Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Nonlinear predictors for systems with bounded trajectories and delayed measurements ⋆

Stefano Battilotti *

* *Dipartimento di Ingegneria Informatica, Automatica e Gestionale "Antonio Ruberti" (e-mail: battilotti@dis.uniroma1.it).*

Abstract: Novel nonlinear predictors are studied for nonlinear systems with delayed measurements without assuming globally Lipschitz conditions or a known predictor map but requiring instead bounded state trajectories. The delay is constant and known. These nonlinear predictors consists of a series of dynamic filters that generate estimates of the state vector (and its magnitude) at different delayed time instants which differ from one another by a small fraction of the overall delay.

1. INTRODUCTION

The problem of accurately reconstructing the unmeasurable state variables by using a system/process model and the available on-line output measurements has been addressed by many authors. Particularly challenging remains the nonlinear observer design problem in the presence of delayed output measurements. Output measurements naturally arise in a variety of engineering applications (system/process to be controlled or monitored but located far from the controller unit, measured output data transmitted through a low-rate communication system, non-negligible time-delays introduced by sensors). In this case it is important to implement some kind of prediction based on the delayed measurements. A nonlinear observer proposed in Marquez et al. [2000] for linearizable by additive output injection systems. A predictor based on a cascade of observers has been introduced with LMI techniques in Besancon et al. [2007]. For globally Lipschitz continuous invertible observability maps (Germani et al. [2011]) the proposed observer consists of a chain of dynamic predictors that reconstruct the unmeasurable state vector at different delayed time-instants within the time-delay window introduced by the output measurements. Hence, the proposed nonlinear observer exhibits a chained structure that explicitly takes into account the magnitude of the output delay. The paper **?**], while adopting a conceptually similar design methodology, aims at overcoming some of the restrictions associated with the above approaches by following a technically different path. Also globally Lipschitz conditions on the system are required in Ibrir [2011]. In all these papers linear predictors are used.

Predictor–based results have been recently obtained in Karafyllis et al. [2013] where a known compact absorbing set (plus some technical facts) is assumed for all the system trajectories. This assumption is much stronger than boundedness of the state trajectories, where the absorbing compact set depends on the initial condition of each state trajectory. On the other hand, these dynamic predictors follow the structure of the ones introduced in Germani et al. [2011] and Kazantzis et al. [2001]. It must be said that the existence of a known compact absorbing set is very much similar to a Lipschitz condition over compact sets, in the sense that after some times all the trajectories stay in some known compact set and over this set the nonlinearities are Lipschitz.

Predictors, which are not implemented as dynamical filters, are designed in Karafyllis et al. [2012b] under the assumption that either a) the expression of the state trajectories is explicitly known or b) the system is globally Lipschitz. In Karafyllis et al. [2012a] the existence of predictor–based observers is shown under the hypothesis that the so-called predictor map is known exactly. Actually, all the above cited results can be implemented only if the predictor map is available (this happens for linear systems, bilinear systems, chains of linear systems with input nonlinearities), except for Karafyllis et al. [2013] where a modified version of the chained predictors, introduced in Germani et al. [2011] and Kazantzis et al. [2001], are used. Further results have been obtained for delays that depend on the delayed states in Bekiaris-Liberis et al. [2013]. Numerical and approximate predictors have been proposed in Karafyllis et al. [2013]. Design of predictors for specific implementation has been proposed in Mazenc et al. [2011].

In this paper we consider the problem of state observation for a class of multi–output systems which satisfy an incremental homogeneity (in the generalized sense) condition with bounded state trajectories. This class of systems includes lower triangular and upper triangular systems and many non-triangular systems. The measurement delay is constant and known. It is not required any globally Lipschitz condition on the system or availability of the predictor map. The additional complexity of the solution with respect to previous contributions with known compact absorbing sets is due to the fact that any trajectory is contained in some compact set which however is not known since it depends on the initial values of the trajectory. An estimation of the maximum delayed state and its magnitude are computed and, using this estimate, a prediction

⋆ This work was supported by MIUR.

is implemented by a chain of nonlinear dynamic predictors that reconstruct the unmeasurable state vector at different delayed time-instants. The novelty of our observers is the use for the first time of nonlinear predictors with saturated estimates where the saturation levels are adapted on-line according to the delayed measurements. This adaptation is needed to estimate the compact absorbing set of each state trajectory (depending on its initial condition) and this is done through the estimation of state magnitude. Our result is also based on the observer design with undelayed measurements proposed in Battilotti [2011]. An important feature of our observer is also the constructive design illustrated by a step-by-step procedure.

2. NOTATION

- $\mathbb{R}^n$ (resp. $\mathbb{R}^{n\times n}$) is the set of n-dimensional real column vectors (resp. $n \times n$ matrices). $\mathbb{R}_\geqslant$ (resp. $\mathbb{R}^n_\geqslant$, $\mathbb{R}^{n\times n}_\geqslant$) denotes the set of real non-negative numbers (resp. vectors in $\mathbb{R}^n$, matrices in $\mathbb{R}^{n\times n}$, with real non-negative entries). $\mathbb{R}_>$ (resp. $\mathbb{R}^n_>$) denotes the set of real positive numbers (resp. vectors in $\mathbb{R}^n$ with real positive entries).
- For any $G \in \mathbb{R}^{p\times n}$ we denote by G_{ij} (or $[G]_{ij}$) the (i, j)–th entry of G and by G_i (or $[G]_i$) the i–th row of G. We retain a similar notation for functions. For any $v \in \mathbb{R}^n$ we denote by diag$\{v\}$ the diagonal $n \times n$ matrix with diagonal elements $v_1, \ldots, v_n$. Also, $\|a\|$ denotes the absolute value of $a \in \mathbb{R}$, $\|a\|$ denotes the euclidean norm of $a \in \mathbb{R}^n$, i.e. $\sqrt{a_1^2 + \ldots + a_n^2}$, and $\langle\!\langle a \rangle\!\rangle$ the column vector of the absolute values of the components of $a \in \mathbb{R}^n$, i.e. $(|a_1| \cdots |a_n|)^T$.
- We denote by $\mathbf{C}^j(\mathscr{X}, \mathscr{Y})$, with $j \geqslant 0$, $\mathscr{X} \subset \mathbb{R}^n$ and $\mathscr{Y} \subset \mathbb{R}^p$, the set of j-times continuously differentiable functions $f : \mathscr{X} \to \mathscr{Y}$, by $\mathbf{D}^j(\mathscr{X}, \mathscr{Z})$, $\mathscr{X}, \mathscr{Z} \subset \mathbb{R}^n$, the set of functions $f \in \mathbf{C}^j(\mathscr{X}, \mathscr{Z})$ with decoupled components, *viz.* $f(x) = (f_1(x_1), \cdots, f_n(x_n))^T$.
- We say that $\sigma^h \in \mathbf{D}^0(\mathbb{R}^n, \mathbb{R}^n)$ is a *saturation function with levels* $h \in \mathbb{R}^n_>$ if for each $i = 1, \ldots, n$, $\sigma^h_i(s_i) = s_i$ for all $s_i : |s_i| \leqslant h_i$ and $\sigma^h_i(s_i) = \text{sign}(s_i)h_i$ for all $s_i : |s_i| > h_i$ (sign(s_i) is 1 if $s_i > 0$ and -1 if $s_i < 0$).
- For any vectors $x, \mathfrak{r} \in \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}^n_>$, we define
$$\varepsilon^{\mathfrak{r}} := (\varepsilon_1^{\mathfrak{r}_1} \cdots \varepsilon_n^{\mathfrak{r}_n})^T,\ \varepsilon^{\mathfrak{r}} \diamond x := (\varepsilon_1^{\mathfrak{r}_1} x_1 \cdots \varepsilon_n^{\mathfrak{r}_n} x_n)^T,$$
viz. $\varepsilon^{\mathfrak{r}} \diamond x$ is the dilation of a vector x with weight $\mathfrak{r}$.

