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Dynamic Programming Framework for Wind Power Maximization

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Abstract: The contribution of this paper is the formulation of the wind farm power maximization problem as a multi-stage dynamic programming problem. This formulation is made possible by introducing a state-space model to describe the evolution of the wind velocity profile as a function of the free-stream wind velocity and the wind turbine control variables. This state-space model is coupled with a multi-stage utility function that quantifies the power to be maximized. The benefits of this approach include: a simple algorithm to determine the turbine optimal controls, as a feedback function of the wind farm state, and a rigorous, yet simple, method to calculate the limit of performance for wind farm power maximization. A one-dimensional cascade of wind turbines is used to illustrate the approach. Also given is an analytical expression for the maximum power that can be extracted from the one-dimensional cascade, which parallels the well-known Betz limit for a single turbine.

Keywords: Wind farm control, dynamic programming, performance limits.

1. INTRODUCTION

Wind farms are arrays of wind turbines that are electrically and aerodynamically coupled. These arrays reduce costs via economies of scale. The wakes that trail behind upstream turbines can diminish the energy production of downstream turbines and increase loading from turbulence, Barthelmie and Khun (2009). U.S. wind farms operating on land have generated 10% to 15% less energy than anticipated, Barthelmie et al. (2009a,b), and have operated less reliably than expected, van Bussel (2001, 2002). Advanced control algorithms have the potential to increase annual energy production (AEP) and decrease the cost of energy.

For high wind speed (Region 3) operation, the wind farm operator distributes power demand to each turbine typically based on load reduction considerations, while for medium to low wind speed (Region 2) operation, maximizing the total power produced by a wind farm is essential for increasing the AEP. Maximizing the individual power output for each turbine in an array does not imply maximization of the entire wind farm power output, Steinbuch et al. (1988). This is primarily due to the wake interactions among the upstream and downstream turbines, as well as the impact of velocity deficit of individual turbines on the efficiency of energy capture. With the ever increasing penetration of wind energy, every percent increase in AEP will lead to more and more benefit for utility sectors. It is thus imperative to develop practical wind farm control strategies that can maximize the wind farm power output under various conditions.

Modeling and control strategies for wind farm power optimization are active areas of research. Pao and Johnson (2009) have provided a tutorial on the dynamics and control of wind turbines and wind farms. In the literature one may find both model-based and model-free strategies for the control of wind farms.

Model-based solutions are found in the works of Schepers and van der Pijl (2007), Johnson and Thomas (2009), Knudsen et al. (2009), Spudic (2010), Madjidian et al. (2011), Soleimanzadeh and Wisniewski (2011), Kristalny and Madjidian (2011), Horvat et al. (2012), Biegel et al. (2013), and Bitar and Seiler (2013). Most of this work assumes that that the available wind power is greater than the demand, and thus the control objective is to minimize the cumulative load for all turbines while satisfying the total power demand. Power maximization as the single control objective is considered in Johnson and Thomas (2009) and Bitar and Seiler (2013).

The actual wind field and turbine characteristics may deviate from the nominal models in practice. This fact may limit the effectiveness of model-based wind farm control. There has been some recent work in model-free control strategies for power optimization. Marden et al. (2013) presented a game theory based cooperative control scheme for maximizing the whole-farm power output. Park et al. (2013) proposed different game theoretic approaches, based on both non-cooperative and cooperative games. More recently, Yang et al. (2013) proposed a nested-loop extremum seeking control scheme for wind farm power optimization.

We formulate the wind farm power maximization problem as a dynamic program, Bellman (2003). This approach facilitates the reduction of a *complex* maximization problem into *a sequence of simpler* optimization problems. Specifically, for a problem with N turbines and m controls per turbine, the maximization problem is reduced from a single problem with Nm optimization variables to a sequence of N problems with m variables each.

