Preprints of the 19th World Congress
The International Federation of Automatic Control
Cape Town, South Africa. August 24-29, 2014

Flat Output Computation for Fractional Linear Systems: Application to a Thermal System

Stéphane VICTOR * **Pierre MELCHIOR** * **Jean LÉVINE** ** **Alain OUSTALOUP** *

* *IMS – UMR 5218 CNRS, IPB/Enseirb-Matmeca – Université Bordeaux 1*
351 cours de la Libération, 33405 Talence cedex – France
Email: {stephane.victor, pierre.melchior, alain.oustaloup}@ims-bordeaux.fr
** *CAS-Mathématiques et Systèmes, MINES-ParisTech*
35 rue Saint-Honoré, 77300 Fontainebleau – France
Email: jean.levine@mines-paristech.fr

Abstract: The flatness property is studied for linear time-invariant fractional systems. In the framework of polynomial matrices of the fractional derivative operator, we give a characterization of fractionally flat outputs and a simple algorithm to compute them. These results are applied to a bidimensional thermal system approximated by a fractional transfer function of order $\frac{1}{2}$. Temperature trajectory planning at a given point is then deduced without integration of the system equations.

1. INTRODUCTION

Fractional models have proved their utility in modeling diffusive phenomena: thermal systems (Oldham and Spanier [1974]), electro-chemical ones (Darling and Newman [1997]), biological systems (Magin [2010]), etc. Integer models may lead to models of too large dimension or poorly reproducing some dynamical aspects. However the motion planning problem in this fractional context, where we want to compute the inputs generating given reference trajectories, has not yet received many contributions.

To this aim, and following Victor et al. [2013], we present an extension to the framework of linear time-invariant fractional systems of the property called differential flatness, or more shortly flatness, introduced by M. Fliess, J. Lévine, Ph. Martin and P. Rouchon (Fliess et al. [1992, 1995, 1999], see also the books Rudolph [2003], Sira-Ramirez and Agrawal [2004], Lévine [2009] and the references therein) and its application to the so-called flatness-based trajectory design (see e.g. Lévine [2009]).

Recall that a system described by ordinary differential equations is said to be differentially flat if, and only if, there exists a variable, called flat output, whose dimension is equal to the dimension of the control, such that all system variables, including the controls, can be expressed as functions of this variable and a finite number of its successive derivatives. Note that, if we restrict to linear time-invariant systems (LTI), differential flatness is equivalent to controllability and flat outputs may be obtained as a by-product of the Brunovský controllability canonical form (see e.g. Sira-Ramirez and Agrawal [2004], Lévine [2009]) or by using a characterization in polynomial matrix form of the so-called *defining matrices* (Lévine [2009]).

The contributions of this paper are: a rigorous algebraic definition of the flatness property for linear fractional systems, and its characterization in the framework of polynomial matrices of the fractional derivative operator; as a by-product, we recover the equivalence between flatness for fractional systems and controllability (see Fliess and Hotzel [1997], Hotzel [1998] on controllability and stabilizability of fractional systems); then a simple algorithm to compute fractionally flat outputs is obtained as well as a specialization of the latter results to the notion of 0-flatness for fractional systems. Flatness is then studied on a bidimensional thermal system approximated by a fractional transfer function of order $\frac{1}{2}$, where the temperature trajectory planning is computed at a given point.

After some recalls on fractional calculus in Section 2, the notion of flatness for fractional linear systems is presented and characterized in Section 3. In Section 4, an application to the temperature control of a metallic 2-dimensional sheet modeled by the heat equation in the right half (x, y)-plane, controlled by the heat density flux at the origin, is presented. Using a suitable series expansion, this system is approximated by a fractional differential equation of order $\frac{1}{2}$, which is proven to be fractionally flat. A planning of the temperature trajectory at a given point of the metallic sheet is deduced and simulations are presented. Some conclusions are drawn in section 5.

2. FRACTIONAL CALCULUS CONTEXT

Let us introduce $\mathbf{D} = \frac{d}{dt}$ the ordinary differentiation operator. A possible approach to define the field $\mathcal{M}$ of fractional derivative operators has been proposed by Fliess and Hotzel [1997], Hotzel [1998], following Mikusiński. It is defined as the field of fractions of the commutative integral domain $\mathcal{C}$ of continuous functions over $[0, \infty[$ endowed with the addition and convolution product (Mikusiński [1983], Fliess and Hotzel [1997], Hotzel [1998]). $\mathcal{M}$ can also be considered as a $\mathbb{R}[s^\gamma]$-module, where s^γ is the Laplace operator associated with the fractional operator [1] $\mathbf{D}^\gamma$, γ being a positive real number. We then denote by $\mathbb{R}[\mathbf{D}^\gamma]$ the set of such $\mathbf{D}^\gamma$-polynomials endowed with the usual addition and multiplication of polynomials.

Note that in $\mathbb{R}[\mathbf{D}^\gamma]$, the properties of polynomial division and Bézout identity, as well as the definitions of greatest common divisor, least common multiple, and prime polynomials, are the same as in the principal ideal domain of formal real polynomials.