3. MAIN ASSUMPTIONS

Consider the system

$$\begin{aligned}
&\dot{x}(t) = f(x(t)) := (A + BF)x(t) + \phi(x(t)),\\
&t \geqslant -\Delta,\ x(-\Delta) = x_0,\\
&y(t) = h(x(t-\Delta)) := Cx(t-\Delta) + \psi(x(t-\Delta)), t \geqslant 0 \quad (1)
\end{aligned}$$

where $\Delta > 0$ is the constant (known) measurement delay, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, y is a function of the state at time $t - \Delta$. A, B, C are in Brunowski canonical form:

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}^T,\ C^T = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}^T,$$

with $F \in \mathbb{R}^{p\times n}$. Moreover, ϕ and ψ are locally Lipschitz continuous with $\phi(0) = 0$, $\psi(0) = 0$, $\frac{\partial\phi}{\partial x}(0) = 0$, $\frac{\partial\psi}{\partial x}(0) = 0$. The vector $A + BF$ represents the linear approximation around zero of the system. The vector of the initial conditions $x(-\Delta)$ is x_0. We will denote by $x(t, x_0)$ the state trajectory of (1) ensuing from x_0 at $t = -\Delta$ (resp. $t = 0$), unique and defined over its maximal right extension interval (theorem 3.7 and proposition 3.10 of Smith [2011]). We identify matrices with linear maps. Moreover, see the appendix for a short review of incremental homogeneity in the upper bound (i.g.h.u.b.). Our assumptions are the following ones:

(A0) (**incremental homogeneity**) ψ and $\phi + BF$ are incrementally homogeneous in the upper bound (i.g.h.u.b.) with quadruple $(\mathfrak{r}, -\mathfrak{g}_1 + \mathfrak{r}_1, \mathfrak{g}, \Psi_0)$ and, respectively, $(\mathfrak{r}, A(-\mathfrak{g}+\mathfrak{r}) + (I - AA^T)(\mathfrak{g}+\mathfrak{r}), \mathfrak{g}, \Phi_0 + BF_0)$ where $\Phi_0(0, 0) = 0$, $\Psi_0(0, 0) = 0$ and $\mathfrak{r}, \mathfrak{g}$ are such that $3\mathfrak{g}_{j+1} - \mathfrak{g}_j \leqslant \mathfrak{r}_{j+1} - \mathfrak{r}_j \leqslant \mathfrak{g}_j + \mathfrak{g}_{j+1}$ for all $j = 1, \ldots, n-1$,

(A1) (**boundedness of trajectories**) for each $x_0 \in \mathbb{R}^n$ there exist a compact set $\mathscr{C}_{x_0} \subset \mathbb{R}^n$ such that $x(t, x_0) \in \mathscr{C}_{x_0}$ $\forall t \geqslant -\Delta$,

(A2) (**incremental observability**) for any $x_0', x_0'' \in \mathbb{R}^n$: $h(x(t, x_0')) = h(x(t, x_0''))$ $\forall t \geqslant -\Delta \Rightarrow x(t, x_0') = x(t, x_0'')$ $\forall t \geqslant -\Delta$.

Remark 1. The notion of (incremental) homogeneity (in a generalized sense) has been introduced in Battilotti [2013], Battilotti [2011] for enlarging the class of homogeneous (in the classical sense: Rosier [1992]) systems in such a way to encompass triangular (non-homogeneous) systems and many other non-triangular systems. On the other hand, homogeneity in the upper bound allows to cope more generally with homogeneous bounds and achieve robustness properties. Assumptions (**A0**) states that ψ and $\phi + BF$ are incrementally homogeneous with a certain relation between degrees and weights. It can be seen that assumptions (**A0**) is satisfied for large classes of systems (1):

(i) with polynomial lower triangular $\phi + BF$ and ψ, i.e. such that for all x
$$|\phi_i(x)| \leqslant \Phi_i^0(x_1, \ldots, x_i),\ i = 1, \ldots, n,$$
$\psi(x) \equiv 0$, $\Phi_0(0) = 0$, $\Psi_0(0) = 0$, for some functions Φ^0 and Ψ_0,

(ii) with polynomial strict upper triangular $\phi + BF$ and ψ, i.e. such that for all x
$$|\phi_i| \leqslant \Phi_i^0(x_{i+2}, \ldots, x_n),\ i = 1, \ldots, n-2,$$
$$|\psi(x)| \leqslant \Psi^0(x_2, \ldots, x_n), \forall x$$
$\phi_n(x) \equiv \phi_{n-1}(x) \equiv 0$, $\Phi_0(0) = 0$, $\Psi_0(0) = 0$, for some functions Φ^0 and Ψ_0,

