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Nonlinear stabilization by dynamic parameter adaptation: Algebraic-Ricatti-Equation-based approach ⋆

S. Ibrir * **M. Bettayeb** ** **C. Y. Su** ***

* *King Fahd University of Petroleum and Minerals, Electrical Engineering Department, KFUPM Box 5038, Dhahran 31261, KSA (e-mail: salimibrir@ieee.org; sibrir@kfupm.edu.sa)*
** *University of Sharjah, Department of Electrical and Computer Engineering, University City, Sharjah, UAE. King Abdulaziz University, Jeddah, KSA. (e-mail: maamar@sharjah.ac.ae)*
*** *Concordia University, Department of Mechanical and Industrial Engineering, Montreal, Canada, (email: cysu@encs.concordia.ca)*

Abstract: A dynamic-gain parameterized controller is proposed to stabilize a class of nonlinear systems subject to norm-bounded uncertainties. The stabilizing controller is designed as a standard linear feedback with polynomial dynamic parameters. The expression of the dynamic parameters is defined from the solution of an Algebraic Ricatti Equation. The pendulum cart system with other examples are given as illustrative case studies to show the simplicity, the straightforwardness and the efficiency of the proposed design.

Keywords: Nonlinear systems, SDRE-based control, Dynamic-gain parameterization feedback, Feedforward nonlinear systems, Pendulum System.

1. INTRODUCTION

Nonlinear control feedback has known a considerable progress since three decades and many approaches have been developed to the control and analysis of inherently nonlinear systems, see e.g. Isidori [1995], Khalil [2001]. Due to the complex nature of the dynamics of nonlinear systems the available results are related to some classes of systems with specific structures. Systems with strict-feedback and feedforward structures are the most popular classes of systems that have been extensively studied since 1990s. For systems in strict feedback form, it was shown that it is possible to dominate the system nonlinearities by applying high-gain feedbacks. However, for systems in feedforward form, the suppression of instabilities can be handled by the construction of bounded-state feedbacks with arbitrary level of saturation, see e.g. Teel [1992], Xudong [2003]. Other seminal works that dealt with analysis and stabilization of feedforward systems are traced in Tall [2011], Sepulchre et al. [1997], Krstic [2004], Lin [1995], Krishnamurthy and Khorrami [2007], Arcak et al. [2001], Xudong [2003], Chen and Huang [2008] and the references therein.

The nested-saturation control of feedforward systems has been widely used to prevent finite-time escape to infinity. A redesign of the nested-saturation feedback algorithm is given in Arcak et al. [2001] where the feedforward system is subject to unmodeled dynamics. Using the fact that the trajectories of feedforward systems can be generally found for a given known input then, it is quite possible to predict the evolution of the system states if a linear feedback controller is conceived to stabilize the linear part of the system dynamics. From this important key observation, the idea of forwarding has been established giving rise to a systematic Lyapunov approach to stabilization of nonlinear feedforward systems, see Sepulchre et al. [1997] for more details. In Krstic [2004] the author showed that some special classes of feedforward systems can be transformed to the chain of integrators if the system dynamics is completely known. The dynamic-scaling approach to stabilization of feedforward systems has shown its usefulness in handling large classes of feedforward systems without any need to saturate the control law, see Krishnamurthy and Khorrami [2007]. In all the proposed algorithms the knowledge of the system input plays a key role in the existence of the stabilizing state-feedback and therefore, eventual uncertainty in the system input may prevent the system dynamics to be transformed to certain suitable/canonical forms and may lead to disastrous escape to infinity in finite time.

For polynomial systems, the State-Dependent-Ricatti-Equation (SDRE) approach has shown its usefulness in controlling complex systems under some conditions of controllability and observability, see e.g., Shamma and Cloutier [2003]. However, the proof of stability is not in general a trivial task. In this paper, we propose a straightforward control procedure for a class of uncertain nonlinear systems where the stability of the closed-loop system is

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assured for arbitrary types of nonlinearities verifying certain prescribed growth conditions. The control action is set as a linear state feedback with time-varying parameters. Irrespective of the number of the uncertain parameters, the feedback formulation is dependent on one unique adaptive parameter. Conceptually, the proposed feedback has the structure of an SDRE-based feedback where the stability is assured by adaptation of one parameter. Illustrative examples are provided to highlight the main features, the strength and the weaknesses of the proposed control approach.