Central to the dynamic programming (DP) framework for wind farm optimization is a control-oriented physics-based state-space model for the wind farm. In this model, the wind velocity is taken to be the *state*, and the turbines are viewed as *actuators* that shape the wind profile and extract energy from it.

To illustrate the DP framework, we derive an analytical solution of the wind farm optimization problem when the power extraction of each turbine is modeled with the so-called actuator disk model. The illustration is restricted to a one-dimensional array as shown in Fig. 2. The solution to this particular case has the following features:

- The optimal control for each turbine is a *linear* feedback of the state, and the optimal wind farm power is *cubic* in the free-stream wind speed.
- The optimal state-feedback gains and power output are computed from a recursion reminiscent of the Riccati difference equation of optimal control.
- The optimal efficiency of the wind farm increases monotonically from the Betz limit for a single turbine to an asymptotic efficiency of 66.66%.

Bitar and Seiler (2013) seem to be the first suggesting dynamic programming for wind farm power maximization. Our results have been developed independently of the work in Bitar and Seiler (2013). We stress the role played by the state (wind velocity) in the formulation of the wind farm power maximization problem as a multi-stage dynamic program. The DP algorithm is provided without assumptions on the turbine's power curves or wake models. Our results, and those in Bitar and Seiler (2013), provide theoretical justification for the nested-loop extremum seeking control scheme in Yang et al. (2013).

2. STATE SPACE MODEL

Motivation for the choice of state variables comes from the actuator disk model (ADM); Manwell et al. (2010) and Burton et al. (2008). Figure 1 depicts a schematic for this one-dimensional model for energy extraction. The ADM can be used to estimate the power P extracted by an ideal turbine rotor, the force F done by the wind on the ideal rotor, and the effect of the rotor operation on the local wind field. This latter effect is characterized by the air velocity V at the disk and the velocity V_1 in the far wake of the rotor. The ADM is given by:

$$P = FV \tag{1a}$$
$$F = \rho A(V_0 - V_1)V \tag{1b}$$
$$V = V_0 - u \tag{1c}$$
$$V_1 = V_0 - 2u \tag{1d}$$

where V_0 is the free-stream velocity upwind of the rotor, and $u \geq 0$ is the reduction in air velocity between the free stream and the rotor plane. The air density and rotor swept area are denoted by ρ and A, respectively. We may think of u as the velocity deficit induced at the disk by the rotor operation. This simple model is valid as long as $u \leq V_0/2$; otherwise $V_1 < 0$, which is a contradiction. The wind energy connoisseur may easily recognize these equations by introducing the axial induction factor a defined by

$$a = \frac{V_0 - V}{V_0}. \tag{2}$$

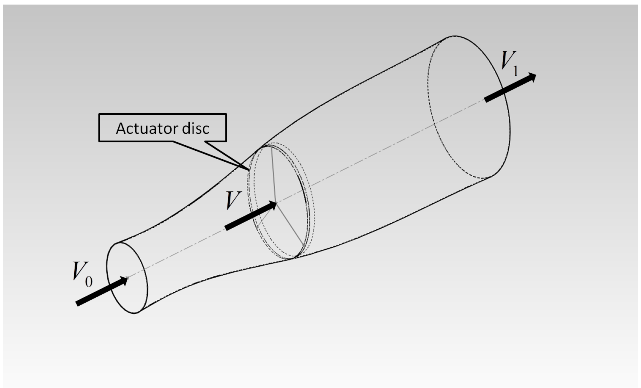

Fig. 1. Actuator disk model and stream-tube model for energy extraction.

If the free-stream velocity V_0 is given, the induced velocity deficit u is the mechanism to adjust the velocities V and V_1, the force F, and hence the power P. Thus, it is natural to think of u as the *control input or variable.* From (1c) and (2) it follows that

$$u = aV_0 \tag{3}$$

which shows that the axial induction factor is the *feedback gain* from the free-stream air velocity V_0 to the control input u.