(iii) for some all equal degrees $\mathfrak{g}_n = \mathfrak{g}_{n-1} = \ldots = \mathfrak{g}_2 = \mathfrak{g}_1 := \mathfrak{g}_0$, with $\mathfrak{g}_0$ being the homogeneity degree of $\phi + BF$ (in the classical sense) and 0 being the homogeneity degree of ψ (in the classical sense) .□

Remark 2. For systems (1) with undelayed measurements a global observer was proposed in Battilotti [2011] under assumptions (**A0**)–(**A2**). (**A0**) is a *local* observability assumption for the undelayed system, stated in terms of incremental homogeneity of f and h. In other words, it guarantees the existence of a local observer for the undelayed state. Note that $\Phi_0(0, 0) = 0$ and $\Psi_0(0, 0) = 0$

is required. This assumption can be relaxed by requiring that the linear approximation of the undelayed system (1) around the origin remains observable under the perturbation terms $\Phi_0(0,0)$ and $\Psi_0(0,0)$.

Assumption (**A2**) is a *global* observability assumption for the undelayed system (1).

Assumption (**A1**) is somewhat restrictive. However, many physical systems have this property (Van Der Pol and Fitzhugh-Nagumo oscillators, the Lorentz equations: see examples in section 7). Note that we do not require the knowledge of a Lyapunov function for the system. □

4. THE STRUCTURE OF THE PREDICTOR

Wherever possible we will omit the dependence of the state trajectories from the initial conditions. The following notation is adopted to denote the following delayed state vectors:

$$x^{(j)}(t) := x(t-\Delta+j\frac{\Delta}{m}),\ t \geqslant -\frac{j\Delta}{m},\ j=0,\dots,m. \quad (2)$$

The nonlinear observer exhibits the following chain structure. A first block is devoted to the estimation of $x^{(0)}(t) := x(t-\Delta)$ (variable $\xi^{(0)}$) and its maximal magnitude $\sup_t \|x(t-\Delta)\|$ (variable $\rho^{(0)}$)

$$\begin{aligned}
&\dot{\xi}^{(0)}(t) = \mathfrak{F}^{\rho^{(0)}(t)}(\xi^{(0)}(t)) + \mathfrak{K}^{\rho^{(0)}(t)}\mathfrak{Y}(t),\\
&\dot{\rho}^{(0)}(t) = \mathfrak{G}^{\rho^{(0)}(t)}(\xi^{(0)}(t))\\
&+(\rho^{(0)}(t))^{1-2|\mathfrak{g}_n|}\|(\rho^{(0)}(t))^{-\mathfrak{r}} \diamond \sigma^{c(\rho^{(0)}(t))^{\mathfrak{r}}}(C^T\mathfrak{Y}(t))\|^2, \quad (3)\\
&t \geqslant 0, \text{ with}
\end{aligned}$$

$$\begin{aligned}
&\mathfrak{F}^{\rho}(\xi) := (A+BF)\xi + \phi(\sigma^{c\rho^{\mathfrak{r}}}(\xi))\\
&\mathfrak{G}^{\rho}(\xi) := \rho^{1-2|\mathfrak{g}_n|}\|\rho^{-\mathfrak{r}} \diamond \sigma^{c\rho^{\mathfrak{r}}}(\xi-\sigma^{c\rho^{\mathfrak{r}}}(\xi))\|^2\\
&\mathfrak{Y} := y - C\xi^{(0)} - \psi(\sigma^{c(\rho^{(0)})^{\mathfrak{r}}}(\xi^{(0)}))\\
&\mathfrak{K}^{\rho} := (I-A^TG^{\rho})^{-1}C^TK_0\rho^{2C\mathfrak{g}}, G^{\rho} := \mathrm{diag}\{\Gamma_0\rho^{2A\mathfrak{g}}\}, \quad (4)
\end{aligned}$$

for some saturation function $\sigma^{c\rho^{\mathfrak{r}}}$ with levels $c\rho^{\mathfrak{r}}$, $c>0$, $K_0>0$ and diagonal positive definite $\Gamma_0 \in \mathbb{R}^{n\times n}$. By assumption (**A1**) the magnitude of $x(t-\Delta)$ is bounded in time but unknown, depending on the initial condition $x(-\Delta)$. The estimates $\xi^{(0)}$ and $\rho^{(0)}$ are used in the block devoted to the estimation of $x^{(1)}(t) := x(t-\Delta+\frac{\Delta}{m})$ (variable $\xi^{(1)}$) and $\rho^{(1)}(t) := \rho^{(0)}(t+\frac{\Delta}{m})$ (variable $\rho^{(1)}$)

$$\begin{aligned}
&\dot{\xi}^{(1)}(t) = \mathfrak{F}^{\rho^{(1)}(t)}(\xi^{(1)}(t)) + \dot{\xi}^{(0)}(t) - \mathfrak{F}^{\rho^{(0)}(t)}(\xi^{(0)}(t))\\
&+\mathfrak{F}^{\rho^{(0)}(t)}(\xi^{(0)}(t)) - \mathfrak{F}^{\rho^{(1)}(t-\frac{\Delta}{m})}(\xi^{(1)}(t-\frac{\Delta}{m}))\\
&\dot{\rho}^{(1)}(t) = \mathfrak{G}^{\rho^{(1)}(t)}(\xi^{(1)}(t)) + \dot{\rho}^{(0)}(t) - \mathfrak{G}^{\rho^{(0)}(t)}(\xi^{(0)}(t))\\
&+\mathfrak{G}^{\rho^{(0)}(t)}(\xi^{(0)}(t)) - \mathfrak{G}^{\rho^{(1)}(t-\frac{\Delta}{m})}(\xi^{(1)}(t-\frac{\Delta}{m})), \quad (5)
\end{aligned}$$