Throughout this paper, $\dot{x}$ denotes the first derivative of x with respect to time. The notation $A > 0$ (resp. $A < 0$) means that the matrix A is positive definite (resp. negative definite). I is the identity matrix of appropriate dimension and A' denotes the matrix transpose of A. $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ stands for the largest and the lowest eigenvalue of A, respectively. We note by $\| \cdot \|$ the Euclidean norm of vector or matrices, and $\delta_{i,j}$ stands for the kronecker symbol.

2. SYSTEM DYNAMICS AND CONTROLLER DESIGN

Consider the feedforward nonlinear system given by:

$$\begin{aligned} \dot{x}_i &= x_{i+1} + \sum_{j=i+2}^{n} \varphi_{i,j}(x,p)\, x_j; \\ &\qquad\qquad 1 \le i \le n-2, \\ \dot{x}_{n-1} &= x_n, \\ \dot{x}_n &= u, \end{aligned} \tag{1}$$

where $x_i = x_i(t)$, $1 \le i \le n$ are the state variables, $u = u(t)$ is the system input, and $u \in \mathbb{R}$ is the unique control input. The scalar state-dependent nonlinearities $\varphi_{i,j}(x,p)$ are corrupted by a set of parameters regrouped in the vector $p = [p_1\ p_2 \cdots p_r]'$. To complete the system description the following assumptions are considered.

Assumption 1. The system nonlinearities $\varphi_{i,j}(x,p)\, x_j$; $1 \le i,j \le n$, are identically null when $x = 0$ and locally Lipschitz for all $x \in \mathbb{R}^n$ with $\varphi_{i,j}(x,p) = 0$ for all $i - j + 1 \ge 0$. Furthermore, we assume that there exists a set of well-known functions $\bar{\varphi}_{i,j}(x_{i+2}, x_{i+3}, \cdots, x_n)$, $1 \le i,j \le n$ such that for all $x \in \mathbb{R}^n$, we have

$$|\varphi_{i,j}(x,p)| \le \bar{\varphi}_{i,j}(x_{i+2}, x_{i+3}, \cdots, x_n),\ 1 \le i,j \le n. \tag{2}$$

Assumption 2. The upper bounds of $(p_i)_{1 \le i \le r}$ are known.

Assumption 3. System (1) is pointwise controllable in the sense of Kalman with respect to the input u.

In matrix form system (1) is rewritten as follows:

$$\dot{x} = A\,x + \varphi(x,p)\,x + Bu, \tag{3}$$

where the entries of the matrices $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times 1}$ are given by: $a_{i,j} = \delta_{i,j-1}$, $b_i = \delta_{i,n}$; $1 \le i,j \le n$. Before presenting the main result of this paper, the result of the following technical Lemma is needed.

Lemma 1. Let $X(\gamma)$ be the solution of the following Algebraic Ricatti Equation:

$$\frac{1}{\gamma}X(\gamma) + A'X(\gamma) + X(\gamma)A - X(\gamma)BB'X(\gamma) = 0, \tag{4}$$

where the entries of the matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n \times 1}$ are given by: $a_{i,j} = \delta_{i,j-1}$, $1 \le i,j \le n$, $b_i = \delta_{i,n}$, $1 \le i \le n$, respectively. Then, the matrix $X(\gamma)$ that solves the matrix equation (4) is symmetric and positive definite for all $\gamma > 0$ and verifies the following properties.

i) Let $X_1 = X(1)$. Then, the matrix $X(\gamma)$ is explicitly given by:

$$X(\gamma) = \frac{1}{\gamma}S^{-1}(\gamma)X_1 S^{-1}(\gamma),\ S(\gamma) = \mathrm{diag}\big(\gamma^{n-i},\ 1 \le i \le n\big). \tag{5}$$

ii) There exist two constants λ_1 and λ_2, independent of γ, such that for $\gamma > 0$;

$$-\frac{\lambda_1}{\gamma}X(\gamma) \le \frac{d}{d\gamma}X(\gamma) \le -\frac{\lambda_2}{\gamma}X(\gamma). \tag{6}$$

iii) The matrix $\dfrac{dX(\gamma^{\alpha})}{d\gamma}$ is also negative definite for any natural number $\alpha \ge 1$ with γ being positive.

