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Active Disturbance Rejection Control for a 2×2 Hyperbolic System with an Input Disturbance

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Abstract: In this paper, active disturbance rejection control (ADRC) approach is used to stabilize a 2×2 system of first-order linear hyperbolic partial differential equations (PDEs) subject to a boundary input disturbance. Disturbance attenuation is achieved with the designed controller, and the resulting closed-loop control system admits a unique solution, which could tend to any arbitrary vicinity of zero.

Keywords: Linear 2×2 hyperbolic systems; Active disturbance rejection control; Boundary control.

1. INTRODUCTION

This paper considers a 2×2 system of first-order linear hyperbolic PDEs with a boundary input disturbance. The control objective is to stabilize the system while attenuating the disturbance, and the control method is active disturbance rejection boundary control.

2×2 hyperbolic systems have wide physical backgrounds, such as oil wells (Landet et al. (2013)), gas flow (Gugat and Dick (2011)), transmission lines (Curró et al. (2011)), road traffic (Goatin (2006)), open channels (Gugat and Leugering (2003)), and so on. And stabilization of these systems has attracted many researchers, see, e.g., Aamo (2013), Bastin and Coron (2010), Vazquez et al. (2011).

Different methods have been employed to deal with particular types of boundary input disturbances, see, e. g., Guo and Guo (2013a), Guo and Guo (2013b). Disturbance rejection/attenuation is usually desirable in system control designs. Application of the ADRC method, firstly proposed by Han in 1990s (see, Han (2009)), has been studied for decades. Recently, this approach has been generalized to distributed parameter systems. For example, it is used to attenuate the more general boundary input disturbance in wave equation (Guo and Jin (2013b)), Euler-Bernoulli beam equation (Guo and Jin (2013a)) and Schr*ö*dinger equation (Guo and Liu (2013)).

This paper is organized as follows. In Section 2, the system to be considered is introduced. In Section 3, previous results from the sliding mode control (SMC) design in Tang and Krstic (2014) are presented. Inspired by the SMC, an active disturbance rejection boundary controller is designed in Section 4. Existence and uniqueness of solutions to the resulting closed-loop system are also proved. Moreover, the solution could tend to any arbitrary vicinity of zero. Effectiveness and performances of both SMC and ADRC are shown from the simulation results in Section 5.

2. PROBLEM STATEMENT

In this paper, we intend to stabilize the following 2×2 system of coupled hyperbolic PDEs:

$$\begin{aligned} u_t(x,t) =& -\varepsilon_1(x)u_x(x,t) + c_1(x)v(x,t) && (1)\\ v_t(x,t) =& \varepsilon_2(x)v_x(x,t) + c_2(x)u(x,t) && (2)\\ u(0,t) =& qv(0,t) && (3)\\ v(1,t) =& U(t) + d(t), && (4)\end{aligned}$$

where $u(x,t), v(x,t)$ are system states with $x \in [0,1]$, $t > 0$; $U(t)$ is control input; $d(t)$ is external disturbance at the control end.

Here are some assumptions:
1. $\varepsilon_1(x), \varepsilon_2(x) \in C^1[0,1]$, $\varepsilon_1(x), \varepsilon_2(x) > 0$,
2. $c_1(x), c_2(x) \in C[0,1]$,
3. $q \neq 0$,
4. $d(t)$ and $\dot{d}(t)$ are uniformly bounded measurable,
5. Initial data $u_0(x), v_0(x) \in L^2[0,1]$.

Following Vazquez et al. (2011), we introduce a backstepping transformation

$$\begin{aligned} \alpha(x,t) =& u(x,t) - \int_0^x K^{uu}(x,\xi)u(\xi,t)d\xi \\ & - \int_0^x K^{uv}(x,\xi)v(\xi,t)d\xi && (5)\\ \beta(x,t) =& v(x,t) - \int_0^x K^{vu}(x,\xi)u(\xi,t)d\xi \\ & - \int_0^x K^{vv}(x,\xi)v(\xi,t)d\xi, && (6)\end{aligned}$$

in which the continuous kernel functions are uniquely determined by the following system of coupled PDEs:

$$\varepsilon_1(x)K^{uu}_x + \varepsilon_1(\xi)K^{uu}_\xi = -\varepsilon_1'(\xi)K^{uu} - c_2(\xi)K^{uv} \quad (7)$$
$$\varepsilon_1(x)K^{uv}_x - \varepsilon_2(\xi)K^{uv}_\xi = \varepsilon_2'(\xi)K^{uv} - c_1(\xi)K^{uu} \quad (8)$$
$$\varepsilon_2(x)K^{vu}_x - \varepsilon_1(\xi)K^{vu}_\xi = \varepsilon_1'(\xi)K^{vu} + c_2(\xi)K^{vv} \quad (9)$$
$$\varepsilon_2(x)K^{vv}_x + \varepsilon_2(\xi)K^{vv}_\xi = -\varepsilon_2'(\xi)K^{vv} + c_1(\xi)K^{vu} \quad (10)$$

with boundary conditions

$$K^{uu}(x,0) = \frac{\varepsilon_2(0)}{q\varepsilon_1(0)}K^{uv}(x,0),\ K^{uv}(x,x) = \frac{c_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)}, \quad (11)$$
$$K^{vu}(x,x) = -\frac{c_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)},\ K^{vv}(x,0) = \frac{q\varepsilon_1(0)}{\varepsilon_2(0)}K^{vu}(x,0). \quad (12)$$

