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A Frequency Domain Interpretation of the Algebraic Differentiators

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Abstract: This work proposes a frequency domain interpretation of the Algebraic Differentiators introduced in "M. Mboup, C. Join and M. Fliess, Numerical Differentiation with annihilators in noisy environment, Numerical Algorithms, vol. 50, pp. 439-457, 2009". This interpretation complements the least squares interpretation in the above reference. In particular, it allows one to explain 1) the behaviour of different types of algebraic differentiators and 2) why it can be preferable to apply *e.g.* two successive first order differentiations in place of one second order differentiation. The frequency domain interpretation also enables the use of the Fourier theory (continuous and discrete transforms) in order to tune and discretize the Algebraic Differentiators.

Keywords: Algebraic Differentiators, Fourier transform

1. INTRODUCTION

This paper revisits the algebraic differentiation method developed in Mboup et al. [2009] (see Liu et al. [2012] for an extension to the fractional order case). The method uses the framework of algebraic estimation which was introduced in the automatic control literature by Fliess and Sira-Ramírez [2003, 2008], Fliess and Join [2013] (see also Fliess et al. [2008, 2003], Trapero et al. [2007] and Fliess et al. [2011] for applications in signal processing and finance respectively). For estimating the n^{th} order, $n \in \mathbb{N}$, derivative of a noisy signal $x(t)$, the algebraic differentiation method considers a pointwise estimation of the parameter $\left.\frac{d^n x(t)}{dt^n}\right|_{t=\tau}$ for $\tau > 0$. For each such given point τ, the signal is represented locally by its Taylor series expansion of order, say $N \geq n$. The method then uses differential elimination in the operational domain to annihilate all terms of the truncated Taylor expansion except the desired parameter $\left.\frac{d^n x(t)}{dt^n}\right|_{t=\tau}$. The process is repeated for each $\tau > 0$ and the derivative estimation follows when the obtained parameters are expressed back in the time domain.

Note that conceptually, the problem of derivative estimation admits a simple solution which derives from the following observation: in an appropriate reproducing kernel Hilbert space, the n^{th} order derivative of a given $x(t)$ takes the form

$$x^{(n)}(\tau) = \langle x(t), \frac{d^n}{d\tau^n}\mathcal{K}(\tau, t)\rangle$$

where $\mathcal{K}(\cdot,\cdot)$ is the reproducing kernel. As mentioned in Mboup et al. [2009], this general observation leaves the problem totally unsolve until an adequate model space is selected. The least squares interpretation in Mboup et al. [2009] (see Mboup [2009] for the most general setting) has shown 1) that the algebraic manipulations above can be cast within this general observation and 2) how adequate is the associated model space. Interesting characterizations of the differentiators then arise, regarding the estimation precision in a noise free context as well as the robustness against additive noises.

Note also that the operational domain analogue of time differentiation is just multiplication by the Laplace variable s. This leads to another conceptually simple solution to derivative estimation: Given an order $n \in \mathbb{N}$, and for any appropriate smoothing filter $H_n(s)$, $s^n H_n(s)$ represents the transfer function of a differentiation filter. But once again, this general principle is more elusive than useful until an adequate associated smoothing filter is devised.

In this paper, we propose a frequency domain interpretation of the algebraic differentiation, in complementary to the least squares interpretation. It turns out that the algebraic manipulations join the frequency domain standpoint of differentiation, outlined above. The corresponding smoothing filters are clearly identified. This permits, on the one hand, to reveal additional interesting properties of the algebraic differentiators and, on the other hand, the use of the classical Fourier theory in order to tune the parameters of the algebraic differentiators. Let us mention that a different frequency analysis was proposed in de A. García Collado et al. [2009] in discrete time. Some interesting features concerning the noise rejection capability of the algebraic differentiators have also been pointed out in the recent and nice frequency domain analysis of Kiltz and Rudolph [2013].

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The paper is organized as follows. Section 2 summarizes the design of the Algebraic Differentiators from Mboup et al. [2009] as well as the associated least squares interpretation. In section 3, the frequency domain interpretation is developed. Sections 4 and 5 specialize the analysis to particular Algebraic Differentiators and the cascade of differentiators respectively. The paper ends with a conclusion in section 6.