Proof. See the Appendix Section.

The stabilization procedure is summarized in the following statement.

Theorem 1. Consider the nonlinear system (1) under Assumptions 1-3. Let $\beta > 0$ be a real constant and let $X(\gamma)$ be the solution of the matrix equation (4) where $X_1 = X(1)$, and $X'_{\mathrm{ch}}X_{\mathrm{ch}}$ being its Choelesky decomposition. Define

$$\begin{aligned} \Gamma &= \mathrm{diag}(n-i+1,\ 1 \le i \le n), \\ W &= (X'_{\mathrm{ch}})^{-1}(\Gamma X_1 + X_1\Gamma - X_1)X_{\mathrm{ch}}^{-1}, \\ \lambda &= \frac{1}{2 + 4\lambda_{\max}(W)}, \end{aligned} \tag{7}$$

and let $\rho(x)$ be a locally Lipschitz and positive state-dependent scalar verifying $\rho(x) \ge 2 \sup_{p \in \Omega} \|X_{\mathrm{ch}}\,\varphi(x,p)X_{\mathrm{ch}}^{-1}\| + \lambda$ with $\rho(0) = \lambda$. Then, under the action of the feedback controller:

$$\begin{aligned} u &= -\theta^{\frac{2\beta}{3}}B'X(\theta^{\frac{2}{3}})x, \\ \dot{\theta} &= -3\lambda\,\theta^{\frac{1}{3}} + \frac{3\rho(x)}{\theta^{\frac{1}{3}}},\ \theta(0) > 1 \end{aligned} \tag{8}$$

the system state trajectories $x(t)$ converge to zero for all bounded initial conditions $x_0 \in \mathbb{R}^n$. Furthermore, the adaptation parameter $\theta(t)$ is globally bounded over the interval of time $t \in [0, +\infty[$.

Remark 1. The dynamics of θ, as defined in (8), is not singular at any instant because $\rho(x)$ is strictly positive even for $x = 0$ and $\theta(0) > 1$. The equilibrium point of "θ dynamics" is unique and equal to one for $x = 0$. Therefore, under all the above assumptions, the trajectories of θ are always strictly positive reals.

Proof. The proof of Theorem 1 is given in the Appendix Section.

Remark that the term $-\theta^{\frac{2\beta}{3}}B'X(\theta^{\frac{2}{3}})x$ is nothing but, a State-Dependent-Ricatti-Equation-based controller. In this paper, the existence of a positive definite matrix $X(\cdot)$ is completely solved whatever the system nonlinearities; which is not always possible if the original system (1) is seen as $\dot{x} = \psi(x,p)\,x + Bu$, where $\psi(x,p) \in \mathbb{R}^{n\times n}$ is the state-dependent matrix. Referring to Shamma and Cloutier [2003], the controllability and the stability issues related to SDRE [1]-based solutions is not in general solvable even though many stabilizing controllers have SDRE structure.

3. CASE STUDIES AND SIMULATION

3.1 Stabilization of feed-forward nonlinear system

Consider the feed-forward nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 + p_1\, x_3^2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= u, \end{aligned} \tag{9}$$

where p_1 is assumed to be unknown time-varying parameter. Note that the following example has served as a historical toy example to illustrate stabilization techniques for feed-forward nonlinear systems. An additional uncertainty constraint is added to the system by imposing a time-varying bounded uncertainty, that is: $p_1 = \sin(3\,t)$. The controller parameters are chosen as: $\beta = 4$, $\rho(x) = 2\|X_{ch}\varphi(x,1)X_{ch}^{-1}\| + \lambda$. In Fig. 1, the history of all the states are recorded where we notice the asymptotic convergence of the whole states to the origin with global boundedness of the adaptive parameter θ. Extensive simulations showed that the peaking of the system states depends on the choice of the parameter β and the initial conditions of the system.

