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Stabilization of the Steady Regime in Stochastic Population Model

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Abstract: The problem addressed is to construct a feedback regulator that stabilizes an equilibrium of the randomly forced nonlinear dynamic system, and synthesizes a required stochastic sensitivity of this equilibrium. The aim of this paper is to demonstrate how elaborated control theory can be used for the protection of the ecological systems against catastrophic shifts. In present paper, a stochastically forced predator-prey model with Allee effect is studied. A probabilistic mechanism of the noise-induced extinction of both species is shown. Using stochastic sensitivity functions technique, we construct confidence ellipses and estimate the threshold value of the intensity for noise generating a transition from the stable coexistence of both species to the extinction. For the stabilization of this population system, a method of the control for confidence ellipses is applied. It is shown that the regulator provides a low level of stochastic sensitivity and prevents unwanted ecological catastrophes.

Keywords: Feedback regulator, stochastic system, noise-induced extinction, confidence ellipses, stabilization.

1. INTRODUCTION

Controlling nonlinear systems forced by stochastic disturbances is an important problem in modern systems engineering. Investigations of the stochastic systems stabilization were started in [Krasovskii and Lidskii (1961)], and continued by many researchers [Kushner (1967); Wonham (1970); Fleming and Rishel (1975); Sun (2006)].

In the theory of stochastic stabilization, a main attention is paid to the designing regulators that provide a mean square stability for equilibrium. To guarantee a uniqueness of the regulator, a problem of optimization with certain criterion was considered usually. In a system with stabilizing regulator, there exists a stable stationary random distribution around the equilibrium.

In many applications, it is necessary to design a regulator that provides assigned probabilistic features for this distribution. Fokker-Planck-Kolmogorov equation gives a full description for this stationary probabilistic distribution. However, for the control problem considered here, it is very difficult to find exact solutions of this equation and especially to connect it with regulator parameters [Guo and Wang (2010)].

In these circumstances, for weak noise, a local analysis based on the first approximation system and stochastic sensitivity function technique can be used. The stochastic sensitivity functions technique was successfully applied to the analysis of various noise-induced phenomena (see [Ryashko and Bashkirtseva (2011a); Bashkirtseva et al. (2010)]).

For control systems, by the corresponding choice of regulator, one can change a stochastic sensitivity of randomly forced attractors and form desired dynamics. This approach was used in [Bashkitseva and Ryashko (2005); Ryashko and Bashkirtseva (2011b); Bashkirtseva et al. (2012, 2013)].

The general approach for synthesis of required sensitivity of stochastic controlled systems was proposed in [Ryashko and Bashkirtseva (2008)]. In Section 2, a brief mathematical background of this theory is presented and discussed.

The aim of this paper is to demonstrate how elaborated control theory can be used for the protection of the ecological systems against possible noise-induced catastrophic shifts.

An analysis of noise-induced transitions is an actively developing area of the investigations in biological systems [Allen (2003)]. Environmental noise is an inevitable attribute of any living system. Abrupt catastrophic shifts in ecosystems can be caused by small deterministic disturbances and stochastic fluctuations [Rietkerk et al. (2004)].

From mathematical point of view, such shifts can be attributed to alternative attractive states of multistable ecosystems, and sensitivity of attraction basin boundaries. Due to random disturbances, a phase trajectory can cross a separatrix between basins of the attraction of coexisting attractors and exhibit a new dynamical regime [Kraut et al. (1999)]. A classic example of such phenomena in ecosystems is a noise-induced extinction in the stochastic population models with an Allee effect [Dennis (2002)].

In present paper, we solve a control problem for the stochastically forced predator-prey model with Allee effect.

In Section 3, main features of the deterministic model are shortly discussed.

In Section 4, a probabilistic mechanism of the noise-induced extinction of both species is shown and investigated on the base of stochastic sensitivity functions technique. We use confidence ellipses for the estimation of the threshold value of the intensity for noise generating a transition from the stable coexistence of both species to the extinction. For the stabilization of this population system, we apply a method of the control by confidence ellipses. It is shown that the regulator constructed on the base of the general control theory from the Section 2, provides a low level of stochastic sensitivity and protect this system from unwanted ecological shifts.