2. AN OVERVIEW OF THE ALGEBRAIC DIFFERENTIATORS

Let us start with a brief overview of the algebraic differentiation method of Mboup et al. [2009]. Consider a signal $x(t)$ measured through an additive noise corruption $\varpi(t)$:

$$y(t) = x(t) + \varpi(t). \tag{1}$$

The goal is to estimate the n^{th} order derivative of x, $n \in \mathbb{N}$, based on the noisy observation y. A Taylor expansion truncated to the N^{th} order, $N \geq n$ is used to represent $x(t)$ on a "small" segment of time around each given point τ. With $x_N(t)$ to denote this truncation, we can set

$$x(\tau - t) \approx x_N(t) = \sum_{i=0}^{N} x^{(i)}(\tau)\frac{(-t)^i}{i!}. \tag{2}$$

The algebraic differentiation method is based on annihilating from (2) the parameters $x^{(i)}(\tau)$ for all i except $i = n$. To simplify the notations, we henceforth redefine the parameters $x^{(i)}(\tau)$ as $(-1)^i x^{(i)}(\tau)$. The annihilation will be achieved by differential elimination which is easier to express in the operational domain. For this, the Laplace transform is applied to (2):

$$\hat{x}_N(s) = \sum_{i=0}^{N} \frac{x^{(i)}(\tau)}{s^{i+1}}. \tag{3}$$

On (3), the application of the annihilator:

$$\Pi^{N,n}_{\kappa,\mu} = \frac{1}{s^{N+\mu+1}}\frac{d^{n+\kappa}}{ds^{n+\kappa}}\frac{1}{s}\frac{d^{N-n}}{ds^{N-n}}s^{N+1}, \quad \kappa, \mu \in \mathbb{N} \tag{4}$$

permits to isolate $x^{(n)}(\tau)$ as follows:

$$\frac{(-1)^{n+\kappa}(n+\kappa)!(N-n)!}{s^{\mu+\kappa+N+n+2}}x^{(n)}(\tau) = \Pi^{N,n}_{\kappa,\mu}\hat{x}_N(s). \tag{5}$$

After isolating $x^{(n)}(\tau)$, equation (5) can be transformed back into the time domain. This can be done by applying the usual rules of operational calculus. Recall that if $\hat{u}(s)$ is the operational analogue of $u(t)$ then, according to the usual rules of operational calculus, the time domain analogue of $\hat{v} = \frac{1}{s^\alpha}\frac{d}{ds^\beta}\hat{u}$ will reads as:

$$v(t) = \frac{1}{(\alpha-1)!}\int_0^t (t-\lambda)^{\alpha-1}(-\lambda)^\beta u(\lambda)d\lambda. \tag{6}$$

Let us consider in (5) the simplest case by putting $N = n$. Then, applying (6), the time domain equivalent of (5) takes the form

$$x^{(n)}(\tau; t, \kappa, \mu) = \int_0^t K(\lambda; t, \kappa, \mu)x_n(\lambda)d\lambda \tag{7}$$

for some $K(\cdot; t, \kappa, \mu, n)$ with parameters t, κ, μ and n. Henceforth we fix the integration time in (7) to $t = T$, a positive constant. Recalling (2) and (1), a derivative estimator is then obtained in terms of convolution product, by substituting $x_n(t)$ by the corresponding actual measurement $y(\tau - t)$:

$$\tilde{x}^{(n)}(\tau; \kappa, \mu) = \int_0^\infty h(\lambda; T, \kappa, \mu, n)y(\tau - \lambda)d\lambda = \{h \star y\}(\tau), \tag{8}$$

with $h(\,\cdot\,; T, \kappa, \mu, n) = -K(\,\cdot\,; T, \kappa, \mu, n)$.

Equation (8), called a *minimal estimator* since it is designed from a minimal Taylor expansion $N = n$, is central in Mboup et al. [2009]. It constitutes in fact the building block to construct more general estimators from (5).