3.2 Semi-global stabilization of the pendulum-cart system

Let us consider the cart-pendulum system whose dynamics is given by:

$$\begin{aligned} (m_1+m_2)\ddot{q}_1 + m_2\, l\, \cos(q_2)\, \ddot{q}_2 &= m_2 l\, \sin(q_2)\dot{q}_2^2 + F, \\ \cos(q_2)\ddot{q}_1 + l\, \ddot{q}_2 &= g\, \sin(q_2), \end{aligned} \tag{10}$$

where F is the applied control input, m_1 and m_2 are the masses of the cart and the pendulum, respectively, l is the length of the pendulum, q_1 is the displacement of the cart, and q_2 is the rotation angle of the pendulum. It has been shown in Tall and Respondek [2005] that for $-\pi/2 < q_2 < \pi/2$, the feedback controller:

$$\begin{aligned} F = &-u\, l(m_1 + m_2 \sin^2(q_2))/\cos(q_2) \\ &+ (m_1+m_2)\, g\, \tan(q_2) - m_2\, l\, \sin(q_2)\, \dot{q}_2^2 \end{aligned} \tag{11}$$

renders the closed-loop dynamics of the cart-pendulum system in the form:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= g\, \tan(x_3) - \frac{l\,u}{\cos(x_3)}, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= u, \end{aligned} \tag{12}$$

where $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$. As proposed in Tall and Respondek [2005], and by setting the transformation $\lambda = \sqrt{l/g}/g$, $z_i = \lambda\,\tilde{x}_i$, $1 \le i \le 4$, $v = \lambda\,\tilde{u}$, where

$$\begin{aligned} \tilde{u} &= \frac{g\,u}{\cos^2(x_3)} + 2g\, x_4^2 \frac{\sin(x_3)}{\cos^3(x_3)}, \\ \tilde{x}_1 &= x_1 + l \int_0^{x_3} \frac{d\,s}{\cos(s)}, \\ \tilde{x}_2 &= x_2 + l \frac{x_4}{\cos(x_3)}, \\ \tilde{x}_3 &= g\, \tan(x_3), \\ \tilde{x}_4 &= g \frac{x_4}{\cos^2(x_3)} \end{aligned}$$

then, the resulting system is given by:

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3 + \frac{z_3 z_4^2}{(1+g z_3^2/l)^{\frac{3}{2}}}, \\ \dot{z}_3 &= z_4, \\ \dot{z}_4 &= v. \end{aligned} \tag{13}$$

For $\beta = 4$, $\lambda_1 = 10$, $g = 9.8$, $l = 1$ and $x(0) = \left[1\ 2\ \frac{\pi}{4}\ 3\right]'$, $1 \le i \le 4$, the feedback is

$$v = -\frac{x_1}{\theta^{\frac{16}{9}}} - 4\,\frac{x_2}{\theta^{\frac{10}{9}}} - 6\,\frac{x_3}{\theta^{\frac{4}{9}}} - 4\,\theta^{\frac{2}{9}} x_4. \tag{14}$$

The dynamics of the adaptation θ is dependent on $\rho(z)$, that is chosen as

$$\rho(z) = 2\sqrt{10}\sqrt{\frac{l^3 {z_3}^2 {z_4}^2}{\left(g{z_3}^2 + l\right)^3}} + \frac{1}{21}. \tag{15}$$

In Fig. 2 the trajectories of z-state system under the feedback $v = -\theta^{\frac{2\beta}{3}}\, B'X(\theta^{\frac{2}{3}})z$, are represented. In order to avoid the singularities of state transformation, the feedback (14) can be only applied when the absolute value of the pendulum angle is less than $\frac{\pi}{2}$.