2. CONTROL OF STOCHASTIC SENSITIVITY

Consider a nonlinear controlled stochastic system

$$\begin{aligned} &dx = f(x, u(x))dt + \varepsilon\sigma(x, u(x))dw(t), \\ &x, f \in R^n,\ u \in R^l, \end{aligned} \tag{1}$$

where $f(x, u)$ is a continuously differentiable n-vector-function, $w(t)$ is a m-dimensional standard Wiener process, $\sigma(x)$ is a $n \times m$-matrix-function characterizing a dependence of disturbances on state and control, ε is a scalar parameter of the noise intensity.

It is supposed that the corresponding deterministic system (1) (with $\varepsilon = 0$ and $u = 0$) has an equilibrium $\bar{x}$ which stability is not supposed.

Consider a class $\mathcal{U}$ of admissible feedbacks $u = u(x)$ satisfying conditions:
(a) $u(x)$ is continuously differentiable and $u(\bar{x}) = 0$;
(b) a feedback $u(x)$ provides an exponential stability of $\bar{x}$ for the closed loop deterministic system

$$dx = f(x, u(x))dt \tag{2}$$

in a certain neighborhood of $\bar{x}$.

The first condition (a) means that $\bar{x}$ remains an equilibrium of the system (2) for any $u \in \mathcal{U}$.

Consider a system of the first approximation for a deviation $z(t) = x(t) - \bar{x}$

$$dz = (F + BK)zdt, \tag{3}$$

where $F = \dfrac{\partial f}{\partial x}(\bar{x}, 0)$, $B = \dfrac{\partial f}{\partial u}(\bar{x}, 0)$, $K = \dfrac{\partial u}{\partial x}(\bar{x})$.

The second condition (b) is equivalent to the exponential stability of a trivial solution of the system (3).

Thus, we can restrict our consideration without loss of generality by more simple regulators in the form of linear feedback

$$u(x) = K(x - \bar{x}). \tag{4}$$

A set of matrices K supplying exponential stability for the trivial solution of the system (3) has the following form

$$\mathbf{K} = \{K|\ Re\lambda_i(F + BK) < 0\}.$$

Here $\lambda_i(F+BK)$ are the eigenvalues of the matrix $F+BK$. Suppose the pair (F, B) is stabilizable. It means that both set $\mathbf{K}$ and class $\mathcal{U}$ are not empty.

Under the small external disturbances ($\varepsilon \neq 0$), the random trajectories $x^\varepsilon(t)$ of the stochastic system (1) leave the equilibrium $\bar{x}$. Due to exponential stability the feedback (4) with a matrix $K \in \mathbf{K}$ allows to localize random states of the system (1),(4) in the neighborhood of the equilibrium $\bar{x}$ and form a stationary distributed solution $\bar{x}^\varepsilon(t)$.

For dynamics of small deviations $z(t) = x^\varepsilon(t) - \bar{x}$, the following stochastic system of the first approximation can be considered

$$dz = (F + BK)zdt + \varepsilon G dw,\ G = \sigma(\bar{x}, 0). \tag{5}$$

The sensitivity of the system (5) solution to noise with intensity ε is characterized by a variable $y = \dfrac{z}{\varepsilon}$ governed by the following equation

$$dy = (F + BK)ydt + Gdw. \tag{6}$$

For the covariance matrix $V(t) = cov(y(t), y(t))$ of an arbitrary solution of the system (6), the equation holds [Wonham (1970)]

$$\dot{V} = (F + BK)V + V(F + BK)^\top + S,\ S = GG^\top. \tag{7}$$

For any $K \in \mathbf{K}$, this equation has a unique stationary solution W which satisfies the matrix algebraic equation

$$(F + BK)W + W(F + BK)^\top + S = 0. \tag{8}$$

In a case of non-singular noises ($\det S \neq 0$) the solution W is positive defined.

Any solution $V(t)$ of the system (7) converges to the corresponding solution W of the system (8)

$$\lim_{t\to\infty} V(t) = W.$$

The matrix W sets the interplay between the noise intensity ε and the covariance matrix $cov(\bar{x}^\varepsilon(t), \bar{x}^\varepsilon(t)) \approx \varepsilon^2 W$ of the solution states $\bar{x}^\varepsilon$ of the system (1). So, the matrix W is a simple quantitative characteristic of a response of the nonlinear system (1) to the small random disturbances in the neighborhood of the equilibrium $\bar{x}$. This matrix W plays a role of the *stochastic sensitivity factor* of the equilibrium $\bar{x}$.

There are practical tasks where not only an equilibrium stability is required but also a small dispersion of random states localized near the equilibrium of a stochastically forced system has to be provided.