$t \geqslant 0$, where $m \geqslant 1$ an integer such that

$$\Delta[\|A+BF\| + n^2c^2] < m. \quad (6)$$

Since

$$x^{(1)}(t) = x^{(0)}(t) + \int_{t-\frac{\Delta}{m}}^{t} [(A+BF)(x^{(1)}(s)) + \phi((x^{(1)}(s)))]ds \quad (7)$$

the predictor for $x^{(1)}(t)$ is obtained by differentiating

$$\xi^{(1)}(t) = \xi^{(0)}(t) + \int_{t-\frac{\Delta}{m}}^{t} \mathfrak{F}^{\rho^{(1)}(s)}(\xi^{(1)}(s))ds \quad (8)$$

which is a copy of (8) except for substituting $\xi^{(1)}$ with its saturated $\sigma^{c(\rho^{(1)})^{\mathfrak{r}}}(\xi^{(1)})$. In general, the estimates $\xi^{(i)}$ and $\varepsilon^{(i)}$, $i=0,\dots,j-1$, are used in the block devoted to the estimation of $x^{(j)}(t) := x(t-\Delta+\frac{j\Delta}{m})$ (variable $\xi^{(j)}$) and $\rho^{(j)}(t) := \rho^{0)}(t+\frac{j\Delta}{m})$ (variable $\rho^{(j)}$)

$$\begin{aligned}
&\dot{\xi}^{(j)}(t) = \mathfrak{F}^{\rho^{(j)}(t)}(\xi^{(j)}(t)) + \sum_{i=0}^{j-1}[\mathfrak{F}^{\rho^{(i)}(t)}(\xi^{(i)}(t))\\
&-\mathfrak{F}^{\rho^{(i+1)}(t-\frac{\Delta}{m})}(\xi^{(i+1)}(t-\frac{\Delta}{m}))] + \dot{\xi}^{(0)}(t) - \mathfrak{F}^{\varepsilon^{(0)}(t)}(\xi^{(0)}(t))\\
&\dot{\rho}^{(j)}(t) = \mathfrak{G}^{\rho^{(j)}(t)}(\xi^{(j)}(t)) + \sum_{i=0}^{j-1}[\mathfrak{G}^{\rho^{(i)}(t)}(\xi^{(i)}(t))\\
&-\mathfrak{G}^{\rho^{(i+1)}(t-\frac{\Delta}{m})}(\xi^{(i+1)}(t-\frac{\Delta}{m}))]\\
&+\dot{\varepsilon}^{(0)}(t) - \mathfrak{G}^{\varepsilon^{(0)}(t)}(\xi^{(0)}(t)),\ j=2,\dots,m, \quad (9)
\end{aligned}$$

$t \geqslant 0$, where the predictor for $x^{(j)}(t)$ is obtained by differentiating a copy of $x^{(j)}(t)$ except for substituting $x^{(j)}(t)$ with its saturated $\sigma^{c(\rho^{(j)}(t))^{\mathfrak{r}}}(x^{(j)}(t))$. The predictor (3)-(5)-(9) consists of $2(m+1)$ filters that generate estimates of the state vector (and its magnitude) at different delayed time instants, which differ from one another by a small fraction of the overall delay Δ, and it is initialized as follows

$$\begin{aligned}
&\rho^{(0)}(0) := \rho_0^{(0)} := \hat{\rho}(-\Delta),\\
&\rho^{(j)}(s) := \rho_0^{(j)}(s) := \hat{\rho}(s-\Delta+\frac{j\Delta}{m}),\\
&\xi^{(0)}(0) := \xi_0^{(0)} := \hat{\xi}(-\Delta),\\
&\xi^{(j)}(s) := \xi_0^{(j)}(s) := \hat{\xi}(s-\Delta+\frac{j\Delta}{m}),\\
&j=1,\dots,m,\ s\in[-\frac{\Delta}{m},0] \quad (10)
\end{aligned}$$

with bounded $\hat{\xi} \in \mathbf{C}^0([-\Delta,0],\mathbb{R}^n)$ and $\hat{\rho} \in \mathbf{C}^0([-\Delta,0],[1,+\infty))$. The vector of the initial conditions $(x_0,\hat{\rho},\hat{\xi})$ will be denoted in what follows by φ_0.

We want to prove that the estimates $\xi^{(j)}(t)$ converge to the actual delayed states $x^{(j)}(t)$ for $j=0,\dots,m-1$, and most importantly, convergence (for $j=m$) of $\xi_m(t)$ to the undelayed actual state $x(t)$ as long as m is chosen sufficiently large. The main result of this paper is the following.

Theorem 3. Assume (**A0**), (**A1**) *and* (**A2**). *There exist* $c, K_0 > 0$ *and diagonal positive definite* $\Gamma_0 \in \mathbb{R}^{n\times n}$ *such that the solution* $x(\cdot,x_0)$, $\xi^{(j)}(\cdot,\varphi_0), \rho^{(j)}(\cdot,\varphi_0)$, $j=0,\dots,m$, *of (1)-(3)-(5)-(9) is defined and bounded for all times and initial conditions* φ_0. *Moreover,* $\lim_{t\to\infty}\|x(t,x_0)-\xi^{(m)}(t,\varphi_0)\| = 0$.

Remark 4. Theorem 3 can be directly extended to systems (1) with inputs and more generally with $A+BF$ replaced by $A+BF+D$ for some diagonal D (see example (22)).

Remark 5. The observer (9) is robust with respect to square integrable output disturbances. Also robustness with respect to non-vanishing output disturbances can be achieved by suitably modifying (9) (this will be the object of future work). ▫

In order to prove theorem 3 we first prove that $\xi^{(0)}(t)$ converges to delayed state $x(t-\Delta)$ and, secondly, that $\xi^{(j)}(t)$ converges to the delayed state $x^{(j)}(t)$ for $j = 1,\ldots,m$.

5. THE OBSERVER FOR $X^{(0)}$

The state $x^{(0)}(\cdot) := x(\cdot - \Delta)$ satisfies the equations

$$\begin{aligned}\dot{x}^{(0)} &= (A+BF)x^{(0)} + \phi(x^{(0)})\\ y &= Cx^{(0)} + \psi(x^{(0)}), t \geqslant 0\end{aligned} \tag{11}$$

which we consider together with (3). The vector of the initial conditions $x_0 := x(-\Delta)$ and $\rho_0^{(0)}, \xi_0^{(0)}$ (see (10)) will be denoted in what follows by $\varphi_0^{(0)}$. The following result can be proved as in Battilotti [2011].