The transformation $(5)-(6)$ is invertible and the inverse is:

$$u(x,t) = \alpha(x,t) + \int_0^x L^{\alpha\alpha}(x,\xi)\alpha(\xi,t)d\xi + \int_0^x L^{\alpha\beta}(x,\xi)\beta(\xi,t)d\xi \quad (13)$$
$$v(x,t) = \beta(x,t) + \int_0^x L^{\beta\alpha}(x,\xi)\alpha(\xi,t)d\xi + \int_0^x L^{\beta\beta}(x,\xi)\beta(\xi,t)d\xi, \quad (14)$$

where the continuous kernel functions are uniquely determined by the following system of coupled PDEs:

$$\varepsilon_1(x)L^{\alpha\alpha}_x + \varepsilon_1(\xi)L^{\alpha\alpha}_\xi = -\varepsilon_1'(\xi)L^{\alpha\alpha} + c_1(x)L^{\beta\alpha} \quad (15)$$
$$\varepsilon_1(x)L^{\alpha\beta}_x - \varepsilon_2(\xi)L^{\alpha\beta}_\xi = \varepsilon_2'(\xi)L^{\alpha\beta} + c_1(x)L^{\beta\beta} \quad (16)$$
$$\varepsilon_2(x)L^{\beta\alpha}_x - \varepsilon_1(\xi)L^{\beta\alpha}_\xi = \varepsilon_1'(\xi)L^{\beta\alpha} - c_2(x)L^{\alpha\alpha} \quad (17)$$
$$\varepsilon_2(x)L^{\beta\beta}_x + \varepsilon_2(\xi)L^{\beta\beta}_\xi = -\varepsilon_2'(\xi)L^{\beta\beta} - c_2(x)L^{\alpha\beta} \quad (18)$$

with boundary conditions

$$L^{\alpha\alpha}(x,0) = \frac{\varepsilon_2(0)}{q\varepsilon_1(0)}L^{\alpha\beta}(x,0),\ L^{\alpha\beta}(x,x) = \frac{c_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)}, \quad (19)$$
$$L^{\beta\alpha}(x,x) = -\frac{c_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)},\ L^{\beta\beta}(x,0) = \frac{q\varepsilon_1(0)}{\varepsilon_2(0)}L^{\beta\alpha}(x,0). \quad (20)$$

The transformation $(5)-(6)$ brings the system $(1)-(4)$ into the following system :

$$\alpha_t(x,t) = -\varepsilon_1(x)\alpha_x(x,t) \quad (21)$$
$$\beta_t(x,t) = \varepsilon_2(x)\beta_x(x,t) \quad (22)$$
$$\alpha(0,t) = q\beta(0,t) \quad (23)$$
$$\begin{aligned}\beta(1,t) =& U(t) + d(t)\\ &- \int_0^1 \alpha(\xi,t)\left(K^{vu}(1,\xi) + \int_\xi^1 K^{vu}(1,\eta)L^{\alpha\alpha}(\eta,\xi)d\eta\right.\\ &\left.+ \int_\xi^1 K^{vv}(1,\eta)L^{\beta\alpha}(\eta,\xi)d\eta\right)d\xi\\ &- \int_0^1 \beta(\xi,t)\left(K^{vv}(1,\xi) + \int_\xi^1 K^{vu}(1,\eta)L^{\alpha\beta}(\eta,\xi)d\eta\right.\\ &\left.+ \int_\xi^1 K^{vv}(1,\eta)L^{\beta\beta}(\eta,\xi)d\eta\right)d\xi. \end{aligned} \quad (24)$$

3. RESULTS FROM SLIDING MODE CONTROL DESIGN

Consider the systems $(1)-(4)$ and $(21)-(24)$ in the state Hilbert space $\mathbf{H} = \left(L^2(0,1)\right)^2$ with an induced norm from the following inner product

$$\begin{aligned}&<(f_1,\ g_1)^T,(f_2,\ g_2)^T>\\ &= \int_0^1 \left(\frac{2-x}{\varepsilon_1(x)}f_1(x)\overline{f_2(x)} + \frac{2q^2(1+x)}{\varepsilon_2(x)}g_1(x)\overline{g_2(x)}\right)dx,\\ &\forall\ (f_1,\ g_1)^T,(f_2,\ g_2)^T \in \mathbf{H}.\end{aligned} \quad (25)$$