The ADM may be written in state-space form by defining the state variable x_k as $x_0 = V_0$ for $k = 0$ and $x_1 = V_1$ for $k = 1$. The result is

$$x_1 = x_0 - 2u \tag{4a}$$
$$y = x_0 - u \tag{4b}$$

Equation (4a) is the *state equation*, which gives the evolution of the air velocity from the upstream location to the down stream location. Equation (4b) is an *output* equation to estimate the air velocity $y = V$ at the rotor plane from the initial state x_0 and the control input u. Note that this elementary state-space model is linear. The power $P = FV$, where the force F and the velocity V are defined in (1), is estimated from the control input u and the output y using the formula

$$P = 2\rho A y^2 u \tag{5}$$

The state-space model (4) is valid if the constraint $0 \leq u \leq x_0/2$ holds.

Let us consider the cascade of wind turbines shown in Fig. 2. The scalar variables x_k and x_{k+1} are the air velocity upstream and downstream of the k-th turbine, respectively. The scalar control input for turbine k is denoted by u_k, and y_k is an output used to estimate the power of turbine k such as $y = V$ for the ADM. We assume that these variables satisfy the equations

$$x_{k+1} = f(x_k, u_k) \tag{6a}$$
$$y_k = h(x_k, u_k) \tag{6b}$$

for $k = 0, 1, \ldots, N-1$, where x_0 is given. The power P_k extracted by turbine k is estimated according to

$$P_k = p(y_k, u_k). \tag{7}$$

No restrictive assumptions are made on the functions $f(\cdot)$, $h(\cdot)$, and $p(\cdot)$.

To illustrate this state-space formulation assume that, for $k = 0, \ldots, N-2$, turbine $k+1$ is in the far wake of the

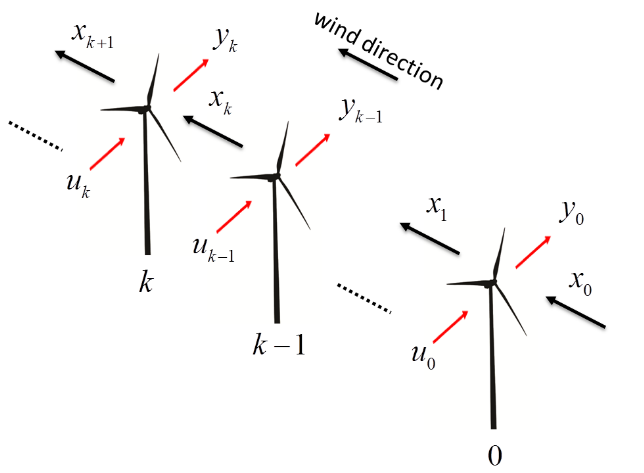

Fig. 2. Cascade of N turbines; $k = 0$ is the index of the most upstream location.

upstream turbine k, and estimate power extraction using the ADM model (1). Then, it follows that

$$f(x_k, u_k) = x_k - 2u_k \tag{8a}$$
$$h(x_k, u_k) = x_k - u_k \tag{8b}$$
$$p(y_k, u_k) = 2\rho A y_k^2 u_k. \tag{8c}$$

To simplify notation, rather than seeking to maximize actual power, we will consider the maximization of a function that does not include any of the constants in the power formula (such as $2\rho A$ in (8c)). We shall denote this *constant-free power function* by $g(x_k, u_k)$, and assume that all turbines in the array produce power according to $g(\cdot)$. For example, for the actuator disk model we define

$$g(x_k, u_k) \doteq y_k^2 u_k = (x_k - u_k)^2 u_k. \tag{9}$$

which results in the following expression for the actual power $P_k = 2\rho A g(x_k, u_k)$.