*Proposition 6. Assume (**A0**), (**A1**) and (**A2**). There exist $c, K_0 > 0$ and diagonal positive definite $\Gamma_0 \in \mathbb{R}^{n\times n}$ such that the solution $x^{(0)}(\cdot,x_0)$, $\xi^{(0)}(\cdot,\varphi_0^{(0)})$ and $\rho^{(0)}(\cdot,\varphi_0^{(0)})$ of (1)-(3) is defined and bounded for all times and initial conditions $\varphi_0^{(0)}$. Moreover, $\lim_{t\to\infty}\|x^{(0)}(t,x_0) - \xi^{(0)}(t,\varphi_0^{(0)})\| = 0$.*

6. THE PREDICTOR FOR $X(T)$

The vector of the initial conditions $x_0, \hat{\rho}, \hat{\xi}$ (see (10)) will be denoted in what follows by φ_0. We set $\|\varphi_0\| = \|x_0\| + \max_{s\in[-\Delta,0]}\|\hat{\rho}(s)\| + \max_{s\in[-\Delta,0]}\|\hat{\xi}(s)\|$. The following notation is adopted to denote the following delayed vectors:

$$\varepsilon^{(j)}(t) := \rho^{(0)}(t + j\frac{\Delta}{m}),\ t \geqslant -\frac{j\Delta}{m},\ j = 0,\ldots,m. \tag{12}$$

We will prove that $\xi^{(j)}(t)$ is an asymptotic estimate of $x^{(j)}(t)$ and, therefore, $\xi^{(m)}(t)$ is an asymptotic estimate of $x^{(m)}(t) = x(t)$, which proves our main theorem 3.

Proposition 7. Let the integer m be chosen in such a way that (6) is satisfied. Then the solutions $x^{(j)}(\cdot,x_0)$, $\xi^{(j)}(\cdot,\varphi_0), \rho^{(j)}(\cdot,\varphi_0)$, $j = 1,\ldots,m$, of (1)-(3)-(5)-(9) are defined and bounded for all times and initial conditions φ_0 and

$$\begin{aligned}&\lim_{t\to+\infty}(x^{(j)}(t,x_0) - \xi^{(j)}(t,\varphi_0)) = 0\\ &\lim_{t\to+\infty}(\rho^{(j)}(t,\varphi_0) - \varepsilon^{(j)}(t,\varphi_0)) = 0,\ j = 1,\ldots,m.\end{aligned} \tag{13}$$

Proof. Throughout the proof we will omit the dependence of the trajectories from the initial conditions and wherever there is no ambiguity we will omit the superscript $^{(j)}$. First, it is easy to show, following , that $\rho^{(j)}(t)$, $j = 1,\ldots,m$, satisfies for all times t in its maximal right extension domain $\rho^{(j)}(t) = \rho^{(j-1)}(t) + \int_{t-\frac{\Delta}{m}}^{t}\mathfrak{G}^{\rho^{(j)}(s)}(\xi^{(j)}(s))ds$. In a similar way we can show that $\xi^{(j)}(t)$, $j = 0,\ldots,m$, satisfies for all times t in its maximal right extension domain $\xi^{(j)}(t) = \xi^{(j-1)}(t) + \int_{t-\frac{\Delta}{m}}^{t}\mathfrak{F}^{\rho^{(j)}(s)}(\xi^{(j)}(s))ds$. Set $e^{(j)} := x^{(j)} - \xi^{(j)}$, $\eta^{(j)} := \varepsilon^{(j)} - \rho^{(j)}$. The proof proceeds by induction. First, we prove the boundedness of $e^{(j)}(t)$ and $\rho^{(j)}(t)$ for all $j = 1,\ldots,m$ and, finally, using invariance theorems we prove their convergence to zero. Assume that

$$\|e^{(j-1)}(t)\|, \|\rho^{(j-1)}(t)\| \leqslant \zeta^{(j-1)}(\|\varphi_0\|) \tag{14}$$

for all $t \geqslant 0$ and for some increasing continuous nonnegative function $\zeta^{(j-1)}$. By the induction hypothesis (14) and on account of (6) and lemma 8

$$\|\rho^{(j)}(t)\| \leqslant \tilde{\zeta}^{(j)}(\|\varphi_0\|) \tag{15}$$

for all $t \geqslant 0$ and for some increasing continuous nonnegative function $\tilde{\zeta}^{(j)}$.

On account of assumption (**A0**) it is easy to see that ϕ is also i.g.h.u.b. with quadruple $(\mathfrak{r}, \mathfrak{g}+\mathfrak{r}, \mathfrak{g}, \Phi_0)$. Using lemma 9, the induction hypothesis (14) and (15) there exists increasing continuous nonnegative functions α, δ such that for all $t \geqslant -\frac{\Delta}{m}$

$$\begin{aligned}&\|e^{(j)}(t)\| \leqslant \zeta^{(j-1)}(\|\varphi_0\|) + \frac{\Delta}{m}[\alpha(\|x_0\|) + \delta\circ\tilde{\zeta}^{(j)}(\|\varphi_0\|)]\\ &+\|A+BF+D\|\int_{t-\frac{\Delta}{m}}^{t}\|e^{(j)}(s)\|ds\end{aligned} \tag{16}$$

On account of (6), by lemma 8 and (15) it follows that

$$\|e^{(j)}(t)\|, \|\rho^{(j)}(t)\| \leqslant \zeta^{(j)}(\|\varphi_0\|) \tag{17}$$

for all $t \geqslant 0$ and for increasing continuous nonnegative function $\zeta^{(j)}$. Since $e^{(0)}(t)$ and $\rho^{(0)}(t) := \varepsilon^{(0)}(t)$ are norm–bounded for all $t \geqslant 0$ by some increasing continuous nonnegative function of $\|\varphi_0\|$, it follows by induction that (17) holds true for all $t \geqslant 0$ and $j = 0,\ldots,m$. Note that, since $\varepsilon^{(j)}(t)$ is norm–bounded for all $t \geqslant 0$ by some increasing continuous nonnegative function of $\|\varphi_0\|$, (17) implies that also $\eta^{(j)}(t)$ is norm–bounded by some increasing continuous nonnegative function of $\|\varphi_0\|$.