In Tang and Krstic (2014), firstly, a sliding surface for the system $(21)-(24)$ is chosen as follows:

$$S_{(\alpha,\beta)^T}(t) = \beta(1,t) = 0, \quad (26)$$

i.e.,

$$S_{(\alpha,\beta)^T} = \{(f,g)^T \in \mathbf{H} \mid g(1) = 0\}. \quad (27)$$

For the system $(1)-(4)$, the sliding surface is

$$S_{(u,v)^T} = \left\{(f,g)^T \in \mathbf{H} \mid g(1) - \int_0^1 K^{vu}(1,\xi)f(\xi)d\xi - \int_0^1 K^{vv}(1,\xi)g(\xi)d\xi = 0\right\}. \quad (28)$$

Secondly, a sliding mode boundary control is designed as

$$U(t) = \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi - K\int_0^t \frac{S_{(u,v)^T}(\tau)}{|S_{(u,v)^T}(\tau)|}d\tau \ \text{ for } S_{(u,v)^T}(t) \neq 0. \quad (29)$$

The following main result is then proved in Tang and Krstic (2014).

Theorem 1. Suppose that d and $\dot{d}$ are bounded measurable in time, then for any initial data $(u(\cdot,0),v(\cdot,0))^T \in \mathbf{H}$, there exists $T_{\max} \geq 0$, depending on initial data, such that the system $(1)-(4)$ with controller (29) admits a unique solution

$$(u(\cdot,t),v(\cdot,t))^T \in C([0,T_{\max}];\mathbf{H}) \quad (30)$$

and

$$S_{(u,v)^T}(t) = v(1,t) - \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi - \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi = 0 \quad (31)$$

for all $t \geq T_{\max}$. Moreover, $S_{(u,v)^T}(t)$ is continuous and monotone in $[0,T_{\max}]$. On the sliding mode surface $S_{(u,v)^T}(t) = 0$, the system $(1)-(4)$ becomes exponentially stable.

4. ACTIVE DISTURBANCE REJECTION CONTROL

Inspired by the controller (29) from SMC design, we implement a to-be-designed controller $U_0(t)$ in $U(t)$:

$$U(t) = \int_0^1 K^{vu}(1,\xi)u(\xi,t)d\xi + \int_0^1 K^{vv}(1,\xi)v(\xi,t)d\xi + U_0(t), \quad (32)$$

that is,

$$\begin{aligned}U(t) = \int_0^1 &\alpha(\xi,t)\left(K^{vu}(1,\xi) + \int_\xi^1 K^{vu}(1,\eta)L^{\alpha\alpha}(\eta,\xi)d\eta\right.\\ &\left.+ \int_\xi^1 K^{vv}(1,\eta)L^{\beta\alpha}(\eta,\xi)d\eta\right)d\xi\\ &+ \int_0^1 \beta(\xi,t)\left(K^{vv}(1,\xi) + \int_\xi^1 K^{vu}(1,\eta)L^{\alpha\beta}(\eta,\xi)d\eta\right.\\ &\left.+ \int_\xi^1 K^{vv}(1,\eta)L^{\beta\beta}(\eta,\xi)d\eta\right)d\xi + U_0(t).\end{aligned} \quad (33)$$

Then the system (21) – (24) becomes

$$\alpha_t(x,t) = -\varepsilon_1(x)\alpha_x(x,t) \tag{34}$$
$$\beta_t(x,t) = \varepsilon_2(x)\beta_x(x,t) \tag{35}$$
$$\alpha(0,t) = q\beta(0,t) \tag{36}$$
$$\beta(1,t) = U_0(t) + d(t), \tag{37}$$

which can be written into the following operator form:

$$\frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \mathscr{A}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \mathscr{B}(U_0(t)+d(t)). \tag{38}$$

Here the operator $\mathscr{A} : D(\mathscr{A})(\subset \mathbf{H}) \to \mathbf{H}$ is defined as follows:

$$\mathscr{A}(f,g)^T = \left(-\varepsilon_1(x)f',\ \varepsilon_2(x)g'\right)^T,\ \forall (f,g)^T \in D(\mathscr{A}), \tag{39}$$

$$D(\mathscr{A}) = \{(f,g)^T \in \left(H^1(0,1)\right)^2 \mid f(0) = qg(0), g(1) = 0\}, \tag{40}$$

and

$$\mathscr{B} = \begin{pmatrix}0\\ 4q^2\delta(x-1)\end{pmatrix}, \tag{41}$$

where $\delta(\cdot)$ denotes the Dirac distribution. In Tang and Krstic (2014), it has been proved that $\mathscr{A}$ generates a C_0-semigroup $e^{\mathscr{A}t}$ of contractions in $\mathbf{H}$ and $\mathscr{B}$ is admissible for $e^{\mathscr{A}t}$.