3. MULTI-STAGE OPTIMAL CONTROL PROBLEM

Consider the state-space model (6) and denote the (constant-free) power of the k-th turbine with $g(x_k, u_k)$. The goal is to determine the optimal controls $u_0^*, \ldots, u_{N-1}^*$ to maximize the cascade's aggregate power

$$G(x_0; u_0, \ldots, u_{N-1}) = \sum_{k=0}^{N-1} g(x_k, u_k). \tag{10}$$

subject to the state equation (6), and the constraints $u_k \in \mathcal{U}(x_k)$, for $k = 0, \ldots, N-1$. This problem is an optimal control problem that can be solved with the dynamic programming algorithm, Bellman (2003).

Consider the cascade in Fig. 2, and assume that the state x_k (air velocity upstream of turbine k) is known. The maximum power $G_k^*(x_k)$ attained by the $N-k$ downstream turbines $k, k+1, \ldots, N-1$ is calculated from

$$G_k^*(x_k) = \max \sum_{i=k}^{N-1} g(x_i, u_i) \tag{11}$$

where the maximization is performed over the controls $u_k \in \mathcal{U}(x_k)$, $u_{k+1} \in \mathcal{U}(x_{k+1})$, ... , $u_{N-1} \in \mathcal{U}(x_{N-1})$ subject to the state-space equation (6). The optimal value function G_k^* depends on x_k but it does not depend on the controls of the turbines upstream of turbine k. The same is true for optimal controls $u_k^*, \ldots, u_{N-1}^*$ attaining the maximum power $G_k^*(x_k)$. Note that $G_k^*(x_k)$ is the optimal power of the sub-array composed by turbines $k, k+1, \ldots N-1$, for a given state x_k.

The dynamic programming (DP) algorithm is a recursion to calculate the optimal sub-array power at stage k given the optimal sub-array power at stage $k+1$. This recursion runs backwards from $k = N-1$ to $k = 0$ according to

$$G_k^*(x_k) = \max \left\{ g(x_k, u_k) + G_{k+1}^*(f(x_k, u_k)) \right\} \tag{12}$$

where the maximization is performed over the control u_k constrained according to $u_k \in \mathcal{U}(x_k)$. When $k = N-1$, the function $G_N^*(\cdot)$ is needed to initialize the algorithm. Note that G_N^* represents the optimal wind power at stage N; since there is no turbine with index $k > N-1$, we initialize the algorithm with $G_N^* = 0$. From Bellman (2003) it follows that, under mild assumptions, the maximum wind farm power satisfies

$$\max_{u_0, \ldots, u_{N-1}} G(x_0, u_0, \ldots, u_{N-1}) = G_0^*(x_0) \tag{13}$$

where $G(\cdot)$ is defined in (10), $G_0^*(\cdot)$ in (12), and the maximization is constrained according to the state-space equation (6) and $u_0 \in \mathcal{U}(x_0)$, ... , $u_{N-1} \in \mathcal{U}(x_{N-1})$. Moreover, an optimal control strategy $u_k^* = \mu_k(x_k)$ is obtained from the solution to (12) for $k = 0, \ldots, N-1$.

Thew following benefits derive from the DP framework:

- Equations (12) and (13) show that the N-dimensional wind farm power maximization problem, over the N scalar controls $u_0, \ldots, u_k, \ldots, u_{N-1}$, is equivalent to solving N *scalar* maximization problems recursively.
- The optimization problem at each stage of the recursion is similar to optimizing the power $g(x_k, u_k)$ of a single turbine plus an additional term that comes from the optimal value function $G_{k+1}^*(f(x_k, u_k))$.
- An optimal control u_k^* attaining the cost $G_k^*(x_k)$ in (12) is obtained in feedback form $u_k^* = \mu_k(x_k)$ (function of the state x_k), which is an appropriate functional form to cope with modeling uncertainty.