If $p, q \in \mathbb{N}$, we call $(\mathbb{R}[\mathbf{D}^\gamma])^{p\times q}$ the set of $\mathbf{D}^\gamma$-polynomial matrices of size $(p \times q)$. When $p = q$, we denote by $GL_p(\mathbb{R}[\mathbf{D}^\gamma])$ the group of unimodular $\mathbf{D}^\gamma$-polynomial matrices, i.e. the set of invertible (square) $\mathbf{D}^\gamma$-polynomial matrices whose inverse is also a $\mathbf{D}^\gamma$-polynomial matrix. We denote by I_p the $p\times p$ identity matrix and by $0_{p\times q}$ the $p \times q$ zero matrix.

$\mathbf{D}^\gamma$-polynomial matrices enjoy the following diagonal decomposition property ([Cohn, 1985, Chap. 8]).

Theorem 1. (Smith diagonal decomposition). For a given matrix $A \in (\mathbb{R}[\mathbf{D}^\gamma])^{p\times q}$, with $p \leq q$ (*resp.* $p \geq q$), there exist two matrices $S \in GL_p(\mathbb{R}[\mathbf{D}^\gamma])$ and $T \in GL_q(\mathbb{R}[\mathbf{D}^\gamma])$ such that:

$$SAT = [\Delta \quad 0_{p,q-p}] \quad (\textit{resp.} = \begin{bmatrix} \Delta \\ 0_{p-q,q} \end{bmatrix}), \tag{1}$$

where $\Delta = \mathrm{diag}\{\delta_1, \ldots, \delta_\sigma, 0, \ldots, 0\}$. Every non zero $\mathbf{D}^\gamma$-polynomial δ_i, for $i = 1, \ldots, \sigma$, is a divisor of δ_j for all $\sigma \geq j \geq i$. The integer σ is called the *rank* of A.

The Smith decomposition over the polynomial ring $\mathbb{R}[\mathbf{D}^\gamma]$ consists in computing a diagonal form by repeatedly using greatest common divisors (e.g., in Gantmacher [1960]).

Definition 1. (Hyper-regularity, Lévine [2009]). Given a matrix $A \in (\mathbb{R}[\mathbf{D}^\gamma])^{p\times q}$, we say that A is hyper-regular if, and only if, in (1), $\Delta = I_p$ (resp. $\Delta = I_q$).

A square matrix $A \in (\mathbb{R}[\mathbf{D}^\gamma])^{p\times p}$ is hyper-regular if, and only if, it is unimodular. Following Antritter and Middeke [2011], we have:

Proposition 2. (Antritter and Middeke [2011]) A matrix $A \in (\mathbb{R}[\mathbf{D}^\gamma])^{p\times q}$ is hyper-regular if, and only if:

(i) if $p < q$, A has a right-inverse, i.e. there exists T in $GL_q(\mathbb{R}[\mathbf{D}^\gamma])$ such that

$$AT = \begin{bmatrix} I_p & 0_{p\times(q-p)} \end{bmatrix}; \tag{2}$$

(ii) if $p \geq q$, A has a left-inverse, i.e. there exists S in $GL_q(\mathbb{R}[\mathbf{D}^\gamma])$ such that

$$SA = \begin{bmatrix} I_q \\ 0_{(p-q)\times q} \end{bmatrix}. \tag{3}$$

[1] The user may use any definition of fractional order differentiation, e.g. Samko et al. [1993].

3. LINEAR FRACTIONALLY FLAT SYSTEMS

3.1 Linear fractional systems

We consider a γ-commensurate [2] linear fractional system

$$Ax = Bu \tag{4}$$

with state or partial state x of dimension n, input u of dimension m, $A \in (\mathbb{R}[\mathbf{D}^\gamma])^{n\times n}$ and $B \in (\mathbb{R}[\mathbf{D}^\gamma])^{n\times m}$. B is assumed to be of rank m, with $1 \leq m \leq n$ [3]. For system (4), we consider (see e.g. Fliess [1990], Polderman and Willems [1998]):

- its *behavior* $\ker[A, -B]$, i.e. the set
$$\left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in (\mathcal{C})^{(n+m)\times 1} | [A, -B] \begin{bmatrix} x \\ u \end{bmatrix} = 0 \right\};$$
- its *system module* $\mathfrak{M}_{A,B}$, the quotient module
$$\mathfrak{M}_{A,B} = \mathbb{R}[\mathbf{D}^\gamma]^{1\times(n+m)} / \mathbb{R}[\mathbf{D}^\gamma]^{1\times n}[A, -B], \tag{5}$$
where $\mathbb{R}[\mathbf{D}^\gamma]^{1\times(n+m)}$ is the set of row vectors of dimension $n+m$ with components in $\mathbb{R}[\mathbf{D}^\gamma]$ and where $\mathbb{R}[\mathbf{D}^\gamma]^{1\times n}[A, -B]$ is the module generated by the rows of the $n \times (n + m)$ matrix $[A, -B]$.

3.2 Flatness for fractional systems

Due to space limitations, we focus only on major results. A more detailed presentation with proofs may be found in Victor et al. [2013]. From now on, it is assumed that the matrix $F \triangleq [A, -B]$ has full (left) row rank.