The adjoint operator of $\mathscr{A}$ is

$$\mathscr{A}^*(\phi,\psi)^T = \left(\varepsilon_1(x)\left(\phi' + \frac{\phi}{x-2}\right), -\varepsilon_2(x)\left(\psi' + \frac{\psi}{x+1}\right)\right)^T, \quad \forall (\phi,\psi)^T \in D(\mathscr{A}^*), \tag{42}$$

$$D(\mathscr{A}^*) = \{(\phi,\psi)^T \in \left(H^1(0,1)\right)^2 \mid \phi(0) = q\psi(0), \phi(1) = 0\}. \tag{43}$$

Choose

$$\begin{pmatrix}\phi\\ \psi\end{pmatrix} = \begin{pmatrix}0\\ x\end{pmatrix} \in D(\mathscr{A}^*) \tag{44}$$

and let

$$y_1(t) = \left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix},\ \begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle = \int_0^1 \frac{2q^2(1+x)}{\varepsilon_2(x)}\beta(x,t)x\,dx, \tag{45}$$

$$y_2(t) = \left\langle \begin{pmatrix}\alpha\\ \beta\end{pmatrix},\ \mathscr{A}^*\begin{pmatrix}\phi\\ \psi\end{pmatrix}\right\rangle = -2q^2\int_0^1 \beta(x,t)(1+2x)\,dx, \tag{46}$$

then we can get

$$\dot{y}_1(t) = 4q^2(U_0(t)+d(t)) + y_2(t). \tag{47}$$

Design the following extended state observer for $y_1(t)$ and $d(t)$:

$$\begin{aligned}\dot{\hat{y}}_\varepsilon(t) &= 4q^2(U_0(t)+\hat{d}_\varepsilon(t)) + y_2(t) + \frac{1}{\varepsilon}(y_1(t)-\hat{y}_\varepsilon(t))\\ \dot{\hat{d}}_\varepsilon(t) &= \frac{1}{4q^2\varepsilon^2}(y_1(t)-\hat{y}_\varepsilon(t)),\end{aligned} \tag{48}$$

where $\varepsilon > 0$ is the small tuning parameter, then the errors

$$\tilde{y}_\varepsilon = y_1 - \hat{y}_\varepsilon,\ \tilde{d}_\varepsilon = d - \hat{d}_\varepsilon \tag{49}$$

satisfy

$$\frac{d}{dt}\begin{pmatrix}\tilde{y}_\varepsilon(t)\\ \tilde{d}_\varepsilon(t)\end{pmatrix} = A\begin{pmatrix}\tilde{y}_\varepsilon(t)\\ \tilde{d}_\varepsilon(t)\end{pmatrix} + B\dot{d}(t), \tag{50}$$

where

$$A = \begin{pmatrix}-\frac{1}{\varepsilon} & 4q^2\\ -\frac{1}{4q^2\varepsilon^2} & 0\end{pmatrix},\ B = \begin{pmatrix}0\\ 1\end{pmatrix}. \tag{51}$$

The eigenvalues of A are

$$\lambda = -\frac{1}{2\varepsilon} \pm \frac{\sqrt{3}}{2\varepsilon}j. \tag{52}$$

The following state feedback controller to (34) – (37) is designed:

$$U_0(t) = -\hat{d}_\varepsilon(t), \tag{53}$$

and then the closed-loop system becomes

$$\alpha_t(x,t) = -\varepsilon_1(x)\alpha_x(x,t) \tag{54}$$
$$\beta_t(x,t) = \varepsilon_2(x)\beta_x(x,t) \tag{55}$$
$$\alpha(0,t) = q\beta(0,t) \tag{56}$$
$$\beta(1,t) = -\hat{d}_\varepsilon(t) + d(t) \tag{57}$$
$$\dot{\hat{y}}_\varepsilon(t) = y_2(t) + \frac{1}{\varepsilon}(y_1(t)-\hat{y}_\varepsilon(t)) \tag{58}$$
$$\dot{\hat{d}}_\varepsilon(t) = \frac{1}{4q^2\varepsilon^2}(y_1(t)-\hat{y}_\varepsilon(t)). \tag{59}$$

Lemma 2. Suppose that d and $\dot{d}$ are uniformly bounded measurable, then for any initial data $(\alpha(\cdot,0),\beta(\cdot,0))^T \in \mathbf{H}$, there exists a unique solution

$$(\alpha(\cdot,t),\beta(\cdot,t))^T \in C([0,\infty);\mathbf{H}) \tag{60}$$

to the closed-loop system (54) – (59). Moreover, the solution tends to any arbitrary vicinity of zero as $t \to \infty, \varepsilon \to 0$.