4. DP WITH ACTUATOR DISK MODEL

We assume that the turbines are modeled using the actuator disk model (ADM) described in section 2. The constraint set for the control input is given by the interval

$$\mathcal{U}(x) = [0, \frac{1}{2}x]. \tag{14}$$

4.1 Initial step of the DP algorithm.

The optimization problem we must solve is (12) with $k = N-1$ and the initial value function set at $G_N^*(x) = 0$ for all x. That is, we seek to solve

$$G_{N-1}^*(x_{N-1}) = \max g(x_{N-1}, u_{N-1}) \tag{15a}$$
$$= \max (x_{N-1} - u_{N-1})^2 u_{N-1} \tag{15b}$$

where the maximization is performed over the control u_{N-1} constrained to the interval $\mathcal{U}(x_{N-1}) = [0, \frac{1}{2}x_{N-1}]$. As shown in the appendix, this maximization problem has a unique solution given by

$$u_{N-1}^* = \frac{1}{3}x_{N-1}. \tag{16}$$

This result corresponds to the so-called *Betz limit*, with optimal induction factor a_{N-1} and maximum power G^*_{N-1} given by

$$a_{N-1} = \frac{1}{3} \tag{17a}$$

$$G^*_{N-1} = \frac{4}{27} x^3_{N-1} \tag{17b}$$

$$P^*_{N-1} = 2\rho A G^*_{N-1} = \frac{8}{27}\rho A x^3_{N-1}. \tag{17c}$$

The solution has the following salient features:

(1) The optimal control (16) is a *linear feedback* of the state variable x_{N-1}.
(2) The optimal power (17b) for a single turbine is a *cubic function* of the state variable (x_{N-1} for the most down-wind turbine).

The algebraic structure of the solution to the initial step is preserved in the subsequent steps of the dynamic programming algorithm. The specific result –Theorem 1– is proven in the appendix.

4.2 Core step of the DP algorithm.

For $k = N-1, N-2, \ldots, 0$, define the scalar variables a_k and Q_k via the following recursions:

$$a_k = \frac{1}{2 + (1 - 6Q_{k+1})^{-1/2}} \tag{18a}$$

$$Q_k = a_k(1-a_k)^2 + (1-2a_k)^3 Q_{k+1} \tag{18b}$$

with boundary condition $Q_N = 0$.

Theorem 1. Consider the wind farm power maximization problem for a cascade with N identical turbines modeled with the ADM (6), (8), and (9). Let x_0 denote the free stream velocity entering the cascade. The optimal control sequence $u^*_0, u^*_1, \ldots, u^*_{N-2}, u^*_{N-1}$ is given by

$$u^*_k = a_k x_k \tag{19}$$

where, for $k = N-1, \ldots, 0$, the state-feedback gain a_k is computed from (18). Moreover, the maximum power produced by the wind farm is given by

$$P^*_0 = 2\rho A Q_0 x^3_0. \tag{20}$$

The backward recursions (18) are reminiscent of the recursions in discrete optimal control problems; see, for example, Kwakernaak and Sivan (1972).

Corollary 2. Under the assumptions of Theorem 1, and given any fixed $\ell \in \{0, \ldots, N-1\}$, the optimal power produced by the $N-\ell$ most down-wind turbines is

$$P^*_\ell = 2\rho A Q_\ell x^3_\ell \tag{21}$$

and the maximum power P^*_ℓ is achieved with the optimal control sequence $u^*_\ell, \ldots, u^*_{N-1}$ calculated from Theorem 1.

From Corollary 2, and noting that x_ℓ is the *free stream velocity* entering the subarray of $N-\ell$ turbines $\ell, \ldots, N-1$, we may define the efficiency η_ℓ of the ℓ-th subarray according to