The control of the dispersion can be implemented by means of synthesis of an assigned matrix W.

Let $\mathbf{M}$ be a set of symmetric and positive defined $n \times n$-matrices. For any $K \in \mathbf{K}$ the regulator (4) forms a corresponding stochastic equilibrium of the system (1). This stochastic equilibrium has a sensitivity matrix W_K which is a solution of the equation (8). Consider the following problem.

Problem of stochastic sensitivity synthesis

For the assigned matrix $W \in \mathbf{M}$, it is necessary to find a matrix $K \in \mathbf{K}$ guaranteeing the equality $W_K = W$, where W_K is a solution of the equation (8).

In some cases, this problem is unsolvable. Therefore we introduce a notion of the attainability and the stochastic controllability.

Definition 1. The element $W \in \mathbf{M}$ is said to be *attainable* for the system (1), under the feedback (4) if the equality $W_K = W$ is true for some $K \in \mathbf{K}$.

Definition 2. A set of all attainable elements

$$\mathbf{W} = \{W \in \mathbf{M} \mid \exists K \in \mathbf{K} \quad W_K = W\}$$

is called *attainability set* for the system (1), (4).

Definition 3. An equilibrium $\bar{x}$ is *completely stochastic controllable* in a system (1),(4) if

$$\forall\, W \in \mathbf{M} \;\; \exists\, K \in \mathbf{K} : \quad W_K \equiv W.$$

Therefore an equilibrium $\bar{x}$ is completely stochastically controllable if and only if the following condition holds

$$\mathbf{W} = \mathbf{M}.$$

Let us describe an attainability set. The connection between the assigned matrix W and the feedback coefficient K follows from the equation (8) which can be rewritten in the form:

$$\begin{aligned} &BKW + WK^\top B^\top + H(W) = 0, \\ &H(W) = S + FW + WF^\top. \end{aligned} \tag{9}$$

Solution of the problem of synthesis of the stochastic sensitivity matrix is given by the following theorem [Ryashko and Bashkirtseva (2008)].

Theorem 1. Let noises in the system (1) be non-singular ($\det S \neq 0$).
(a) If the matrix B is quadratic and non-singular ($\mathrm{rank} B = n = l$) then $\mathbf{W} = \mathbf{M}$ and for any matrix $W \in \mathbf{M}$, the equation (9) has a solution

$$\begin{aligned} &K = \bar{K} + B^{-1} Z \bar{W}^{-1} \in \mathbf{K}, \\ &\bar{K} = -B^{-1}\left(\frac{1}{2} S W^{-1} + F\right), \end{aligned} \tag{10}$$

where Z is an arbitrary skew-symmetric $n \times n$-matrix.
(b) If $\mathrm{rank}(B) < n$ then the element $W \in \mathbf{M}$ is attainable if and only if the matrix W is a solution of the equation

$$P_2 H(W) P_2 = 0. \tag{11}$$

Under these conditions for any matrix $W \in \mathbf{M}$ satisfying (11) the equation (9) has a solution

$$\begin{aligned} &K = \bar{K} + C \in \mathbf{K}, \\ &\bar{K} = B^+ H(W) \left(\frac{1}{2} P_1 - I\right) W^{-1}, \end{aligned} \tag{12}$$

where C is an arbitrary $l \times n$-matrix satisfying the condition

$$BCW + WC^\top B^\top = 0. \tag{13}$$

Here a sign "+" means a pseudoinversion, $P_1 = BB^+$ and $P_2 = I - P_1$ are projective matrices.

Note that if $\mathrm{rank} B = 1$, the equation (9) has a unique solution $K = \bar{K}$.

3. DETERMINISTIC POPULATION MODEL

Consider a predator-prey system described by two differential equations

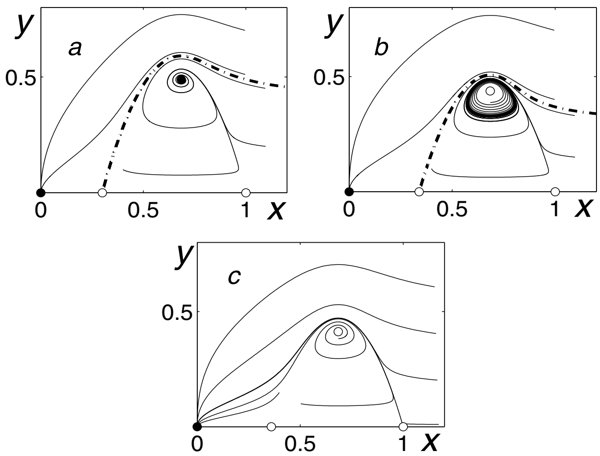

Fig. 1. Phase portraits of the deterministic system with separatrices (dash-dotted) for a) $\beta = 0.3$; b) $\beta = 0.34$; c) $\beta = 0.36$.