System (4) reads:

$$F \begin{bmatrix} x \\ u \end{bmatrix} = 0. \tag{6}$$

Definition 2. (Flatness and defining matrices). The system (6) is called fractionally flat if, and only if, there exist matrices $P \in (\mathbb{R}[\mathbf{D}^\gamma])^{m\times(n+m)}$ and $Q \in (\mathbb{R}[\mathbf{D}^\gamma])^{(n+m)\times m}$, called defining matrices, such that

$$Q(\mathcal{C})^m = \ker F \quad \text{and} \quad PQ = I_m. \tag{7}$$

In other words, there exist matrices P and Q over the ring $\mathbb{R}[\mathbf{D}^\gamma]$ such that, for all (x, u) satisfying $F \begin{bmatrix} x \\ u \end{bmatrix} = 0$, we have $y = P \begin{bmatrix} x \\ u \end{bmatrix}$ and $\begin{bmatrix} x \\ u \end{bmatrix} = Qy$. The variable y, taking its values in $(\mathcal{C})^m$, is called *fractionally flat output*.

Theorem 3. We have the following equivalences (see [Victor et al., 2013, Theorem 3.1]):

(i) system (6) is fractionally flat;
(ii) the system module $\mathfrak{M}_{A,B}$ is free;
(iii) the matrix F is hyper-regular over $\mathbb{R}[\mathbf{D}^\gamma]$.

For linear controllable systems, a set of flat outputs may be obtained via Brunovský's canonical form and do not depend on the input u. This property is called *fractional 0-flatness*. It reads: there exist $P_1 \in (\mathbb{R}[\mathbf{D}^\gamma])^{m\times n}$ and

[2] When all differentiation orders are integer multiples of a real positive γ, the system is said commensurate of order γ or γ-commensurate

[3] This assumption is only made to discard trivial cases of flatness: if $n < m$, x completed by $m - n$ arbitrary components of u can be directly chosen as flat output

$Q_1 \in (\mathbb{R}[\mathbf{D}^\gamma])^{n\times m}$ such that $y = P_1 x$, $x = Q_1 y$, and $P_1 Q_1 = I_m$.

Definition 3. A system is said fractionally 0-flat if $P\begin{bmatrix}0_{n,m}\\ I_m\end{bmatrix} = 0_m$ and y is called *fractional* 0*-flat output.*

Fractional 0-flatness is thus equivalent to the existence of P and Q such that $P = [P_1\ 0_{m,n}]$ with $P_1 \in (\mathbb{R}[\mathbf{D}^\gamma])^{m\times n}$ and $P_1 Q_1 = I_m$ where $Q_1 \triangleq [I_n\ 0_{n,m}]\,Q$.

Fractionally 0-flat outputs can be computed as follows:

Lemma 4. (Elimination). If B is hyper-regular, there exists a unimodular matrix $M \in (\mathbb{R}[\mathbf{D}^\gamma])^{n\times n}$ such that $MB = \begin{bmatrix}I_m\\ 0_{(n-m)\times m}\end{bmatrix}$. Moreover, there exist matrices $\tilde{F} \in (\mathbb{R}[\mathbf{D}^\gamma])^{(n-m)\times n}$ and $R \in (\mathbb{R}[\mathbf{D}^\gamma])^{m\times n}$ such that System (4) is equivalent to $Rx = u$, $\tilde{F}x = 0$.

Theorem 5. If B is hyper-regular, the following statements are equivalent (see [Victor et al., 2013, Theorem 3.3]):

(i) System (4) is fractionally 0-flat;
(ii) the system module
$$\mathfrak{M}_{\tilde{F}} \triangleq (\mathbb{R}[\mathbf{D}^\gamma])^{1\times n} / (\mathbb{R}[\mathbf{D}^\gamma])^{1\times(n-m)}\,\tilde{F}$$
is free, with $\tilde{F}$ defined in Lemma 4;
(iii) the matrix $\tilde{F}$ is hyper-regular over $\mathbb{R}[\mathbf{D}^\gamma]$.

Algorithm 1. (Computation of fractionally 0-flat output).

***Input*:** The matrices A and B of System (4) with B hyper-regular.
***Output*:** Defining matrices P and Q, i.e. satisfying (7), with $P = [P_1\ 0_{m,n}]$, $P_1 \in (\mathbb{R}[\mathbf{D}^\gamma])^{m\times n}$, $Q_1 \triangleq [I_n\ 0_{n,m}]\,Q$ and $P_1 Q_1 = I_m$.
***Procedure*:**
(1) Check, using row-reduction, if B is hyper-regular. If not, return "fail".
(2) Else, find $M \in GL_n(\mathbb{R}[\mathbf{D}^\gamma])$ such that $MB = \begin{bmatrix}I_m\\ 0_{(n-m)\times m}\end{bmatrix}$.
(3) With M, obtain $R \in (\mathbb{R}[\mathbf{D}^\gamma])^{m\times n}$ and $\tilde{F} \in (\mathbb{R}[\mathbf{D}^\gamma])^{(n-m)\times n}$, according to Lemma 4, by:
$$MA = \begin{bmatrix}R\\ \tilde{F}\end{bmatrix}.$$
(4) Find $W \in GL_n(\mathbb{R}[\mathbf{D}^\gamma])$, according to Proposition 2 (i), such that $\tilde{F}W = \left[I_{n-m}, 0_{(n-m)\times m}\right]$. We get:
$$Q_1 = W\begin{bmatrix}I_{n-m}\\ 0_{m\times(n-m)}\end{bmatrix}, P_1 = \left[I_{n-m}, 0_{(n-m)\times m}\right]W^{-1}$$
(5) Set $P = [P_1, 0]$ and $Q = \begin{bmatrix}Q_1\\ RQ_1\end{bmatrix}$.