Set $q = N - n$ and consider the set of minimal estimators:

$$\mathcal{S}_{\kappa,\mu,q} = \Big\{\tilde{x}^{(n)}(\tau; \kappa+q, \mu), \cdots, \tilde{x}^{(n)}(\tau; \kappa+q-l, \mu+l), \cdots, \tilde{x}^{(n)}(\tau; \kappa, \mu+q)\Big\},\ q \leq n+\kappa, l \in [0, q]. \tag{9}$$

It was shown in Mboup et al. [2009] that any affine combination of elements of (9) defines a derivative estimator. In particular [Mboup et al., 2009, Equation (30)] establishes that any derivative estimator $\tilde{x}^{(n)}(\tau; \kappa, \mu, N)$ obtained from (5) with a Taylor series expansion of order $N > n$ is of the form

$$\tilde{x}^{(n)}(\tau; \kappa, \mu, N) = \sum_{l=0}^{N-n} \lambda_l \tilde{x}^{(n)}(\tau; \kappa_l, \mu_l), \tag{10}$$

where $\kappa_l = \kappa + q - l$ and $\mu_l = \mu + l$ and for some set of coefficients $\lambda_l \in \mathbb{Q}$, satisfying $\sum_{l=0}^{N-n}\lambda_l = 1$ such that $\lambda_\ell < 0$ for at least one $\ell \in [0, N-n]$.

3. FREQUENCY DOMAIN INTERPRETATION

The exact expression of $h(\,\cdot\,; T, \kappa, \mu, n)$ is given in Mboup et al. [2009] by:

$$h(t; T, \kappa, \mu, n) = \gamma_{\kappa,\mu,n,T}\frac{d^n}{dt^n}w_{\{\kappa,\mu,n,T\}}(t), \tag{11}$$

where

$$w_{\{\kappa,\mu,n,T\}}(t) = \begin{cases} t^{\kappa+n}(T-t)^{\mu+n}, & 0 < t < T \\ 0 & \text{else} \end{cases} \tag{12}$$

and

$$\gamma_{\kappa,\mu,n,T} = \frac{(-1)^n}{T^{\kappa+\mu+2n+1}}\frac{(\kappa+\mu+2n+1)!}{(\kappa+n)!(\mu+n)!}. \tag{13}$$

Denote by $\tilde{X}^{(n)}(\omega)$ and $Y(\omega)$ the Fourier transforms of $\tilde{x}^{(n)}(\tau; \kappa, \mu)$ and $y(t)$ respectively. Using (11), equation (8) then translates in the frequency domain as:

$$\tilde{X}^{(n)}(\omega) = (i\omega)^n H_{\kappa,\mu,n}(\omega)Y(\omega), \tag{14}$$

with

$$H_{\kappa,\mu,n}(\omega) = \gamma_{\kappa,\mu,n,T}\int_0^T t^{\kappa+n}(T-t)^{\mu+n}e^{-i\omega t}dt. \tag{15}$$

Equation (14) shows that the Algebraic Differentiator corresponds to a smoothing filter followed by an n^{th} order frequency domain differentiation. As mentioned in Kiltz and Rudolph [2013], the integral in (15) can be expressed in terms of the confluent hypergeometric Kummer functions. Instead, we use the following relation:

Lemma 3.1. The smoothing filter in (15) satisfies the recurrence relation on the differentiation order:

$$H_{\kappa,\mu,n+1}(\omega) = \varrho_{\kappa,\mu,n}\left(Ti\frac{d}{d\omega}H_{\kappa,\mu,n}(\omega) + \frac{d^2}{d\omega^2}H_{\kappa,\mu,n}(\omega)\right), \tag{16}$$

with

$$\varrho_{\kappa,\mu,n} = -\frac{(\kappa+\mu+2n+2)(\kappa+\mu+2n+3)}{(\kappa+n+1)(\mu+n+1)T^2}.$$

Proof 3.2. The recurrence $\gamma_{\kappa,\mu,n+1,T} = \varrho_{\kappa,\mu,n}\gamma_{\kappa,\mu,n,T}$ is easy to obtain from (13). Also, we deduce directly from (12) that $w_{\{\kappa,\mu,n+1,T\}}(t)$ is obtained by multiplying $w_{\{\kappa,\mu,n,T\}}(t)$ by $Tt-t^2$. Now this translates in the Fourier domain into the differential operator $Ti\frac{d}{d\omega}+\frac{d^2}{d\omega^2}$.

The recurrence (16) permits to iteratively compute the frequency response of the smoothing filter corresponding to an n^{th} order minimal Algebraic Differentiator. Then the frequency response corresponding to non-minimal estimators can be computed using (10).

