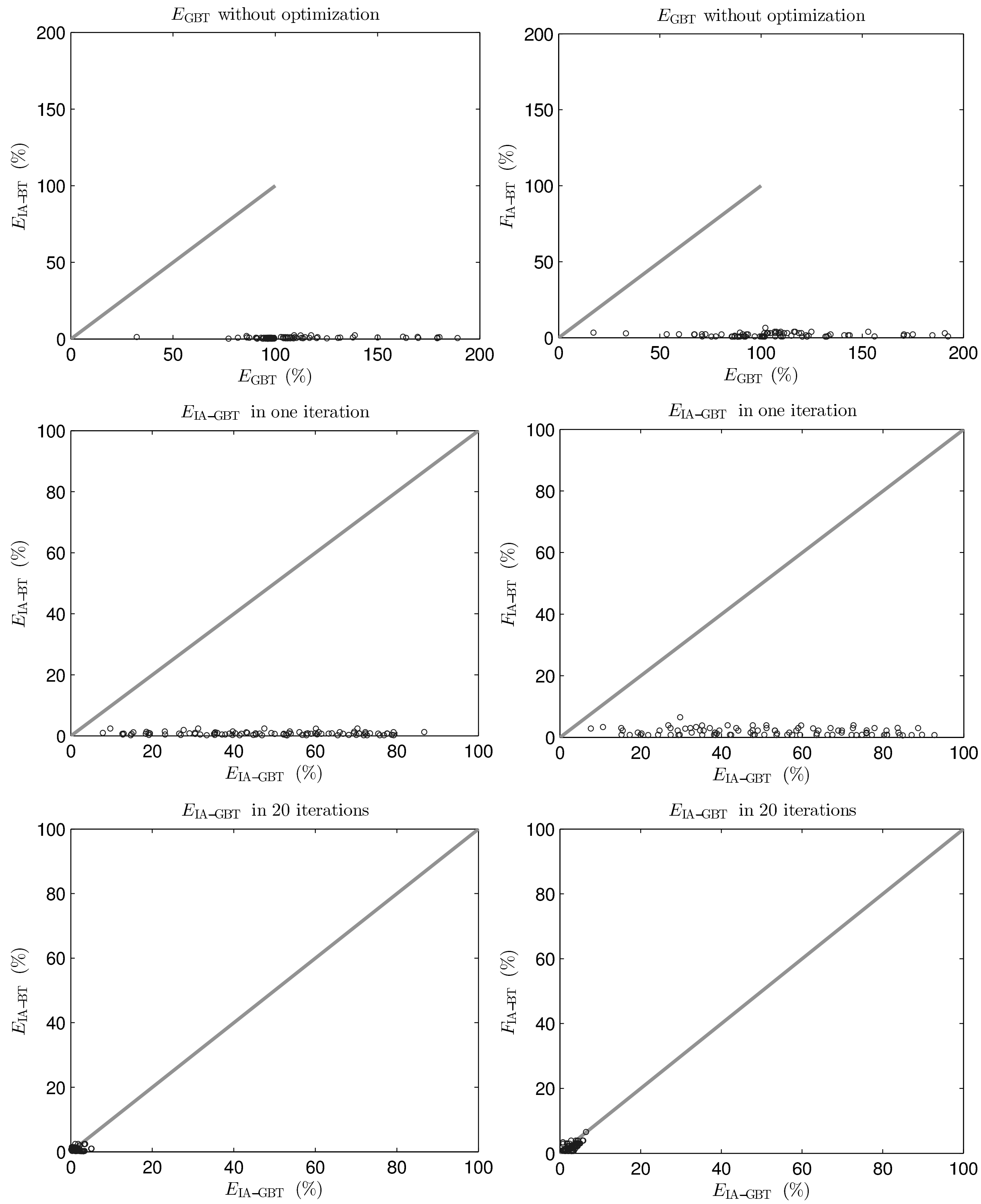

Fig. 1. Experimental results in Example 2 (left: the CT case; right: the DT case). All the results on $E_{\text{IA-BT}}$ is obtained in one iteration.