4. A THERMAL BIDIMENSIONAL APPLICATION

4.1 Heat equation

A two dimension heated metallic sheet is considered (see Figure 1). The medium is considered as a homogeneous metallic semi-infinite plane of diffusivity α, conductivity

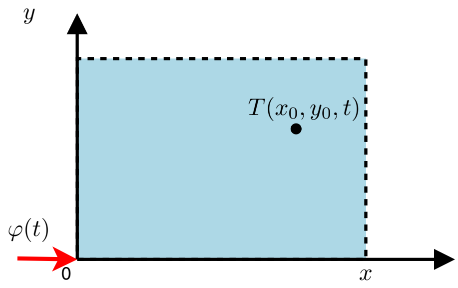

Figure 1. Heated metallic sheet

λ, isolated and without heat losses. The temperature $T(x, y, t)$ is controlled by the heat density flux $\varphi(t)$ across the y-axis at the origin.

Our aim is to plan a temperature trajectory at a given point of the sheet, of coordinates (x_0, y_0) and obtain the open-loop control that generates it.

As it is well-known, the temperature diffusion satisfies the following scalar heat equation:
$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{\alpha}\frac{\partial}{\partial t}\right)T(x, y, t) = 0, \tag{8}$$
in the open half-space $x \in [0, \infty[$, $y \in [0, \infty[$ and $t > 0$, with boundary condition
$$-\lambda\left.\frac{\partial T(x, y, t)}{\partial x}\right|_{x=0, y=0} = \varphi(t), \quad \forall t > 0 \tag{9}$$
and Cauchy condition
$$T(x, y, 0) = 0 \text{ for all } 0 \le x < \infty \text{ and } 0 \le y < \infty \tag{10}$$
meaning that the initial temperature vanishes at any point of the sheet for all $t \le 0$.

Applying the Laplace transform to (8), we get:
$$\frac{s}{\alpha}\hat{T}(x, y, s) = \frac{\partial^2\hat{T}(x, y, s)}{\partial x^2} + \frac{\partial^2\hat{T}(x, y, s)}{\partial y^2}. \tag{11}$$

Let us compute its solution $\hat{T}(x, y, s)$ using the well-known method of separation of variables:
$$\hat{T}(x, y, s) = \hat{T}_x(x, s)\hat{T}_y(y, s). \tag{12}$$

Equation (11) reads
$$\frac{s}{\alpha}\hat{T}_x(x, s)\hat{T}_y(y, s) = \hat{T}_y(y, s)\frac{\partial^2\hat{T}_x(x, s)}{\partial x^2} + \hat{T}_x(x, s)\frac{\partial^2\hat{T}_y(y, s)}{\partial y^2}.$$

Dividing by $\hat{T}(x, y, s)$, we get
$$\frac{s}{\alpha} = \frac{1}{\hat{T}_x(x, s)}\frac{\partial^2\hat{T}_x(x, s)}{\partial x^2} + \frac{1}{\hat{T}_y(y, s)}\frac{\partial^2\hat{T}_y(y, s)}{\partial y^2}.$$

Grouping the functions of (x, s) and (y, s) respectively, the latter equation can be split into the two independent ones:
$$\begin{cases}\dfrac{s}{2\alpha} = \dfrac{1}{\hat{T}_x(x, s)}\dfrac{\partial^2\hat{T}_x(x, s)}{\partial x^2},\\[2ex] \dfrac{s}{2\alpha} = \dfrac{1}{\hat{T}_y(y, s)}\dfrac{\partial^2\hat{T}_y(y, s)}{\partial y^2}\end{cases}$$

whose solutions are given by:
$$\begin{cases}\hat{T}_x(x, s) = K_x e^{\sqrt{\frac{s}{2\alpha}}x} + L_x e^{-\sqrt{\frac{s}{2\alpha}}x}\\ \hat{T}_y(y, s) = K_y e^{\sqrt{\frac{s}{2\alpha}}y} + L_y e^{-\sqrt{\frac{s}{2\alpha}}y}.\end{cases}$$

The limit conditions for $x \to \infty$ and for $y \to \infty$ lead to $K_x = 0$ and $K_y = 0$.

Thus

$$\hat{T}(x,y,s) = \hat{T}_x(x,s)\hat{T}_y(y,s) = L_x L_y e^{-\sqrt{\frac{s}{2\alpha}}(x+y)} \quad (13)$$

and the flux density is given by

$$\hat{\varphi}(x,y,s) = -\lambda \frac{\partial \hat{T}_x(x,s)}{\partial x} = \lambda \sqrt{\frac{s}{2\alpha}} L_x L_y e^{-\sqrt{\frac{s}{2\alpha}}(x+y)}. \quad (14)$$

The heat density flux at the origin is thus

$$\hat{\varphi}(s) \triangleq \hat{\varphi}(0,0,s) = \lambda \sqrt{\frac{s}{2\alpha}} L_x L_y. \quad (15)$$

Therefore, the unique solution of (11) for every given flux density $\varphi \in L^2(0,\infty)$ reads:

$$\hat{T}(x,y,s) = \frac{1}{\lambda\sqrt{\frac{s}{2\alpha}}} e^{-\sqrt{\frac{s}{2\alpha}}(x+y)} \hat{\varphi}(s). \quad (16)$$