Proof. The system (54) – (59) is equivalent to the following system:

$$\alpha_t(x,t) = -\varepsilon_1(x)\alpha_x(x,t) \tag{61}$$
$$\beta_t(x,t) = \varepsilon_2(x)\beta_x(x,t) \tag{62}$$
$$\alpha(0,t) = q\beta(0,t) \tag{63}$$
$$\beta(1,t) = \tilde{d}_\varepsilon(t) \tag{64}$$
$$\dot{\tilde{y}}_\varepsilon(t) = -\frac{1}{\varepsilon}\tilde{y}_\varepsilon(t) + 4q^2\tilde{d}_\varepsilon(t) \tag{65}$$
$$\dot{\tilde{d}}_\varepsilon(t) = -\frac{1}{4q^2\varepsilon^2}\tilde{y}_\varepsilon(t) + \dot{d}(t). \tag{66}$$

Firstly, the $\left(\tilde{y}_\varepsilon(t),\tilde{d}_\varepsilon(t)\right)^T$-subsystem (65) – (66) can be solved separately:

$$\begin{pmatrix}\tilde{y}_\varepsilon(t)\\ \tilde{d}_\varepsilon(t)\end{pmatrix} = e^{At}\begin{pmatrix}\tilde{y}_\varepsilon(0)\\ \tilde{d}_\varepsilon(0)\end{pmatrix} + \int_0^t e^{A(t-\tau)}B\dot{d}(\tau)\,d\tau. \tag{67}$$

From (52), it can be derived that

$$\left(\tilde{y}_\varepsilon(t),\tilde{d}_\varepsilon(t)\right) \to 0 \quad \text{as} \quad t\to\infty, \varepsilon \to 0. \tag{68}$$

Secondly, the $(\alpha,\beta)^T$-subsystem (61) – (64) can be written as

$$\frac{d}{dt}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \mathscr{A}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} + \mathscr{B}\tilde{d}_\varepsilon(t). \tag{69}$$

Since from Tang and Krstic (2014), it can be proved that for the system

$$\frac{d}{dt}\begin{pmatrix}\gamma\\ \delta\end{pmatrix} = \mathscr{A}\begin{pmatrix}\gamma\\ \delta\end{pmatrix},\ \begin{pmatrix}\gamma(\cdot,0)\\ \delta(\cdot,0)\end{pmatrix} \in \mathbf{H}, \tag{70}$$

there exists a unique solution

$$(\gamma(\cdot,t),\delta(\cdot,t))^T \in C([0,\infty);\mathbf{H}), \tag{71}$$

and

$$\left\|(\gamma(\cdot,t),\delta(\cdot,t))^T\right\|_{\mathbf{H}} \leqslant e^{-a/2t}\left\|(\gamma(\cdot,0),\delta(\cdot,0))^T)\right\|_{\mathbf{H}}, \tag{72}$$

where

$$a = \frac{1}{2}\min_{x\in[0,1]}\{\varepsilon_1(x),\varepsilon_2(x)\} > 0, \tag{73}$$

then it can be derived that the C_0-semigroup $e^{\mathscr{A}t}$ generated by $\mathscr{A}$ is exponentially stable. Moreover, since $\mathscr{B}$ is admissible for $e^{\mathscr{A}t}$, there exists a unique solution to the system (69):

$$\begin{pmatrix} \alpha(\cdot,t) \\ \beta(\cdot,t) \end{pmatrix} = e^{\mathscr{A}t} \begin{pmatrix} \alpha(\cdot,0) \\ \beta(\cdot,0) \end{pmatrix} + \int_0^t e^{\mathscr{A}(t-\tau)} \mathscr{B} \tilde{d}_\varepsilon(\tau) d\tau. \quad (74)$$

From (68), for any given $\varepsilon_0 > 0$, there exist $t_0 > 0$ and $\varepsilon_1 > 0$ such that $|\tilde{d}_\varepsilon(t)| < \varepsilon_0$ for all $t > t_0$, $0 < \varepsilon < \varepsilon_1$. Thus,

$$\begin{aligned} \begin{pmatrix} \alpha(\cdot,t) \\ \beta(\cdot,t) \end{pmatrix} =& e^{\mathscr{A}t} \begin{pmatrix} \alpha(\cdot,0) \\ \beta(\cdot,0) \end{pmatrix} + \int_{t_0}^t e^{\mathscr{A}(t-\tau)} \mathscr{B} \tilde{d}_\varepsilon(\tau) d\tau \\ &+ e^{\mathscr{A}(t-t_0)} \int_0^{t_0} e^{\mathscr{A}(t_0-\tau)} \mathscr{B} \tilde{d}_\varepsilon(\tau) d\tau. \end{aligned} \quad (75)$$