3.3 Stabilization of a system of arbitrary structure

Consider the nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 + x_3^2, \\ \dot{x}_2 &= x_3 + \frac{x_1}{\sqrt{1+x_1^2}} x_4^2, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= u, \end{aligned} \tag{16}$$

where $u \in \mathbb{R}$ is the control input. The system dynamics (16) is neither in feedback canonical form nor in feedforward form. The nonlinearities of system (16) verify all the required Assumptions 1-3. Thus, by choosing $\beta = 6$, $\lambda_1 = 10$, and $\rho(x)$ as

$$8\, \max\left(|x_3| + \frac{\left|x_3\sqrt{{x_1}^2+1} - 3\,x_1 x_4\right|}{\sqrt{{x_1}^2+1}}, \frac{|x_1 x_4|}{\sqrt{{x_1}^2+1}}\right) + \frac{1}{42} \tag{17}$$

the controller is then given by:

$$\begin{aligned} u &= -\frac{x_1}{\theta^{\frac{4}{3}}} - 4\,\frac{x_2}{\theta^{\frac{2}{3}}} - 6\,x_3 - 4\,\theta^{\frac{2}{3}} x_4, \\ \dot{\theta} &= -3\lambda\,\theta^{\frac{1}{3}} + \frac{3\rho(x)}{\theta^{\frac{1}{3}}},\ \theta(0) > 1. \end{aligned} \tag{18}$$

For the initial condition $x_0 = [-4\ 3\ 0.5\ 1]'$ and $\theta(0) = 1.1$, we have recorded the history of the states in Fig. 3. By performing extensive simulations with different initial

[1] State Dependent Ricatti Equation.

conditions, we have seen that the amount of peaking of x_1 and x_2 is dependent on the initial conditions of x_3 and x_4 and the selected value of β. For large initial conditions of $x_3(0)$ and $x_4(0)$ the state x_1 converge to zero after a long period of time with a slow peaking.

4. CONCLUSION

This paper gives a straightforward method for the stabilization of a class of nonlinear systems subject to unknown state uncertainties. It has been shown that systems with feedforward structure fall in the studied class system and the efficiency of the control feedback is confirmed by numerical simulations. One of the advantages of the proposed control strategy is that all the uncertain parameters are handled by adaptation of only one parameter. This in turn reduces the number of the control-state variables to be used in feedback, and makes the feedback design similar to well-known stabilization techniques of linear systems. The proposed design can be also extended to other types of nonlinear systems, where the system nonlinearities $\varphi_{i,j}$; $i-j+1<0$ involve the whole state vector. Connection between the proposed approach and previous designs of SDRE-based controllers is highlighted.

Appendix A. Proof of Lemma 1.

Let

$$E(\gamma) = -A - \frac{1}{2\gamma}I. \tag{19}$$

Note that for all $\gamma > 0$ the matrix $X^{-1}(\gamma)$ verifies the following Lyapunov equation:

$$X^{-1}(\gamma)E'(\gamma) + E(\gamma)X^{-1}(\gamma) = -BB'. \tag{20}$$

Since the matrix $E(\gamma)$ is Hurwitz for all $\gamma > 0$ then, from the Lyapunov theory, there exists a symmetric and positive definite matrix $X^{-1}(\gamma)$ that solves the Lyapunov equation.

i) Note that X_1 verifies the following ARE:

$$X_1 + A'X_1 + X_1A - X_1BB'X_1 = 0. \tag{21}$$

To prove the first item of Lemma 1, let us pre- and post multiplying the ARE (21) by $S^{-1}(\gamma)$. This yields

$$\begin{aligned} &S^{-1}(\gamma)X_1S^{-1}(\gamma) + S^{-1}(\gamma)A'X_1S^{-1}(\gamma) \\ &+ S^{-1}(\gamma)X_1AS^{-1}(\gamma) - S^{-1}(\gamma)X_1BB'X_1S^{-1}(\gamma) = 0. \end{aligned} \tag{22}$$

Using the fact that $S^{-1}(\gamma)A' = \gamma A'S^{-1}(\gamma)$, $S^{-1}(\gamma)B = B$ then, (22) takes the form:

$$\begin{aligned} &S^{-1}(\gamma)X_1S^{-1}(\gamma) + \gamma A'S^{-1}(\gamma)X_1S^{-1}(\gamma) \\ &+ \gamma S^{-1}(\gamma)X_1S^{-1}(\gamma)A \\ &- S^{-1}(\gamma)X_1S^{-1}(\gamma)BB'S^{-1}(\gamma)X_1S^{-1}(\gamma) = 0. \end{aligned} \tag{23}$$