$$\begin{cases} \dfrac{dp}{d\tau} = g(p) - f(p)z, \\[2ex] \dfrac{dz}{d\tau} = af(p)z - bz, \end{cases} \tag{14}$$

where p and z are densities of prey and predator at time τ. The function $g(p)$ presents the intrinsic prey growth, $f(p)$ is the predator trophic response, a is the food utilization coefficient, b is the predator mortality. In this paper, the predator trophic response is Holling type II and for the function $f(p)$ the following parameterization is used [Holling (1965)]

$$f(p) = \frac{Ap}{B+p},$$

where B is the half-saturation prey density and A is the maximum predation rate.

We consider the prey population dynamics with the strong Allee effect and use for the function $g(p)$ the following standard parametrization

$$g(p) = cp(p - p_0)(K - p),$$

where c is the prey growth rate, K is the prey carrying capacity, p_0 is the prey survival threshold. The value p_0, $0 < p_0 < K$, is the control parameter for the Allee effect intensity. For dimensionless variables $x = p/K, y = z/(aK), t = (AaK/B)\tau$, it follows from (14) that

$$\begin{cases} \dfrac{dx}{dt} = \gamma x(x - \beta)(1 - x) - \dfrac{xy}{1 + \alpha x}, \\[2ex] \dfrac{dy}{dt} = \dfrac{xy}{1 + \alpha x} - \delta y, \end{cases} \tag{15}$$

where $\alpha = K/B, \beta = p_0/K, \gamma = cKB/(aA), \delta = bB/(aAK)$ are dimensionless parameters.

The deterministic system (15) possesses four equilibria $M_1(0,0)$, $M_2(\beta,0)$, $M_3(1,0)$, $M_4(\bar{x},\bar{y})$, where

$$\bar{x} = \frac{\delta}{1 - \alpha\delta} > 0, \; \bar{y} = \gamma(1 + \alpha\bar{x})(\bar{x} - \beta)(1 - \bar{x}) > 0$$

for $\beta < \dfrac{\delta}{1 - \alpha\delta} < 1$. The trivial equilibrium M_1 is stable and equilibria M_2, M_3 are unstable for any parameters.

In this paper, we fix the following set [Petrovskii et al. (2005)] of parameters: $\alpha = 0.5$, $\gamma = 3$, $\delta = 0.51$ and vary the Allee parameter $\beta \in [0.3,\ 0.38]$. In this β-interval, the local and global bifurcations occur, and the system (15) exhibits three different regimes of dynamics (see Fig. 1).

The non-degenerate equilibrium $M_4(\bar{x}, \bar{y})$ corresponding to the coexistence of prey and predator is stable for $0.3 < \beta < \beta_* = 0.3271$. As the parameter β gets over the value β_* from left to right, the equilibrium $M_4(\bar{x}, \bar{y})$ loses stability and the system (15) demonstrates auto-oscillations with the limit cycle γ.

The value β_* is the Andronov-Hopf bifurcation point. In Fig. 1a, for $\beta = 0.3 < \beta_*$, a separatrix (dash-dotted) detaches basins of the attraction of the stable equilibrium $M_1(0,0)$ and the stable equilibrium M_4. For $\beta = 0.34 > \beta_*$ (see Fig. 1b), separatrix detaches basins of the attraction of the stable equilibrium $M_1(0,0)$ and the stable limit cycle γ. Here, β_* is the point of a local bifurcation. As β increases, limit cycle enlarges and for $\beta^* = 0.35529$ cycle is destroyed: a lower part of the cycle coalesces with the line $y = 0$, upper part adheres to the separatrix, and heteroclinic orbit is born. Here, $\beta^* = 0.35529$ is the point of a global bifurcation. In Fig. 1c, the phase portrait with the single stable equilibrium M_1 for $\beta = 0.36 > \beta^*$ is shown.