With admissibility of $\mathscr{B}$, it can be derived that

$$\begin{aligned} &\left\| \int_0^{t_0} e^{\mathscr{A}(t_0-\tau)} \mathscr{B} \tilde{d}_\varepsilon(\tau) d\tau \right\|_{\mathbf{H}}^2 \\ &\leqslant C_{t_0} \left\| \tilde{d}_\varepsilon \right\|_{L^2_{loc}(0,t_0)}^2 \leqslant C_{t_0} t_0^2 \left\| \tilde{d}_\varepsilon \right\|_{L^\infty(0,t_0)}^2, \forall \tilde{d}_\varepsilon \in L^\infty(0,\infty), \end{aligned} \quad (76)$$

where C_{t_0} is a constant that is independent of $\tilde{d}_\varepsilon$. With exponential stability of $e^{\mathscr{A}t}$, it can be derived that

$$\begin{aligned} \left\| \int_{t_0}^t e^{\mathscr{A}(t-\tau)} \mathscr{B} \tilde{d}_\varepsilon(\tau) d\tau \right\|_{\mathbf{H}} =& \left\| \int_0^t e^{\mathscr{A}(t-\tau)} \mathscr{B} (0 \underset{t_0}{\diamond} \tilde{d}_\varepsilon)(\tau) d\tau \right\|_{\mathbf{H}} \\ \leqslant& N \left\| \tilde{d}_\varepsilon \right\|_{L^\infty(0,\infty)} \leqslant N\varepsilon_0, \end{aligned} \quad (77)$$

where N is a constant independent of $\tilde{d}_\varepsilon$, and $d_1 \underset{s}{\diamond} d_2$ denotes the s-concatenation of d_1 and d_2. Since $\|e^{\mathscr{A}t}\| \leqslant e^{-a/2t}$, then from (75), (76) and (77),

$$\begin{aligned} \left\| \begin{pmatrix} \alpha(\cdot,t) \\ \beta(\cdot,t) \end{pmatrix} \right\|_{\mathbf{H}} \leqslant& e^{-a/2t} \left\| \begin{pmatrix} \alpha(\cdot,0) \\ \beta(\cdot,0) \end{pmatrix} \right\|_{\mathbf{H}} \\ &+ C_{t_0} e^{-a/2(t-t_0)} \left\| \tilde{d}_\varepsilon \right\|_{L^\infty(0,t_0)} + N\varepsilon_0. \end{aligned} \quad (78)$$

With arbitrariness of ε_0, the proof is completed.

By equivalence between the transformations $(5)-(6)$ and $(13)-(14)$, we summarize our closed-loop construction in the following main theorem.

Theorem 3. Suppose that d and $\dot{d}$ are uniformly bounded measurable, then for any initial data $(u(\cdot,0), v(\cdot,0))^T \in \mathbf{H}$, there exists a unique solution

$$(u(\cdot,t), v(\cdot,t))^T \in C([0,\infty); \mathbf{H}) \quad (79)$$

to the following closed-loop system:

$$u_t(x,t) = -\varepsilon_1(x) u_x(x,t) + c_1(x) v(x,t) \quad (80)$$
$$v_t(x,t) = \varepsilon_2(x) v_x(x,t) + c_2(x) u(x,t) \quad (81)$$
$$u(0,t) = qv(0,t) \quad (82)$$
$$v(1,t) = U(t) + d(t) \quad (83)$$
$$\dot{\hat{y}}_\varepsilon(t) = y_2(t) + \frac{1}{\varepsilon}(y_1(t) - \hat{y}_\varepsilon(t)) \quad (84)$$
$$\dot{\hat{d}}_\varepsilon(t) = \frac{1}{4q^2\varepsilon^2}(y_1(t) - \hat{y}_\varepsilon(t)), \quad (85)$$

where the control is

$$U(t) = \int_0^1 K^{vu}(1,\xi) u(\xi,t) d\xi + \int_0^1 K^{vv}(1,\xi) v(\xi,t) d\xi - \hat{d}_\varepsilon(t), \quad (86)$$

and

$$\begin{aligned} y_1(t) =& \int_0^1 \frac{2q^2(1+x)}{\varepsilon_2(x)} x \left(v(x,t) - \int_0^x K^{vu}(x,\xi) u(\xi,t) d\xi \right. \\ &\left. - \int_0^x K^{vv}(x,\xi) v(\xi,t) d\xi \right) dx, \end{aligned} \quad (87)$$

$$\begin{aligned} y_2(t) =& -2q^2 \int_0^1 (1+2x) \left(v(x,t) - \int_0^x K^{vu}(x,\xi) u(\xi,t) d\xi \right. \\ &\left. - \int_0^x K^{vv}(x,\xi) v(\xi,t) d\xi \right) dx. \end{aligned} \quad (88)$$

Moreover, the solution tends to any arbitrary vicinity of zero as $t \to \infty, \varepsilon \to 0$.