Let us define the *thermal impedance* by:

$$\tilde{H}(x,y,s) \triangleq \frac{\hat{T}(x,y,s)}{\hat{\varphi}(s)} = \frac{e^{-\sqrt{\frac{s}{2\alpha}}(x+y)}}{\lambda\sqrt{\frac{s}{2\alpha}}} = \frac{\sqrt{2\alpha}}{\lambda\sqrt{s}} \frac{e^{-\sqrt{\frac{s}{2\alpha}}\frac{(x+y)}{2}}}{e^{\sqrt{\frac{s}{2\alpha}}\frac{(x+y)}{2}}} \quad (17)$$

4.2 Approximate fractional heat transfer

After series expansion in the variable $(x+y)$ around the origin, truncated at a given order K, we get:

$$\tilde{H}_K(x,y,s) = \frac{\sqrt{2\alpha}}{\lambda} \frac{\sum_{k=0}^{K} a_k s^{k\gamma}}{\sum_{k=0}^{K} |a_k|\, s^{(k+1)\gamma}}, \quad (18)$$

where the commensurate order is $\gamma = \frac{1}{2}$ and $a_k = (-1)^k \frac{(x+y)^k}{k!(8\alpha)^{k\gamma}}$.

For fixed arbitrary x and y, the fractional transfer function $\tilde{H}_K$ is thus proper and it can be verified that it fastly converges to $\tilde{H}$ as K tends to infinity. Moreover, $\tilde{H}_K$ can be represented, in the time domain, by a fractional differential system of order $\frac{1}{2}$:

$$\begin{aligned} AX &= B\varphi \\ T_K(x,y,t) &= CX \end{aligned} \quad (19)$$

where $T_K(x,y,t)$ is the truncated (approximate) temperature at the position (x,y) and time t and with [4]

$$X \triangleq \begin{bmatrix} X_K \\ \vdots \\ X_0 \end{bmatrix}, B \triangleq \begin{bmatrix} 1 \\ 0_{K\times 1} \end{bmatrix}, \quad C \triangleq \frac{\sqrt{2\alpha}}{\lambda} \left[\, a'_K \ \cdots \ a'_0 \,\right]$$

$$A \triangleq \begin{bmatrix} \mathbf{D}^{\frac{1}{2}} + |a'_{K-1}| & |a'_{K-2}| & \dots & |a'_0| & 0 \\ -1 & \mathbf{D}^{\frac{1}{2}} & 0 & \dots & 0 \\ 0 & -1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ 0 & & \dots & 0 & -1 & \mathbf{D}^{\frac{1}{2}} \end{bmatrix}, \quad (20)$$

and where we have denoted $a'_k = a_k/|a_K|$.

[4] the component X_k of X may be interpreted as the derivative of order $k/2$ of the heat flux $\varphi(t)$.

4.3 Flat output computation

Let us now apply Algorithm 1. Since B is hyper-regular and already in its Smith form, we immediately find that $M = I_{K+1} \in GL_{K+1}\left(\mathbb{R}\left[\mathbf{D}^{\frac{1}{2}}\right]\right)$.

$R \in \left(\mathbb{R}\left[\mathbf{D}^{\frac{1}{2}}\right]\right)^{1\times(K+1)}$ and $\tilde{F} \in \left(\mathbb{R}\left[\mathbf{D}^{\frac{1}{2}}\right]\right)^{K\times(K+1)}$ are thus given by: $\begin{bmatrix} R \\ \tilde{F} \end{bmatrix} = MA = A$ with

$$R = \left[\mathbf{D}^{\frac{1}{2}} + |a'_{K-1}|, |a'_{K-2}|, \dots, |a'_0|, 0\right]$$

and

$$\tilde{F} = \begin{bmatrix} -1 & \mathbf{D}^{\frac{1}{2}} & & 0 \\ & \ddots & \ddots & \\ 0 & & -1 & \mathbf{D}^{\frac{1}{2}} \end{bmatrix}.$$

$\tilde{F}$ is hyper-regular, and thus System (20) is 0-flat. We next compute $W \in \left(\mathbb{R}\left[\mathbf{D}^{\frac{1}{2}}\right]\right)^{(K+1)\times(K+1)}$ satisfying $\tilde{F}W = [I_K \ 0_{K\times 1}]$. We get the upper triangular matrix

$$W = \begin{bmatrix} -1 & -\mathbf{D}^{\frac{1}{2}} & -\mathbf{D}^{1} & \dots & -\mathbf{D}^{\frac{K}{2}} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & -\mathbf{D}^{1} \\ & & & \ddots & -\mathbf{D}^{\frac{1}{2}} \\ 0 & & & & -1 \end{bmatrix},$$

and its inverse, also upper triangular,

$$W^{-1} = \begin{bmatrix} -1 & \mathbf{D}^{\frac{1}{2}} & & 0 \\ & \ddots & \ddots & \\ & & \ddots & \mathbf{D}^{\frac{1}{2}} \\ 0 & & & -1 \end{bmatrix}.$$

Therefore

$$Q_1 = W \begin{bmatrix} 0_{K\times 1} \\ 1 \end{bmatrix} = \begin{bmatrix} -\mathbf{D}^{\frac{K}{2}} \\ \vdots \\ -\mathbf{D}^{\frac{1}{2}} \\ -1 \end{bmatrix},$$

$$P_1 = [0_{1\times K} \ 1]\, W^{-1} = [0_{1\times K} \ -1].$$

and we indeed have $P_1 Q_1 = 1$.