Next, we proceed again by induction. Assume that for some $j \geqslant 2$

$$\begin{aligned}&\lim_{t\to+\infty}[x^{(i)}(t) - \xi^{(i)}(t)] = 0,\\ &\lim_{t\to+\infty}[\varepsilon^{(i)}(t) - \rho^{(i)}(t)] = 0,\ i = 0,\ldots,j-1.\end{aligned} \tag{18}$$

Since $(x^{(0)}(t),\ldots,x^{(j)}(t),\xi^{(0)}(t),\ldots,\xi^{(j)}(t))$ and $(\varepsilon^{(0)}(t), \ldots,\varepsilon^{(m)}(t),\ \rho^{(0)}(t),\ldots,\rho^{(j)}(t))$ are bounded for all $t \geqslant 0$, its Ω-limit set is non-empty, compact and invariant (corollary 5.6 of Smith [2011]). Moreover, since $x^{(i)}(t) - \xi^{(0)}(t + i\frac{\Delta}{m}) \to 0$ for all $i = 0,\ldots,j$ as $t\to+\infty$ and by claim #2 of proposition 6, it also follows that

$$\lim_{t\to+\infty}[\sigma^{c(\varepsilon^{(i)})^{\mathfrak{r}}(t)}(x^{(i)}(t))) - x^{(i)}(t)] = 0,\ i = 0,\ldots,j. \tag{19}$$

It follows by invariance of the Ω-limit set, the induction hypothesis (18) and lemmas 9 and 10 that the trajectories

$(\hat{x}^{(0)}(t), \ldots, \hat{x}^{(j)}(t),\ \hat{\xi}^{(0)}(t), \ldots, \hat{\xi}^{(j)}(t), \hat{\varepsilon}^{(0)}(t), \ldots, \hat{\varepsilon}^{(j)}(t), \hat{\rho}^{(0)}(t), \ldots, \hat{\rho}^{(j)}(t))$ inside the Ω-limit set must satisfy for all $t \geqslant 0$

$$\|\hat{x}^{(j)}(t) - \hat{\xi}^{(j)}(t)\| \leqslant [\|A + BF + D\| + \gamma(\|\varphi_0\|)] \cdot \\ \cdot \int_{t-\frac{\Delta}{m}}^{t} [\|\hat{x}^{(j)}(s) - \hat{\xi}^{(j)}(s)\| + |\hat{\varepsilon}^{(j)}(s) - \hat{\rho}^{(j)}(s)|]ds \qquad (20)$$

In a similar way, using lemmas 9 and 10 it is seen that for some increasing continuous nonnegative function χ and for all $t \geqslant 0$

$$|\hat{\varepsilon}^{(j)}(t) - \hat{\rho}^{(j)}(t)| \leqslant \chi(\|\varphi_0\|) \int_{t-\frac{\Delta}{m}}^{t} [\|\hat{x}^{(j)}(s) - \hat{\xi}^{(j)}(s)\| \\ + |\hat{\varepsilon}^{(j)}(s) - \hat{\rho}^{(j)}(s)|]ds \qquad (21)$$

Summing (20) and (21) and using lemma 8 we conclude that $\hat{x}^{(j)}(t) = \hat{\xi}^{(j)}(t)$ and $\hat{\varepsilon}^{(j)}(t) = \hat{\rho}^{(j)}(t)$ for all $t \geqslant 0$. It follows that $\lim_{t\to+\infty}[x^{(j)}(t) - \xi^{(j)}(t)] = 0$ and $\lim_{t\to+\infty}[\varepsilon^{(j)}(t) - \rho^{(j)}(t)] = 0$. Since by construction

$$\lim_{t\to+\infty}[x^{(0)}(t) - \xi^{(0)}(t)] = 0,\ \lim_{t\to+\infty}[\varepsilon^{(0)}(t) - \rho^{(0)}(t)] = 0$$

we infer by induction that $\lim_{t\to+\infty}[x^{(j)}(t) - \xi^{(j)}(t)] = 0$ and $\lim_{t\to+\infty}[\varepsilon^{(j)}(t) - \rho^{(j)}(t)] = 0$ for all $j = 1, \ldots, m$.

7. EXAMPLES

A simulation has been worked out (Fig. 1(a), 1(b) and 1(c)) with $\Delta = 1$, $x(-\Delta) = (3, 2, 1)^T$ and $\hat{\xi}(s) = (0, 0, 0)^T$ for all $s \in [-\Delta, 0]$ for the forced Lorentz equation

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) - x_1(t) \\ \dot{x}_2(t) &= -x_1(t)x_3(t) - x_2(t) + 2x_1(t) \\ \dot{x}_3(t) &= -x_3(t) + x_1(t)x_2(t) + \sin t \end{aligned} \qquad (22)$$

If $\Delta = 1$ sec. then the integer m is chosen in such a way that $m \geqslant [\|A+BF+D\|+n^2c^2]\Delta$. For example, $m = 3$ and we readily obtain (according to proposition 6) an observer for the state $x(t - \Delta)$ of the Lorentz system.

8. CONCLUSIONS

Novel nonlinear predictors are studied for nonlinear systems with delayed measurements without assuming globally Lipschitz conditions but requiring bounded trajectories. The delay is constant and known. Further developments will be studied for the case of unknown delay and unstable systems.

Appendix A. INCREMENTAL HOMOGENEITY IN THE GENERALIZED SENSE: A REVIEW

The notion of (incremental) homogeneity in the generalized sense has been introduced (in a slightly different but equivalent form) in Battilotti [2013], Battilotti [2011] for enlarging the class of homogenous systems (in the classical sense: Rosier [1992]) in such a way to encompass triangular (non-homogeneous) systems and many other non-triangular systems.