5. NUMERICAL SIMULATIONS

5.1 Example 1

Consider the system $(1)-(4)$ with $\varepsilon_1(x) = 0.1$, $\varepsilon_2(x) = 0.2$, $c_1(x) = 0.03$, $c_2(x) = 0.04$, $q = 1/4$ and disturbance $d(t) = 10\sin t$. Set initial data $u(x,0) = \frac{5}{2}(1-x)$, $v(x,0) = 10(1-x)$.

Take the time length, steps of time and space as 50, 0.01 and 0.01, then open-loop response and and closed-loop response with ADRC (choosing $\varepsilon = 0.001$) of the $(u,v)^T-$system are shown in Fig. 1 and Fig. 3, respectively. The control kernel functions $K^{vu}(1,\xi), K^{vv}(1,\xi)$ are depicted in Fig. 2. Also, for the closed-loop $(u,v)^T-$system with SMC (choosing $K = 20$), a figure very similar to Fig. 3 is obtained, which is not shown here due to the page limit. As can be seen from these figures, the designed ADRC, as well as SMC, has stabilized the $(u,v)^T-$system to very small vicinities of zero.

Remark 4. From the above simulation results, it's worth noting that the first order SMC designed in Tang and Krstic (2014) exhibits a non-chattering character and could achieve satisfactory results as well as ADRC. However, if we set the time step to be bigger (0.05), a figure very similar to Fig. 3 is obtained for the closed-loop $(u,v)^T-$system with ADRC, while Fig. 4 is obtained for that with SMC. Thus, considering SMC's higher requirement to integrators than ADRC, ADRC is a better choice than SMC.

5.2 Example 2

Consider the system $(1)-(4)$, where system parameters, disturbance and initial data are chosen to be the same as those in Example 1 except that $c_1(x) = 0.3$, $c_2(x) = 0.4$.

Take the time length, steps of time and space as 50, 0.01 and 0.01, then open-loop response and and closed-loop response with ADRC (choosing $\varepsilon = 0.001$) of the system are shown in Fig. 5 and Fig. 7, respectively. The control kernel functions $K^{vu}(1,\xi), K^{vv}(1,\xi)$ are depicted in Fig. 6. If choosing $K = 20$, a figure very similar as Fig. 7 is obtained for the closed-loop system with SMC. Thus, although the open-loop system blows up, ADRC, as well as SMC, has still successfully stabilized them.

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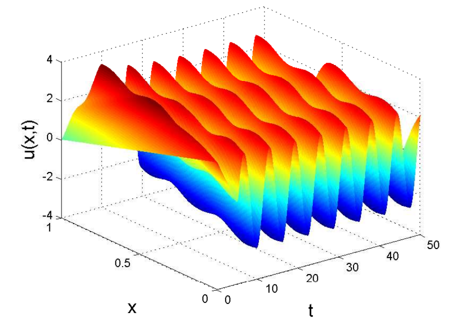

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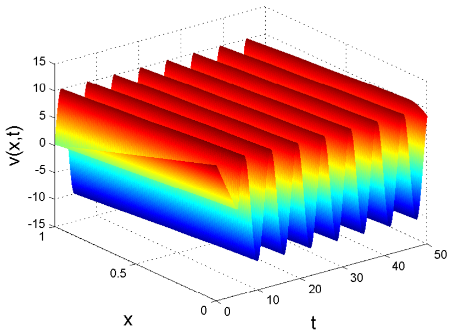

(b)

Fig. 1. Simulation results for open-loop $(u,v)^T-$systems (time step=0.01, space step=0.01)

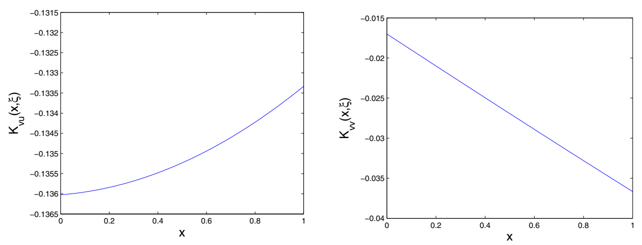

Fig. 2. Control kernel functions $K^{vu}(1,\xi)$ (left) and $K^{vv}(1,\xi)$ (right)

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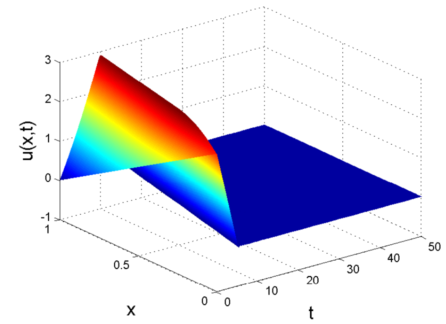