Finally, the defining matrices P and Q read:

$$P = [P_1 \ 0] = [0_{1\times K} \ -1 \ 0],$$

$$Q = \begin{bmatrix} Q_1 \\ RQ_1 \end{bmatrix} = \begin{bmatrix} -\mathbf{D}^{\frac{K}{2}} \\ \vdots \\ -\mathbf{D}^{\frac{1}{2}} \\ -1 \\ -\sum_{k=0}^{K} |a'_k|\, \mathbf{D}^{\frac{k+1}{2}} \end{bmatrix}$$

which proves that System (19) is fractionally 0-flat and that a fractional flat output Y is given by

$$\begin{aligned} &Y = P_1 X = -X_0, \\ &T_K = CX = CQ_1 Y = -\frac{\sqrt{2\alpha}}{\lambda}\sum_{k=0}^{K} a'_k \mathbf{D}^{\frac{k}{2}} Y, \\ &\varphi = RQ_1 Y = -\sum_{k=0}^{K} |a'_k|\, \mathbf{D}^{\frac{k+1}{2}} Y. \end{aligned} \tag{21}$$

4.4 Trajectory planning

Let us define a rest-to-rest temperature trajectory at the point $x_0 = 0.005$m and $y_0 = 0.002$m located on the metallic sheet (see Fig. 1). The desired temperature should rise of $30°C$ over the ambient temperature at time t_f and arrive at rest:

$$T(x_0, y_0, 0) \triangleq T_0 = 0,\ \dot{T}(x_0, y_0, 0) = 0,\ \ddot{T}(x_0, y_0, 0) = 0,$$
$$T(x_0, y_0, t_f) \triangleq T_f = 30,\ \dot{T}(x_0, y_0, t_f) = 0,\ \ddot{T}(x_0, y_0, t_f) = 0.$$

We first translate these constraints on the fractionally flat output Y, assuming that we can identify $T(x, y, t)$ with $T_K(x, y, t)$ and that the Y-trajectory is given by a polynomial of t:

$$Y(t) \triangleq \sum_{j=0}^{r} \eta_j \left(\frac{t}{t_f}\right)^j 1_{[0,t_f]}(t) \tag{22}$$

with the integer r and the real coefficients η_j, $j = 0, \ldots, r$, to be determined, and where $1_{[0,t_f]}(t)$ is the indicator function of the interval $[0, t_f]$. We compute $Y(0)$ and $Y(t_f)$ using (21). To this aim, recall the following formula, for all $k = 0, \ldots, K$, $j = 0, \ldots, r$, and $t \geq 0$:

$$\mathbf{D}^{\frac{k}{2}} t^j = \frac{\Gamma(j+1)}{\Gamma(j+1-\frac{k}{2})} t^{j-\frac{k}{2}}.$$

The desired temperature trajectory at (x_0, y_0) and the heat density flux φ are then deduced from (21): replacing these expressions in (21), we get:

$$T(t) = -\frac{\sqrt{2\alpha}}{\lambda}\sum_{j=0}^{r}\left(\sum_{k=0}^{K} a'_k \frac{\Gamma(j+1)}{\Gamma(j+1-\frac{k}{2})} t^{j-\frac{k}{2}}\right)\left(\frac{\eta_j}{t_f^j}\right).$$

Differentiating this expression with respect to time and using the identity $\Gamma(j+1-\frac{k}{2}) = (j-\frac{k}{2})\Gamma(j-\frac{k}{2})$, yields:

$$\dot{T}(t) = -\frac{\sqrt{2\alpha}}{\lambda}\sum_{j=0}^{r}\left(\sum_{k=0}^{K} a'_k \frac{\Gamma(j+1)}{\Gamma(j-\frac{k}{2})} t^{j-\frac{k}{2}-1}\right)\left(\frac{\eta_j}{t_f^j}\right)$$

and

$$\ddot{T}(t) = -\frac{\sqrt{2\alpha}}{\lambda}\sum_{j=0}^{r}\left(\sum_{k=0}^{K} a'_k \frac{\Gamma(j+1)}{\Gamma(j-\frac{k}{2}-1)} t^{j-\frac{k}{2}-2}\right)\left(\frac{\eta_j}{t_f^j}\right).$$

In order to satisfy the conditions $T(0) = T_0$ and $\dot{T}(0) = \ddot{T}(0) = 0$, the constant term (for $j = k = 0$) being equal to $-\frac{\sqrt{2\alpha}}{\lambda} a'_0 \eta_0$, we get $\eta_0 = -\frac{\lambda T_0}{a'_0\sqrt{2\alpha}} = 0$ and the coefficients η_j corresponding to the exponents satisfying $j - \frac{k}{2} - 2 < 0$ must vanish. Hence $\eta_j = 0$ for all $j < \frac{K}{2} + 2$ and T reads:

$$T(t) = -\frac{\sqrt{2\alpha}}{\lambda}\sum_{j=\lceil \frac{K}{2}+2\rceil}^{r}\left(\sum_{k=0}^{K} a'_k \frac{\Gamma(j+1)}{\Gamma(j+1-\frac{k}{2})} t^{j-\frac{k}{2}}\right)\left(\frac{\eta_j}{t_f^j}\right).$$