In addition to (16), the following relation can be verified

Lemma 3.3.

$$H_{\kappa+1,\mu+1,n}(\omega) = -H_{\kappa,\mu,n+1}(\omega). \tag{17}$$

Proof 3.4. The proof is by direct inspection and it is left to the reader.

This relation points out that the same smoothing filter (15) is involved in different order Algebraic Differentiators. More interestingly, (17) shows that as the filter order n is increased its spectrum spreads out. In order to illustrate this claim let us examine the figure 1 representing the spectrum of three filters from (15) with $n=1$. Remark

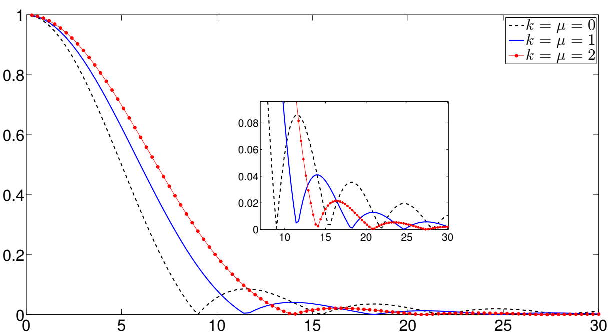

Fig. 1. First order derivative smoothing filter. Influence of increasing κ and μ.

that as κ and μ are increased the filter frequency response spreads out. Then according to (17), as n is increased the spectrum of the smoothing filter $H_{\kappa,\mu,n}(\omega)$ spreads out. More generally, for higher values of κ and μ, let us notice that the smoothing filter impulse response (11) contains terms of the form $(T-t)^{\mu}t^{\kappa}$. Thus, as κ and μ are increased the "effective" support (see figure 2) of the filter is reduced and consequently, its spectrum spreads out.

Moreover, an algebraic numerical differentiator can be obtained by taking discrete values for ω and evaluating a discrete filter coefficients from (15) for each value of ω. In addition, the Discrete Fourier transform together with a spectrum discretization can also be used to design algebraic numerical differentiators.

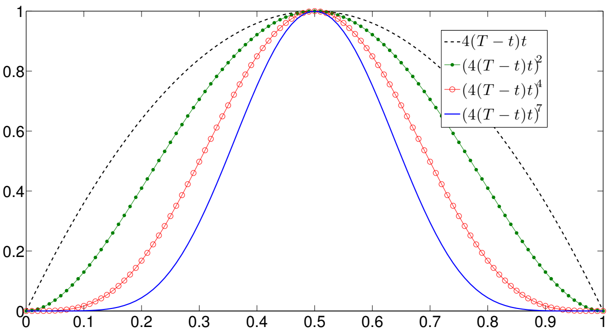

Fig. 2. Increasing κ and μ reduces the "effective" support of the smoothing filter.

Let us discuss in the sequel two particular cases for the minimal estimator (8): $\kappa=\mu$ and $\kappa\neq\mu$.

3.1 Minimal estimators: The case $\kappa=\mu$

With $\kappa=\mu$, the polynomial $[(T-t)t]^{\kappa}$ is symmetric with respect to $t=\frac{T}{2}$, consequently the phase of the smoothing filter (15) is linear and its spectrum is real.

The least squares interpretation of the estimators given in Mboup et al. [2009] revealed that the estimator (8) induces an estimation delay. In particular this delay is equal to $\frac{T}{2}$ whenever $\kappa=\mu$. This claim can be demonstrated also from the frequency analysis since the phase shift of (8) with $\kappa=\mu$ is given by $\mathrm{Arg}(H_{\kappa=\mu,n}(\omega)) = -\frac{T}{2}\omega$.

3.2 Minimal estimators: The case $\kappa\neq\mu$

If $\kappa\neq\mu$ the smoothing filter (15) do not admit a symmetry and consequently its phase is nonlinear. Typical gain and phase diagrams are shown on the figures 3 and 4. One can notice that the attenuation of high frequencies decreases as κ increases. Thus, κ should be kept equal to μ. Note finally that the same conclusion applies if one chooses to increase μ instead of κ since the following gain relation from (15) is satisfied:

$$|H_{\kappa,\mu,n}(\omega)| = |H_{\mu,\kappa,n}(\omega)|.$$

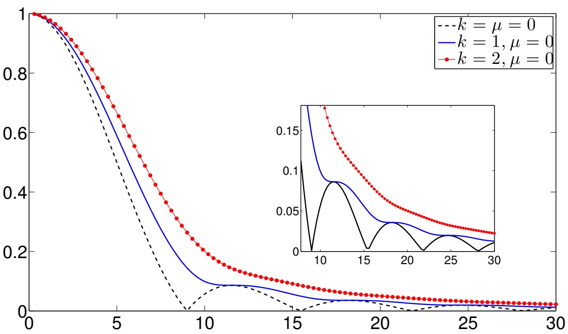

Fig. 3. Minimal estimator: Influence of increasing κ and μ ($\kappa\neq\mu$). Gain diagram.