It follows from this analysis that for $0.3 \le \beta < \beta^*$ the extinction zone is restricted by separatrix from below, and for $\beta^* < \beta \le 0.38$ the extinction zone coincides with the whole first quadrant $x > 0, y > 0$. So, for $0.3 \le \beta < \beta^*$, the separatrix serves as the boundary between a zone of the positive coexistence of prey with predator and a zone of the total extinction.

In Fig. 2a, for $0.3 \le \beta \le 0.38$, attractors of the deterministic system (15) are presented. Here, values of y-coordinate for the stable equilibria M_1, M_4 and extreme values of variable y for the stable limit cycle are plotted by solid lines. Dashed line corresponds to the y-coordinate of the unstable equilibrium M_4. Here a splitting of upper graph marks the bifurcation point β_*. For $\beta = \beta^*$ the limit cycle disappears and for $\beta > \beta^*$ the equilibrium M_1 remains as a single attractor.

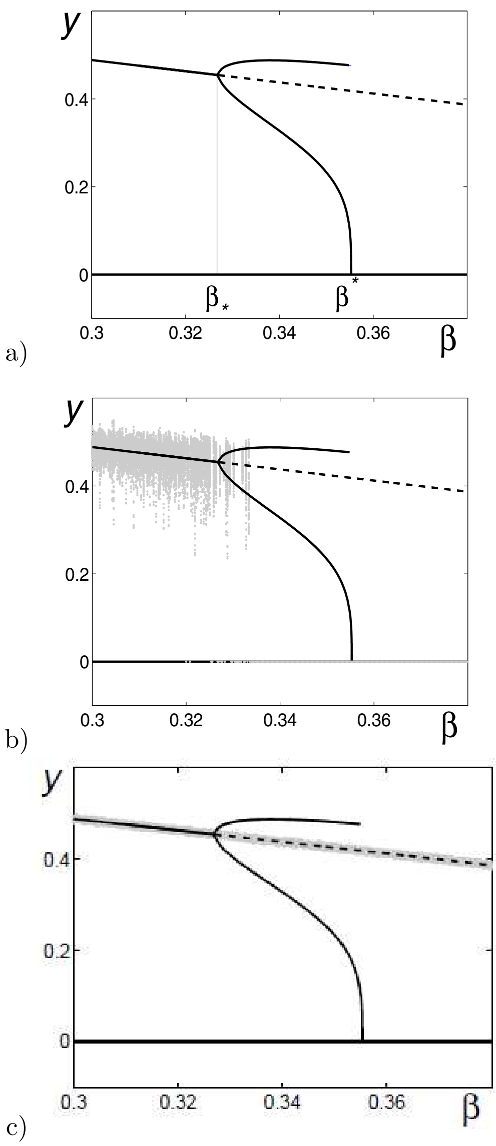

Fig. 2. a) Bifurcation diagram of the deterministic system; b) random states (grey) of the system without control and $\varepsilon = 0.01$, c) random states (grey) of the system with control and $\varepsilon = 0.01, w = 0.1$.

4. STOCHASTIC POPULATION MODEL WITH CONTROL

Along with deterministic system (15) consider a stochastically forced system

$$\begin{cases} \dot{x} = \gamma x(x-\beta)(1-x) - \dfrac{xy}{1+\alpha x} + u_1 + \\ \qquad + \sigma_1 \gamma x(1-x)\dot{w}_1, \\ \dot{y} = \dfrac{xy}{1+\alpha x} - \delta y + u_2 + \sigma_2 y \dot{w}_2. \end{cases} \quad (16)$$

Here u_1, u_2 are control inputs, w_1, w_2 are uncorrelated standard Wiener processes, σ_1, σ_2 are intensities of multiplicative noises modeling random disturbances of parameters β and δ correspondingly.

At first consider a system (16) without control ($u_1 = u_2 = 0$). Random trajectories of the forced system (16) leave a deterministic attractor and form a corresponding stochastic attractor. Here we consider $\sigma_1 = \sigma_2 = \varepsilon$.

For weak noise and $0.3 < \beta < \beta_* = 0.3271$, random states are concentrated near M_4 (see Fig. 3a). For small stochastic disturbances and $0.3271 < \beta < \beta^* = 0.35529$, random states are distributed around the orbit of the stable limit cycle (see Fig. 4a).