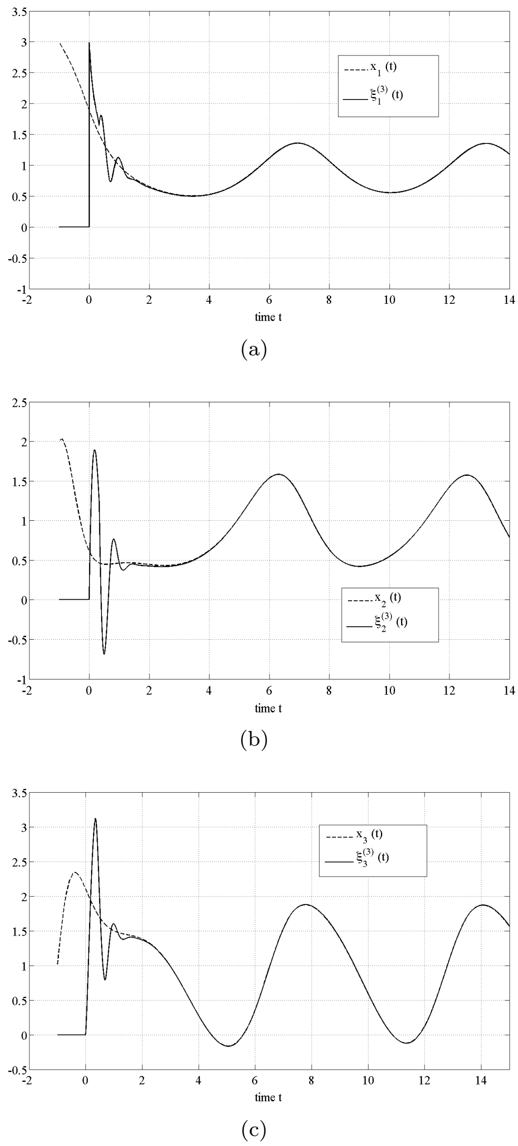

Fig. 1. The states $x(t - \frac{2\Delta}{3})$, $x(t - \frac{\Delta}{3})$ and $x(t)$ for (22) and their estimates $\xi^{(j)}(t)$, $j = 1, 2, 3$.

A function $\phi \in \mathbf{C}^0(\mathbb{R}_> \times \mathbb{R}^n, \mathbb{R}^l)$ is said to be incrementally homogeneous in the generalized sense (i.g.h.) with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \Phi_0)$ if there exist $\mathfrak{d} \in \mathbb{R}^l$, $\mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$ and $\Phi_0 \in \mathbf{C}^0(\mathbb{R}^{2n}, \mathbb{R}^{l\times n})$ such that

$$\phi_i(\varepsilon, \varepsilon^{\mathfrak{r}} \diamond w) - \phi_i(\varepsilon, \varepsilon^{\mathfrak{r}} \diamond z) \\ = \varepsilon^{\mathfrak{d}_i} \sum_{j=1}^{n} \varepsilon^{\mathfrak{h}_j} [\Phi_0]_{ij}(w, z)(w_j - z_j) \qquad (A.1)$$

for all $i = 1, \ldots, l$, $\varepsilon > 0$ and $w, z \in \mathbb{R}^n$.

The function $\phi(x) := x_1 + x_2^3$ is i.g.h. with quadruple $(\mathfrak{r}, 0, \mathfrak{h}, \Phi_0)$, where $\mathfrak{r} := (\mathfrak{r}_1, \mathfrak{r}_2)^T$, $\mathfrak{h} := (\mathfrak{r}_1, 3\mathfrak{r}_2)^T$ and $\Phi_0(w, z) = (1, w_2^2 + z_2^2 + z_2 w_2)$. The function $\phi(\varepsilon, x) := \varepsilon(x_1 + x_2^3)$ is i.g.h. with quadruple $(\mathfrak{r}, 1, \mathfrak{h}, \Phi_0)$.

A function $\phi \in \mathbf{C}^0(\mathbb{R}_> \times \mathbb{R}^n, \mathbb{R}^l)$ is said to be homogeneous in the generalized sense (g.h.) with quadruple $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \Phi_0)$ if there exist $\mathfrak{d} \in \mathbb{R}^l$, $\mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$ and $\Phi_0 \in \mathbf{C}^0(\mathbb{R}^n, \mathbb{R}^{l\times n})$ such that

$$\phi_i(\varepsilon, \varepsilon^{\mathfrak{r}} \diamond w) = \varepsilon^{\mathfrak{d}_i} \sum_{j=1}^{n} \varepsilon^{\mathfrak{h}_j} [\Phi_0]_{ij}(w) w_j \qquad (A.2)$$

for all $i = 1, \ldots, l$, $\varepsilon > 0$ *and* $w \in \mathbb{R}^n$.

I.g.h. implies g.h. and g.h. generalizes the classical notion of homogeneity (Rosier [1992]). Note that for homogeneity in the generalized sense we have two degrees $(\mathfrak{d}, \mathfrak{h})$ instead of only one $\mathfrak{d}$ in the classical sense and the function may depend on the dilating parameter ε itself. This clearly allows more structure in the nonlinearity of ϕ. In some sense we can say that we have a homogeneity degree $\mathfrak{d}_i$ for each component ϕ_i and one degree $\mathfrak{h}_j$ for each direction w_j.

There are functions, like $\sin x$, which are not i.g.h. but their absolute value is bounded by the absolute value of a function which is i.g.h.

A function $\phi \in \mathbf{C}^0(\mathbb{R}_> \times \mathbb{R}^n, \mathbb{R}^l)$ *is said to be incrementally homogeneous in the upper bound in the generalized sense (i.g.h.u.b.) with quadruple* $(\mathfrak{r}, \mathfrak{d}, \mathfrak{h}, \Phi_0)$ *if there exist* $\mathfrak{d} \in \mathbb{R}^l, \mathfrak{h} \in \mathbb{R}^n$, $\mathfrak{r} \in \mathbb{R}^n_>$, $\Phi_0 \in \mathbf{C}^0(\mathbb{R}^{2n}, \mathbb{R}^{l \times n}_{\geqslant})$ *and* $\varepsilon_0 > 0$ *such that*

$$\begin{aligned} &|\phi_i(\varepsilon, \varepsilon^{\mathfrak{r}} \diamond w) - \phi_i(\varepsilon, \varepsilon^{\mathfrak{r}} \diamond z)| \\ &= \varepsilon^{\mathfrak{d}_i} \sum_{j=1}^{n} \varepsilon^{\mathfrak{h}_j} [\Phi_0]_{ij}(w, z) |w_j - z_j| \end{aligned} \qquad (A.3)$$

for all $i = 1, \ldots, l$, $\varepsilon \geqslant \varepsilon_0$ *and* $w, z \in \mathbb{R}^n$.