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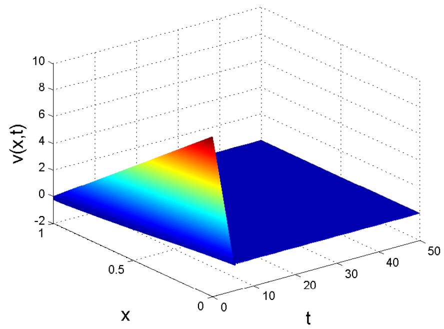

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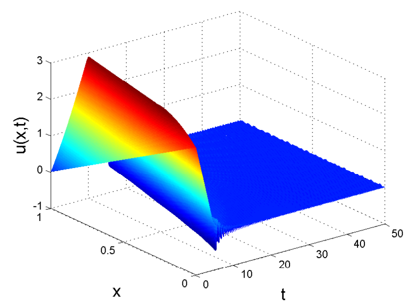
Fig. 3. Simulation results for closed-loop $(u,v)^T-$systems with ADRC (time step=0.01, space step=0.01)



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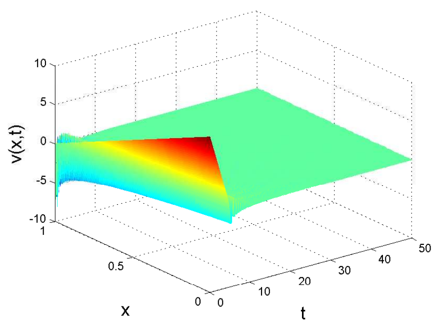
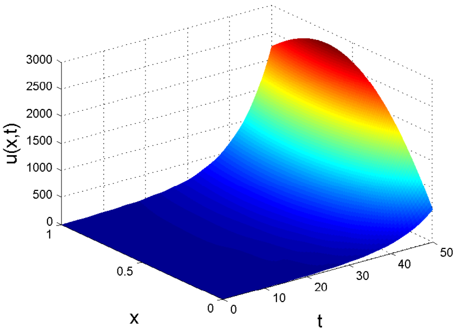


(b)

Fig. 4. Simulation results for closed-loop $(u,v)^T-$systems with SMC (time step=0.05, space step=0.01)



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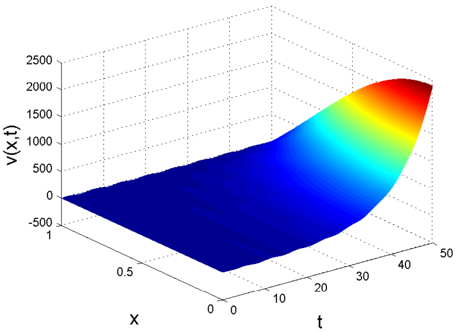

(b)

Fig. 5. Simulation results for open-loop $(u,v)^T$ −systems (time step=0.01, space step=0.01)

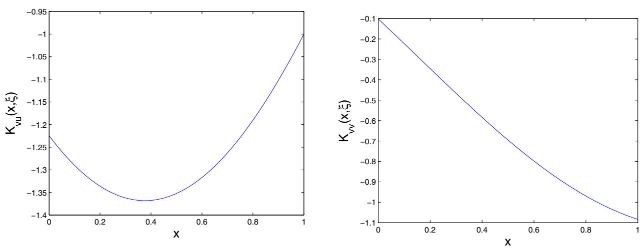

Fig. 6. Control kernel functions $K^{vu}(1,\xi)$ (left) and $K^{vv}(1,\xi)$ (right)

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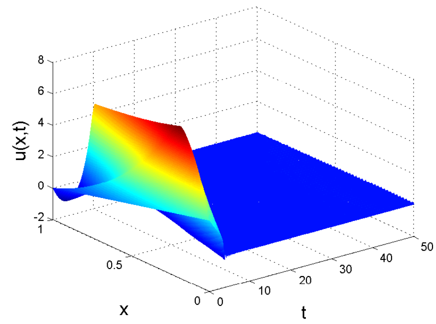

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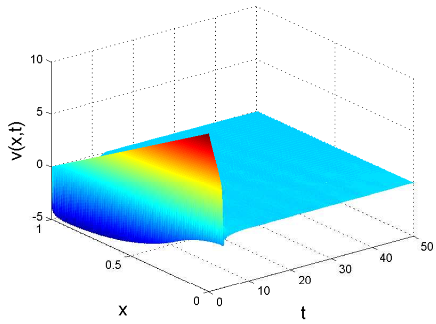

(b)

Fig. 7. Simulation results for closed-loop $(u,v)^T$ −systems with ADRC (time step=0.01, space step=0.01)

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