where $\lceil \frac{K}{2} + 2\rceil$ denotes the ceiling of $\frac{K}{2} + 2$. Note that it readily implies that $T(0) = \dot{T}(0) = \ddot{T}(0) = 0$. Since there are 3 final conditions left for $T(t_f)$, $\dot{T}(t_f)$ and $\ddot{T}(t_f)$, r must satisfy $r \geq \lceil \frac{K}{2} + 2\rceil + 2$. We therefore have to solve the linear system in the coefficients η_j, $j = \lceil \frac{K}{2} + 2\rceil, \ldots r$:

$$\begin{cases} -\dfrac{\sqrt{2\alpha}}{\lambda}\displaystyle\sum_{j=\lceil \frac{K}{2}+2\rceil}^{r}\left(\sum_{k=0}^{K}\frac{a'_k}{t_f^{\frac{k}{2}}}\frac{\Gamma(j+1)}{\Gamma(j+1-\frac{k}{2})}\right)\eta_j = T_f \\ -\dfrac{\sqrt{2\alpha}}{\lambda}\displaystyle\sum_{j=\lceil \frac{K}{2}+2\rceil}^{r}\left(\sum_{k=0}^{K}\frac{a'_k}{t_f^{\frac{k}{2}+1}}\frac{\Gamma(j+1)}{\Gamma(j-\frac{k}{2})}\right)\eta_j = 0 \\ -\dfrac{\sqrt{2\alpha}}{\lambda}\displaystyle\sum_{j=\lceil \frac{K}{2}+2\rceil}^{r}\left(\sum_{k=0}^{K}\frac{a'_k}{t_f^{\frac{k}{2}+2}}\frac{\Gamma(j+1)}{\Gamma(j-\frac{k}{2}-1)}\right)\eta_j = 0 \end{cases} \tag{23}$$

and plug its solution in (22) to obtain the required reference trajectory of $Y(t)$. The required temperature trajectory at the point (x_0, y_0) and the corresponding heat density flux φ are then deduced from (21) without integration of the system equations.

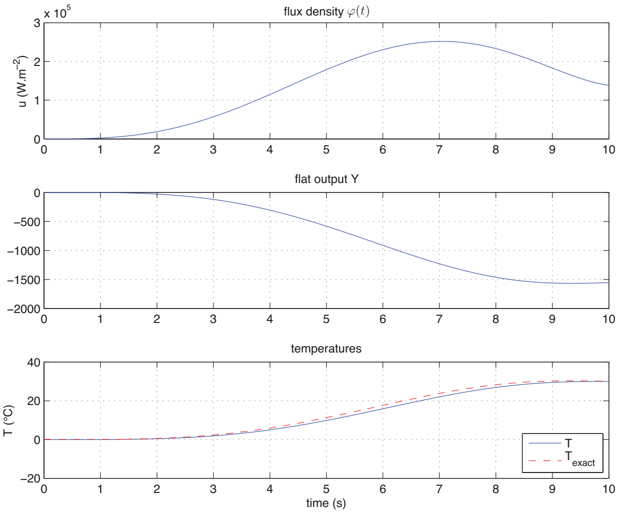

Figure 2. 2D thermal system simulation with $t_f = 10s$: flux density (control) $\varphi(t)$, flat output Y, desired temperature T and temperature obtained from (16) by inverse Laplace transform T_{exact}.

4.5 Simulations

In the next simulations, we have chosen $K = 3$, and thus $\lceil \frac{K}{2} + 2\rceil = 4$, and $r = 6$, with $a'_0 = 328.89$, $a'_1 = -86.58$, $a'_2 = 11.40$, $a'_3 = -1$ (recall that $a_k = (-1)^k \frac{(x+y)^k}{k!(8\alpha)^{k\gamma}}$ and $a'_k = \frac{a_k}{|a_3|} = (-1)^k \left(\frac{3!}{k!}\right)\left(\frac{2\sqrt{2\alpha}}{(x+y)}\right)^{(3-k)}$, $k = 0, \ldots, 3$).

Simulations have been carried out with $\alpha = 8.83 \times 10^{-5} m^2.s^{-1}$, $\lambda = 210 W.m^{-1}.K^{-1}$ and for the following durations: $t_f = 10s$, and $t_f = 100s$. For each duration, the coefficients η_j, $j = 4, 5, 6$, solutions of System (23) are:

- for $t_f = 10s$ (Fig. 2), $\eta_4 = -2.62 \times 10^4$, $\eta_5 = 4.28 \times 10^4$, $\eta_6 = -1.82 \times 10^4$,
- for $t_f = 100s$ (Fig. 3), $\eta_4 = -2.30 \times 10^4$, $\eta_5 = 3.70 \times 10^4$, $\eta_6 = -1.55 \times 10^4$.