$$\eta_\ell \doteq \frac{P_\ell}{\frac{1}{2}\rho A x^3_\ell}. \tag{22}$$

From Corollary 2 it follows that the optimal efficiency is

$$\eta^*_\ell = 4Q_\ell. \tag{23}$$

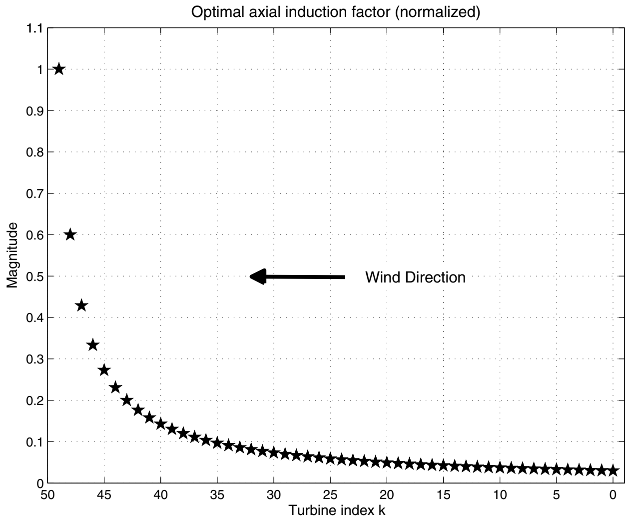

Fig. 3. Variation of optimal induction factor a_k with turbine index k. Plotted is a_k normalized by $a_{49} = \frac{1}{3}$.

4.3 Numerical example.

Figure 3 depicts the optimal induction factors calculated from (18) when $N = 50$ turbines. Note that the horizontal axis runs from the most down-wind turbine ($k = 49$) to the most up-wind turbine ($k = 0$). Table 1 shows the numerical values of the optimal induction factors a_k. Note that $a_{49} = 1/3$, which is the optimal value for a cascade of length one (e.g., a single isolated turbine). The optimal induction factors decrease monotonically as one moves upwind in the cascade. These results are intuitive; e.g., only the most down-wind turbine should operate at its own optimal induction factor $a_{49} = 1/3$.

Table 1. Parameters of optimal cascade

index (k)	$a_k/0.3333$	η^*_k (%)	cascade length
49	1.00	59.26	1
48	0.60	64.00	2
47	0.43	65.31	3
46	0.33	65.84	4
45	0.27	66.12	5
⋮	⋮	⋮	⋮
0	0.03	66.66	50

Also shown in Table 1 is the optimal efficiency, calculated from (23), as function of the length of the cascade (number of turbines in the cascade). For a cascade of length one, the efficiency is the Betz limit

$$\eta^*_{49} = \frac{16}{27} \approx 0.5926. \tag{24}$$

Intuitively, the larger the number of turbines the higher the optimal efficiency. Fig. 4 shows the optimal efficiency as a function of the cascade length. The asymptotic behavior of the graph and Table 1 suggest that

$$\lim_{N\to\infty} \eta^*_0 = \frac{2}{3} \approx 0.6666. \tag{25}$$

This result is easy to verify from the recursion (18). From Table 1 it follows that four turbines suffice to be within 1% of the highest possible efficiency 66.66%.

Fig. 4 also shows the efficiency η^g when the induction factors are all set at

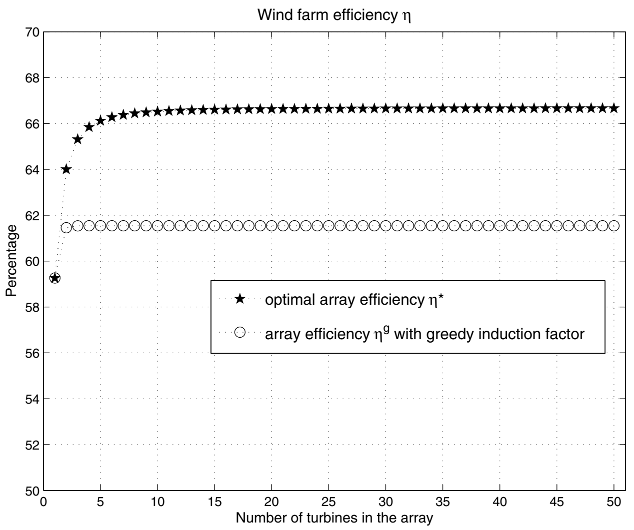

Fig. 4. Variation of optimal efficiency η^* (from (23)), and efficiency η^g, with the number of turbines.