3.3 Non-minimal differentiators and filters cascades

Let us now investigate two interesting cases: the non-minimal estimators and the cascade of several filters.

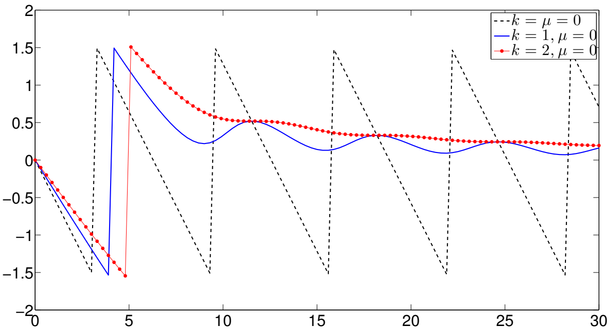

Fig. 4. Minimal estimator: Influence of increasing κ and μ ($\kappa \neq \mu$). Phase diagram.

The analysis will be carried for first and second order algebraic differentiators. Generalizations to higher orders are pointed out.

4. NON-MINIMAL DIFFERENTIATORS

The simplest non-minimal first order Algebraic Differentiator is calculated from (10) by setting $q = 1$ (*i.e* $N = n + 1$). The corresponding smoothing filter is given by $G(\omega) = \lambda_1 H_{1,0,1}(\omega) + \lambda_2 H_{0,1,1}(\omega)$ where $\lambda_1 = 3$ and $\lambda_2 = -2$ (the detailed computation of λ_1 and λ_2 can be found in Mboup et al. [2009] or Riachy et al. [2011]).

In addition, notice that the weights of the affine combination in (10) are restricted to $\mathbb{Q}$. Quite effective differentiators can be obtained by extending the set of parameters λ_l in (10) to $\mathbb{R}$ instead of $\mathbb{Q}$. In particular the differentiator obtained with the smoothing filter given by $F(\omega) = -0.618H_{1,0,1}(\omega) + 1.618H_{0,1,1}(\omega)$ is studied since it admits a higher order least squares interpretation (the detailed explanation of the least squares interpretation can be found in Mboup et al. [2009], the computation of $F(\omega)$ can be found in Riachy et al. [2011] section 7.2.1).

The spectra of the smoothing filter corresponding to: 1) the minimal estimator $H_{0,0,1}(\omega)$, 2) $G(\omega)$ and 3) $F(\omega)$ are plotted in figures 5 and 6. Let us briefly recall the main features of these differentiators, that was established in Mboup et al. [2009] using the least squares interpretation:

- The non-minimal differentiator associated with $G(\omega)$, which follows directly from (5), does not undergo a time delay. However, performance loss compared to even the minimal case was observed.
- Any point in the $\mathbb{R}$-affine hull of $\mathcal{S}_{\kappa,\mu,q}$ as in (10) corresponds to an algrebraic differentiator with a given delay. The non-minimal differentiator associated with $F(\omega)$ is such a point, with a judiciously chosen delay.

The frequency domain interpretation provides a simple demonstration of these facts as illustrated in the next figures. Indeed, $G(\omega)$ considerably reduces the phase shift and consequently the estimation time delay, as compared to $H_{0,0,1}(\omega)$ (compare the blue solid-line with the black dashed-line curves in Fig. 6). However, $G(\omega)$ amplifies by 150% the low-frequency content of the signal (see blue solid-line curve in Fig. 5) which may not be tolerable in practice. Note that while the performance improvement due to a deliberate introduction of a properly selected delay was clearly established in Mboup et al. [2009], the very bad performance of the non-minimal non-delayed algebraic differentiator was so far unexplained. Meanwhile, the estimator $F(\omega)$ not only reduces the phase shift compared to the minimal estimator (green dot-solid-line, in Fig. 6) but also, it offers a flat region within the bandwidth as indicated on the gain plot (green dot-solid-line, in Fig. 5). This is indeed a very interesting algebraic differentiator, as already known from the least squares interpretation of Mboup et al. [2009].