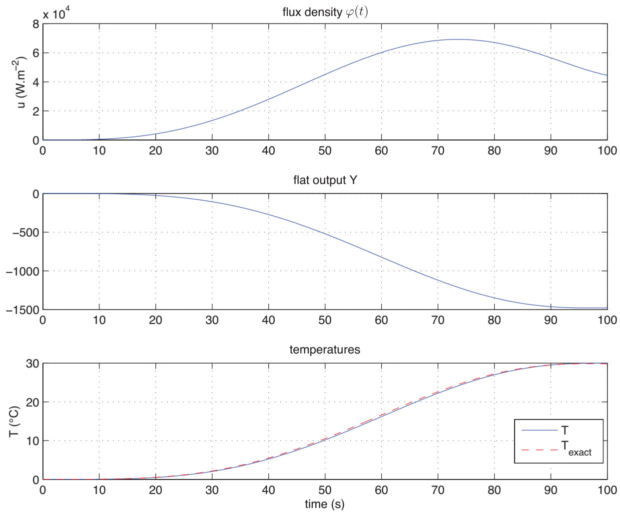

Figure 3. 2D thermal system simulation $t_f = 100s$: flux density (control) $\varphi(t)$, flat output Y, desired temperature T and temperature obtained from (16) by inverse Laplace transform T_{exact}.

Moreover, for comparison, the exact solution of the heat equation, computed by applying the inverse Laplace transform to (16), and denoted by T_{exact} in the figures, is plotted in dashed lines. Note that the error between the computed reference trajectory T and T_{exact} remains small during the transient and converges to 0, which confirms the validity of our fractional approximation.

It can be also noticed that the smaller the duration t_f, the greater the flux density, which is indeed natural.

Furthermore, the fractional flat output is obtained as a simple polynomial, i.e. with finite degree, thus avoiding the use of Gevrey functions as in Laroche et al. [1998], Sedoglavic [2001].

5. CONCLUSION

We have presented an extension of the notion of differential flatness to fractional linear systems, leading to a simple and effective algorithm to compute flat outputs. These results have been applied to a fractional approximation of order $\frac{1}{2}$ of the heat equation, corresponding to a model of a heated 2D metallic sheet. Simulations show that the obtained trajectories almost coincide with the solution of the heat equation computed by inverse Laplace transform. For future works, feedback controllers such as CRONE type controllers (Oustaloup [1995]) will be studied in order, in particular, to improve the robustness of the trajectory tracking versus various perturbations and errors such as modeling approximation errors.

REFERENCES

F. Antritter and J. Middeke. A toolbox for the analysis of linear systems with delays. In *CDC/ECC 2011*, pages 1950–1955, Orlando, USA, 2011.

P. M. Cohn. *Free rings and Their relations*. London, Academic Press, 1985.

R. Darling and J. Newman. On the short behavior of porous intercalation electrodes. *Journal of the Electrochemical Society*, 144(9):3057–3063, 1997.

M. Fliess. Some basic structural properties of generalized linear systems. *Systems Control Letters*, 15:391–396, 1990.

M. Fliess and R. Hotzel. Sur les systèmes linéaires à dérivation non entière. *Comptes Rendus de l'Académie des Sciences - Series IIB - Mechanics-Physics-Chemistry-Astronomy*, 324(2):99 – 105, 1997. ISSN 1251-8069. doi: DOI: 10.1016/S1251-8069(99)80013-X.

M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Sur les systèmes non linéaires différentiellement plats. I-315: 619–624, 1992.

M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. Flatness and defect of nonlinear systems: introductory theory and examples. *International Journal of Control*, 61:1327–1361, 1995.

M. Fliess, J. Lévine, Ph. Martin, and P. Rouchon. A Lie-Bäcklund approach to equivalence and flatness of nonlinear systems. *IEEE Transaction of Automatic Control*, 44:922–937, 1999.

F.R. Gantmacher. *Theory of Matrices*. Chelsea, New York, 1960.

R. Hotzel. *Contribution à la théorie structurelle et la commande des systèmes linéaires fractionnaires*. PhD thesis, Université Paris-sud, Paris, France, 1998.

B. Laroche, Ph. Martin, and P. Rouchon. Motion planning for a class of partial differential equations with boundary control. *IEEE-CDC*, pages 3494–3497, Tampa 1998.

J. Lévine. *Analysis and Control of Nonlinear Systems A Flatness-based Approach*. Springer, 2009.

R. L. Magin. Fractional calculus models of complex dynamics in biological tissues. *Comput. Math. Appl.*, 59:1586–1593, March 2010. ISSN 0898-1221.

J. Mikusiński. *Operational Calculus*. Pergamon, Oxford & PWN, Varsovie, 1983.

K.B. Oldham and J. Spanier. *The fractional calculus - Theory and Applications of Differentiation and Integration to Arbitrary Order*. Academic Press, New-York and London, 1974.

A. Oustaloup. *La dérivation non-entière*. Hermès, Paris, 1995.

J.W. Polderman and J.C. Willems. *Introduction to Mathematical System Theory: a behavioral approach*. Springer-Verlag, Berlin, 1998.

J. Rudolph. *Flatness Based Control of Distributed Parameter Systems*. Shaker Verlag, Aachen, Germany, 2003.

S.G. Samko, A.A. Kilbas, and O.I. Marichev. *Fractional integrals and derivatives: theory and applications*. Gordon and Breach Science, 1993.

A. Sedoglavic. *Méthodes seminumériques en algèbre différentielle - applications à l'étude des propriétés structurelles de systèmes différentiels algébriques en automatique*. PhD thesis, École polytechnique, 2001.

H. Sira-Ramirez and S.K. Agrawal. *Differential flat systems*. Marcel Dekker, New York, 2004.

S. Victor, P. Melchior, J. Lévine, and A. Oustaloup. Flatness for fractional systems : a bidimensional thermal application. *arXiv:1311.3805 [physics.comp-ph]*, 2013. URL `www.arxiv.org`.