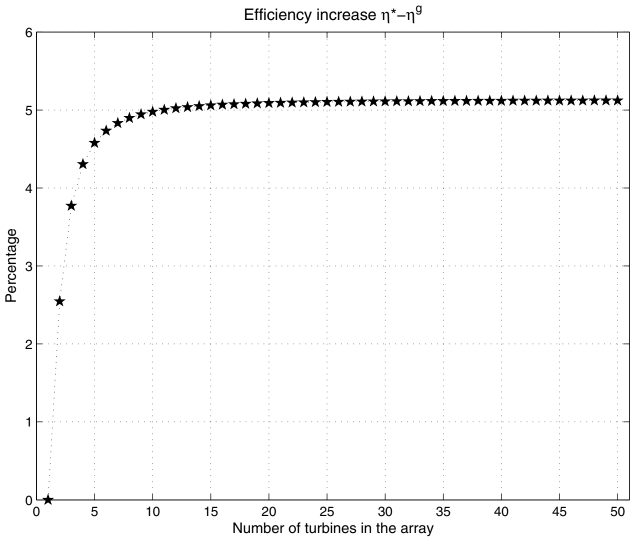

Fig. 5. Efficiency increase of the optimal induction factor a_k in (18) with respect to $a_k^g = 1/3$.

$$a_{49}^g = a_{48}^g = \ldots = a_0^g = \frac{1}{3}. \tag{26}$$

This case corresponds to setting each turbine induction factor at the value that maximizes the power coefficient of that particular turbine. As in Marden et al. (2013), we refer to this control strategy as the *greedy* control strategy since each turbine is seeking to maximize its own power without consideration of the other turbines in the array. Fig. 5 shows the optimal efficiency increment with respect to the greedy strategy. The increment is 5%, or greater, with ten or more turbines.

5. CONCLUSIONS

In this paper we provided a multi-stage dynamic program (DP) for the wind farm power maximization problem. The DP framework reduces a *complex* maximization problem to *a sequence of simpler* optimization problems. For a problem with N turbines and m controls per turbine, the power maximization problem is reduced from Nm optimization variables to a sequence of N problems with m variables each. The solution of each optimization problem in the sequence is obtained as a feedback function of the state of the wind farm, which is a functional form known to be robust to modeling uncertainty.

When the power extraction of each turbine is modeled with the actuator disk model, the DP framework is used to show that the optimal control for each turbine is a *linear* feedback of the state. The optimal state-feedback gains and the optimal wind farm power output are computed using a recursion reminiscent of the Riccati difference equation of optimal control; see (18). The optimal efficiency of the wind farm increases monotonically from the Betz limit (for a single turbine) to the optimal asymptotic efficiency 66.66%.

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Appendix A. PROOF OF THEOREM 1

We shall first prove the result for the initial step, equation (16). Calculating $g(\cdot)$ at the boundary values of the constraint set $\mathcal{U}(x_{N-1}) = [0, \frac{1}{2}x_{N-1}]$ we obtain

$$g(x_{N-1}, 0) = 0 \qquad \text{(A.1a)}$$

$$g(x_{N-1}, \frac{1}{2}x_{N-1}) = \frac{1}{8}x_{N-1}^3 \qquad \text{(A.1b)}$$

The values of $u_{N-1} \in (0, \frac{1}{2}x_{N-1})$ with vanishing derivative are obtained from the solutions u to the equation

$$\frac{\partial g(x_{N-1}, u)}{\partial u} = (x_{N-1} - u)(x_{N-1} - 3u) = 0 \qquad \text{(A.2)}$$