As noise intensity increases, the system (16) can exhibit qualitative changes of stochastic dynamics. Under the random disturbances, the stochastic trajectory can exit from the neighborhood of the stable attractor, cross the separatrix and pass to M_1 (see Figs. 3b, 4b). In Fig. 2b, random states of the system (16) with $\varepsilon = 0.01$ are presented by grey color. Here, for $t = 0$, trajectories started from the equilibrium M_4 and after transient time period $[0, 200]$ were plotted for $t \in [200, 300]$. The corresponding time

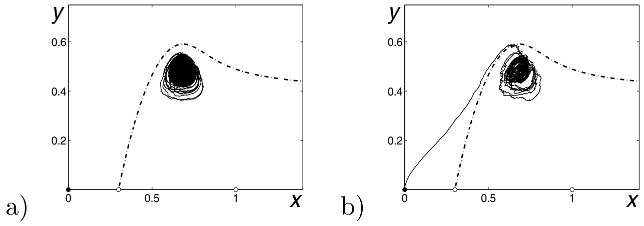

Fig. 3. Stochastic trajectories for $\beta = 0.3$: a) non-exit for $\varepsilon = 0.01$; b) noise-induced exit for $\varepsilon = 0.02$.

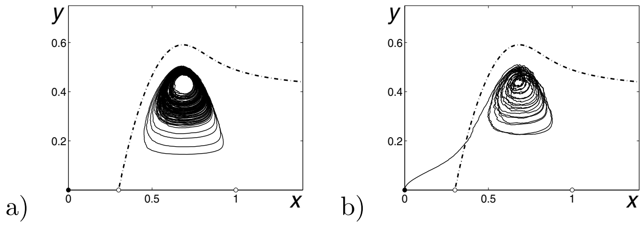

Fig. 4. Stochastic trajectories for $\beta = 0.34$: a) non-exit for $\varepsilon = 0.005$; b) noise-induced exit for $\varepsilon = 0.01$.

series for coordinate $y(t)$ are presented in Fig. 6 by grey color.

This transitions can be interpreted as a noise-induced extinction of both species in this population.

Consider further in detail a case $0.3 < \beta < \beta_* = 0.3271$ when the equilibrium M_4 is stable.

The quantitative parametrical analysis of the stochastic sensitivity of this equilibrium can be done with help of stochastic sensitivity matrix W. This matrix is a solution of the equation (8) with $K = 0$. The eigenvalues μ_1 and μ_2 of this matrix are convenient scalar characteristics of stochastic sensitivity. Stochastic sensitivity function technique allows to construct confidence ellipses [Ryashko and Bashkirtseva (2011a)]. Confidence ellipse is a simple geometrical model for the description of a configurational arrangement of random states near stable equilibrium.

For sufficiently small noise intensity, the confidence ellipses localized near the stable equilibrium M_4 entirely belong to the basin of the attraction of M_4 (see Fig. 5a). Corresponding random trajectories are concentrated near M_4 (see Fig. 3a).

As the noise intensity increases, the confidence ellipses begin to expand and after intersection of the separatrix occupy a basin of attraction of M_1. This occupation means that random trajectories of noisy system with high probability can leave the basin of attraction of M_4 and go to the trivial equilibrium M_1 (see Fig. 3b). Noise intensity that corresponds to the intersection of confidence ellipse with separatrix can be used as an estimation of the threshold value ε^*. Here $\varepsilon^* \approx 0.015$.

To protect the population system from such type unwanted noise-induced ecological shifts leading to the extinction, it is necessary to construct the feedback

$$\begin{aligned} u_1 &= k_{11}(x - \bar{x}) + k_{12}(y - \bar{y}), \\ u_2 &= k_{21}(x - \bar{x}) + k_{22}(y - \bar{y}). \end{aligned} \tag{17}$$