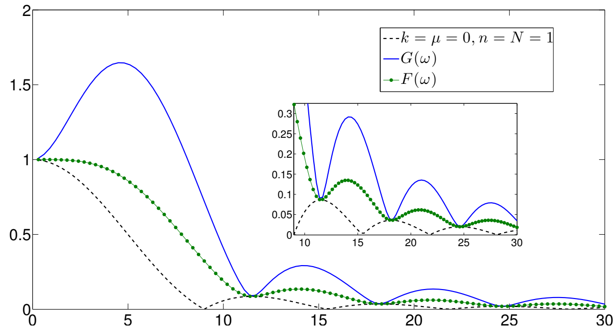

Fig. 5. Non-minimal estimators. Gain diagram.

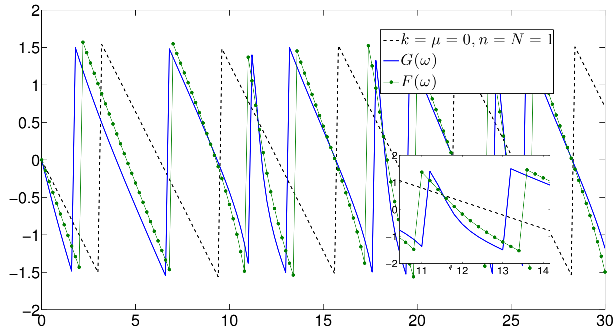

Fig. 6. Non-minimal estimators. Phase diagram.

5. CASCADE OF DIFFERENTIATORS

When estimating high order derivatives of a noisy signal, a natural question arises whether is it better to apply a high order algebraic differentiator or to cascade several first order algebraic differentiators. Moreover, if cascading several first order differentiators gives a superior result, is it better to use same values for $\kappa = \mu$ for the cascaded filters or not?

Note that this aspect has not been addressed in Mboup et al. [2009] since an analysis via the least squares interpretation is not straightforward. The frequency domain interpretation provides simple and immediate answers to these questions.

Let us provide some possible answers by examining the gain diagrams on figure 7 corresponding to the smoothing filters of second order differentiators.

One can immediately notice the poor filtering capabilities of the smoothing filter $H_{0,0,2}(\omega)$ corresponding to the second order algebraic differentiator (see the black dashed line in figure 7). An increased attenuation of high frequencies can be achieved by cascading two first order differentiators with smoothing filters $H_{0,0,1}(\omega) \times H_{0,0,1}(\omega)$ as it can be noticed from the red dot-solid-line on the figure 7. Note finally that an additional attenuation of

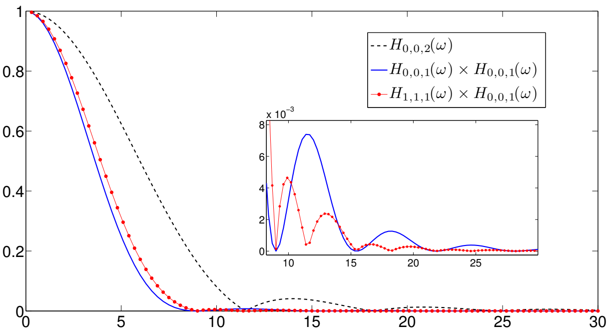

Fig. 7. Second order differentiation: Cascade of first order differentiators.

high frequencies can be obtained by using the differentiator whose smoothing filter is given by $H_{0,0,1}(\omega) \times H_{1,1,1}(\omega)$ instead of $H_{0,0,1}(\omega) \times H_{0,0,1}(\omega)$ since its gain diagram crosses zero twice as much as $H_{0,0,1}(\omega) \times H_{1,1,1}(\omega)$ (see the zoom in figure 7).

6. CONCLUSION

A frequency domain interpretation is provided for the Algebraic Differentiators of Mboup et al. [2009]. It complements the least squares interpretation given in Mboup et al. [2009] and reveals additional properties of these differentiators. Moreover, it permits the use of the Fourier theory in order to choose and tune the parameters of an Algebraic Differentiator.

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