The function $\phi(x) := \varepsilon \begin{pmatrix} x_2 & x_2^2 \sin x_1 \end{pmatrix}^T$ is i.g.h.u.b. with triple $(\mathfrak{r}, 1, \mathfrak{h}, \Phi_0)$, where

$$\begin{aligned} &\mathfrak{r} := \begin{pmatrix} \mathfrak{r}_1 \\ \mathfrak{r}_2 \end{pmatrix}, \ \mathfrak{h} = \begin{pmatrix} \mathfrak{r}_2 - \mathfrak{r}_1 \\ \mathfrak{r}_2 + \mathfrak{r}_1 \end{pmatrix}, \\ &\Phi_0(w, z) := \begin{pmatrix} 0 & 1 \\ z_2^2 \frac{|\sin w_1 - \sin z_1|}{|w_1 - z_1|} & |w_2 + z_2| \end{pmatrix}. \end{aligned} \qquad (A.4)$$

Without loss of generality one can assume $\varepsilon_0 > 1$, otherwise rescale z and w as $z' = \varepsilon_0^{\mathfrak{r}} \diamond z$ and, respectively, $w' = \varepsilon_0^{\mathfrak{r}} \diamond w$ and define new functions $[\Phi']_j(w', z') := \varepsilon_0^{\mathfrak{d}_i + \mathfrak{r}_i - \mathfrak{r}_j + \mathfrak{h}_j} \Phi_j(\varepsilon_0^{-\mathfrak{r}} \diamond w', \varepsilon_0^{-\mathfrak{r}} \diamond z')$ (resp. $[\Phi']_{ij}(w, z) := \varepsilon_0^{\mathfrak{d}_i + \mathfrak{r}_i - \mathfrak{r}_j + \mathfrak{h}_j} \Phi_{ij}(\varepsilon_0^{-\mathfrak{r}} \diamond w', \varepsilon_0^{-\mathfrak{r}} \diamond z')$). Homogeneous in the upper bound in the generalized sense (g.h.u.b.) functions are also defined in a similar way.

Some key properties of incremental homogeneity can be found and proved in Battilotti [2013], Battilotti [2011].

Appendix B. AUXILIARY RESULTS

The following result follows from the Grownwall-Bellman inequality.

Lemma 8. Assume that $s \in \mathbf{C}^0([-\frac{\Delta}{m}, +\infty), [0, +\infty))$ *is such that* $s(t)$ *is bounded for all* $t \in [-\frac{\Delta}{m}, 0]$ *and* $s(t) \leqslant k_0 \int_{t-\frac{\Delta}{m}}^{t} s(\tau) d\tau + k_1$ *for all* $t \geqslant 0$ *and for some* $k_0, k_1 \geqslant 0$. *If* m *is such that* $\frac{\Delta}{m} k_0 < 1$ *there exists* $c > 0$ *such that* $s(t) \leqslant c$ *for all* $t \geqslant 0$. *If, in addition,* $k_1 = 0$ *then* $s(t) = 0$ *for all* $t \geqslant \frac{\Delta}{m}$.

A saturation function has the following properties.

Lemma 9. If $\sigma^h \in \mathbf{D}^0(\mathbb{R}^n, \mathbb{R}^n)$ *is a saturation function with levels* h, *there exist* $c_1, c_2 > 0$ *such that* $\langle\langle \sigma^h)(w) - \sigma^h(z)) \rangle\rangle \leqslant c_1 \sigma^h(\langle\langle w - z \rangle\rangle) \leqslant c_2 \langle\langle w - z \rangle\rangle$ *for all* $w, z \in \mathbb{R}^n$.

In a similar way we can prove the following related result.

Lemma 10. If $\sigma^h, \sigma^k \in \mathbf{D}^0(\mathbb{R}^n, \mathbb{R}^n)$ *are saturation functions with levels* h *and* k, *there exist* $c > 0$ *such that* $\langle\langle \sigma^h(x) - \sigma^k(x) \rangle\rangle \leqslant c \langle\langle k - h \rangle\rangle$ *for all* $x \in \mathbb{R}^n$.

REFERENCES

S. Battilotti. Incremental generalized homogeneity, observer design and semiglobal stabilization. *Int. Journ. Asian Contr.*, 2013.

H. Smith. *An introduction to delay differential equations.* Text in Applied Math., Springer Series 57, 2011.

S. Battilotti. Generalized incremental homogeneity, incremental observability and global observer design. *Proc. IEEE Conf. Dec. Contr.*, Orlando, USA, 2011.

L.A. Marquez–Martinez, C. Moog, M. Velasco–Villa. Observability and observers with time–delay. *Proc. 2nd IFAC Workshop Time Delay Systems (LTDS)*, Ancona (Italy), pages 52–57, 2000.

F. Mazenc, M. Malisoff. Stabilization of a chemostat model with Haldane growth functions and a delay in the measurement. *IEEE Trans. on Automat. Contr.*, pages 1428–1436, 2010.

A. Germani, C. Manes, P. Pepe. A new approach to state observation of nonlinear systems with delayed output. *IEEE Trans. on Automat. Contr.*, 47, pages 96–101, 2002.

S. Ibrir. Observer based control of a class of time delay nonlinear systems. *Automatica*, 47, pages 388-394, 2011.

M. Kristic. *Delay compensation for nonlinear adaptive and PDE systems.* Birkhauser, Boston, 2009.

I. Karafyllis, M. Kristic. Predictor–based output feedback for nonlinear delay systems. *Preprint*, arXiv:1108.4499, 2012.

I. Karafyllis, M. Kristic. Nonlinear stabilization under sampled and delayed measurements and with inputs subject to delay and zero–order hold. *IEEE Trans. Autom. Contr.*, **57**, pages 1141–1154, 2012.

I. Karafyllis, M. Kristic, T. Ahmed–Ali, F. Lamnabhi–Lagariggue. Global stabilization of nonlinear delay systems with a compact absorbing set. *Preprint*, 2013.

I. Karafyllis. Stabilization by means of approximate predictors for systems with dealyed input. *SIAM Journ. Optim. Contr.*, 49, pages 1141–1154, 2012.

N. Kazantzis. Stabilization by means of approximate predictors for systems with delayed input. *SIAM Journ. Optim. Contr.*, 49, pages 1141–1154, 2012.

N. Kazantzis, R. A. Wright. Nonlinear observer design in the presence of delayed output measurements. *Syst. Contr. Lett.*, 54, pages 11001123, 2001.

N. Bekiaris-Liberis, M. Kristic. Nonlinear control under delays that depend on delayed states. *Europ. Journ. Contr.*, 19, pages 389398, 2013.

G. Besancon, D. Georges, Z. Benayache. Asymptotic state prediction for continuous time systems with delayed input output and application to control. *Proc. Europ. Contr. Conf.*, pages 552–557, 2007.

L. Rosier. Homogeneous Lyapunov function for homogeneous continuous vector field. *Syst. Contr. Lett.*, 19, pages 467–473, 1992.