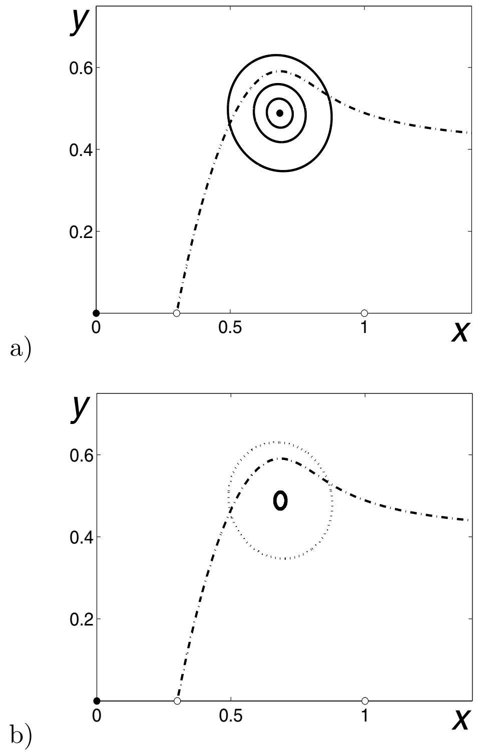

Fig. 5. Confidence ellipses for $\beta = 0.3$ and $P = 0.9$: a) for the system without control and $\varepsilon = 0.005$ (small), $\varepsilon = 0.01$ (middle), $\varepsilon = 0.02$ (large); b) for the system with $\varepsilon = 0.02$ without control (dotted) and with control (solid) providing $w = 0.1$.

in such a way as to localize a confidence ellipse to be inside the basin of attraction of the equilibrium M_4. As shown above, the confidence ellipse for the uncontrolled system with $\varepsilon = 0.02$ (see Fig. 5a) is too wide. Indeed, this confidence ellipse contains the separatrix and partly occupies the basin of attraction of the trivial equilibrium. It results in the noise-induced extinction of the population. Here, the control problem will be reduced to decreasing the size of the confidence ellipse.

A size of the confidence ellipse is defined by the noise intensity and the stochastic sensitivity matrix W. For the uncontrolled system (16) with $b = 0.3$, we have

$$W = \begin{bmatrix} 8.82 & -0.44 \\ -0.44 & 4.73 \end{bmatrix}.$$

To decrease the stochastic sensitivity, we assign

$$\bar{W} = \begin{bmatrix} w & 0 \\ 0 & w \end{bmatrix}$$

with a sufficiently small value w. Note that for the system (16), any matrix W is attainable (see Theorem 1, case a).

The matrix

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}$$

of the feedback coefficients of the regulator (17) synthesizing the assigned $\bar{W}$, can be found by the formula (10) in Theorem 1.

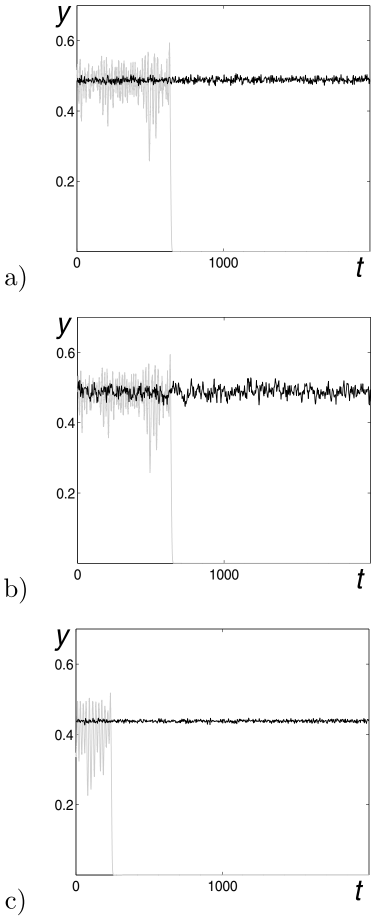

Fig. 6. Time series for the system without control (grey) and with control (black): a) for $\beta = 0.3, \varepsilon = 0.02, w = 0.1$; b) for $\beta = 0.3, \varepsilon = 0.02, w = 0.5$; c) for $\beta = 0.34, \varepsilon = 0.01, w = 0.1$.

Put $w = 0.1$. It is quite sufficient to synthesize small ellipse that is entirely arranged in the basin of attraction of the equilibrium M_4 (see Fig. 5b, solid line).

Results of the direct numerical simulation of solutions of the stochastic closed-loop system (16),(17) are presented in Fig. 6. Here, by black color time series of this closed-loop system are plotted. As one can see, the regulator (17) provides small stochastic sensitivity w and stabilizes the population system near non-trivial deterministic equilibrium M_4.

By change of the value w, one control the dispersion of the small-amplitude stochastic oscillations of the system (16),(17) near the equilibrium M_4 (compare Fig. 6a and Fig. 6b). It is worth noting that this regulator allows us to stabilize even the unstable equilibrium (see Fig. 6c for $b = 0.34$).

Moreover, this regulator solves an important problem of the structural stabilization for the whole parametrical interval $0.3 < \beta < 0.38$ (Fig. 2c) and provides a small uniform dispersion of random states around M_4.

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