The only feasible solution is $u^* = \frac{1}{3}x_{N-1}$. Calculating $g(x_{N-1}, u^*)$ we obtain

$$g(x_{N-1}, u^*) = \frac{4}{27}x_{N-1}^3. \qquad \text{(A.3)}$$

Comparing with (A.3) with (A.1) we conclude that (16) gives the solution to the initial step, and the maximum power of the most down-wind turbine is given by

$$G_{N-1}^*(x_{N-1}) = g(x_{N-1}, u^*) = \frac{4}{27}x_{N-1}^3. \qquad \text{(A.4)}$$

Defining the coefficient $Q_{N-1} \doteq 4/27$, it follows that the optimal power is *a cubic* function of the state; i.e,

$$G_{N-1}^*(x_{N-1}) = Q_{N-1}x_{N-1}^3. \qquad \text{(A.5)}$$

To complete the proof of Theorem 1 we proceed by induction. That is, we assume that the result for the index $k = N-1$ (initial step) also holds true for the index $k+1$ and prove the result for the index k. Under this assumption, the optimal power G_{k+1}^* for the sub-array of turbines $k+1, k+2, \ldots, N-1$ is of the form

$$G_{k+1}^*(x_{N-1}) = Q_{k+1}x_{k+1}^3 \qquad \text{(A.6)}$$

where Q_{k+1} is a nonnegative constant. Following the dynamic programming algorithm (12), we compute the optimal control u_k^* from

$$u_k^* = \arg\max\left\{(x_k - u_k)^2 u_k + Q_{k+1}(x_k - 2u_k)^3\right\} \qquad \text{(A.7)}$$

where the maximization is performed over the interval $u_k \in [0, \frac{1}{2}x_k]$. Define the function $G_k(x_k, u_k)$ by

$$G_k(x_k, u_k) \doteq (x_k - u_k)^2 u_k + Q_{k+1}(x_k - 2u_k)^3 \qquad \text{(A.8)}$$

We now seek the values of $u_k \in (0, \frac{1}{2}x_k)$ that null the partial derivative of G_k with respect to u_k. A simple calculation yields

$$\frac{\partial G_k}{\partial u_k} = (1 - 6Q_{k+1})x_{k+1}^2 - u_k^2 \qquad \text{(A.9)}$$

where $x_{k+1} = x_k - 2u_k$. Thus, $\frac{\partial G_k}{\partial u_k}$ vanishes if and only if

$$(1 - 6Q_{k+1})x_{k+1}^2 = u_k^2 \qquad \text{(A.10)}$$

which implies that $(1 - 6Q_{k+1}) \geq 0$ and

$$\sqrt{1 - 6Q_{k+1}}\,x_{k+1} = u_k \qquad \text{(A.11)}$$

for both $u_k > 0$ and $x_{k+1} > 0$. Solving for u_k we obtain

$$u_k^* = a_k x_k \qquad \text{(A.12)}$$

$$a_k = \frac{1}{2 + (1 - 6Q_{k+1})^{-1/2}}. \qquad \text{(A.13)}$$

It can be shown that the boundary values $u_k = 0$ or $u_k = \frac{1}{2}x_k$ yield lower values of G_k. Thus, the solution to the dynamic programming equation (12) is given by (19).

To complete the proof we must obtain the equation for the optimal value function

$$\begin{aligned} G_k^*(x_k) &= G_k(x_k, u_k^*) && \text{(A.14)} \\ &= (x_k - u_k^*)^2 u_k^* + Q_{k+1}(x_k - 2u_k^*)^3. \end{aligned}$$

Since u_k^* is linear in x_k it follows from (A.14) that G_k^* is of the form

$$G_k^*(x_k) = Q_k x_k^3 \qquad \text{(A.15)}$$

where Q_k is a nonnegative constant. Equations (A.14) and (A.15) hold for any $x_k \geq 0$ if and only if Q_{k+1} and Q_k satisfy (18). This completes the proof of the theorem. ■