By dividing Eq. (23) by γ^2 we conclude immediately that $X(\gamma) = S^{-1}(\gamma)X_1S^{-1}(\gamma)/\gamma$ is the solution of (4).

ii) Let $\gamma_1 > \gamma_2 > 0$, and let $X(\gamma_1)$ and $X(\gamma_2)$ be the solutions of (4) for $\gamma = \gamma_1$, and $\gamma = \gamma_2$, respectively. Then,

$$\begin{aligned} &-\gamma_1^{-1}X^{-1}(\gamma_1) + \gamma_2^{-1}X^{-1}(\gamma_2) - \big(X^{-1}(\gamma_1) - X^{-1}(\gamma_2)\big)A' \\ &- A\big(X^{-1}(\gamma_1) - X^{-1}(\gamma_2)\big) = 0. \end{aligned} \tag{24}$$

The last matrix equation can be rewritten as $(X^{-1}(\gamma_1) - X^{-1}(\gamma_2))E'(\gamma_1) + (X^{-1}(\gamma_1) - X^{-1}(\gamma_2))E(\gamma_1) = -(\gamma_2^{-1} - \gamma_1^{-1})X^{-1}(\gamma_2)$. Since the matrix $(\gamma_2^{-1} - \gamma_1^{-1})X^{-1}(\gamma_2) > 0$ and the matrix $E(\gamma_1)$ is Hurwitz then, we conclude that the matrix $X^{-1}(\gamma_1) - X^{-1}(\gamma_2) > 0$; which means that $X(\gamma_1) < X(\gamma_2)$. Consequently, the ratio:

$$\frac{X(\gamma_1) - X(\gamma_2)}{\gamma_1 - \gamma_2} < 0. \tag{25}$$

If we take $\gamma_1 = \gamma + \delta\gamma$ and $\gamma_2 = \gamma$ it can be concluded that

$$\lim_{\delta\gamma \to 0} \frac{X(\gamma + \delta\gamma) - X(\gamma)}{\delta\gamma} = \frac{dX(\gamma)}{d\gamma} < 0. \tag{26}$$

Let $\Upsilon(\gamma) = S^{-1}(\gamma)/\gamma$. Using the result of item i) then,

$$\begin{aligned} \frac{dX(\gamma)}{d\gamma} &= \frac{1}{\gamma^2}S^{-1}(\gamma)X_1S^{-1}(\gamma) \\ &+ \left(\frac{d\Upsilon(\gamma)}{d\gamma}\right)X_1S^{-1}(\gamma) + S^{-1}(\gamma)X_1\left(\frac{d\Upsilon(\gamma)}{d\gamma}\right). \end{aligned}$$

Noting that

$$\frac{d\Upsilon(\gamma)}{d\gamma} = -\frac{\Gamma}{\gamma^2}S^{-1}(\gamma), \tag{27}$$

where Γ is defined as in the statement of Theorem 1. Consequently,

$$\frac{dX(\gamma)}{d\gamma} = \Upsilon(\gamma)\big(X_1 - \Gamma X_1 - X_1\Gamma\big)\Upsilon(\gamma). \tag{28}$$

As a result, the matrix $X_1 - \Gamma X_1 - X_1\Gamma < 0$ which implies that the matrix $W_0 = \Gamma X_1 + X_1\Gamma - X_1 > 0$. By taking $\lambda_1 = \lambda_{\max}\big(X'^{-1}_{\mathrm{ch}}W_0X^{-1}_{\mathrm{ch}}\big)$, $\lambda_2 = \lambda_{\min}\big(X'^{-1}_{\mathrm{ch}}W_0X^{-1}_{\mathrm{ch}}\big)$; we arrive to the result

$$-\frac{\lambda_1}{\gamma}X(\gamma) \le \frac{dX(\gamma)}{d\gamma} \le -\frac{\lambda_2}{\gamma}X(\gamma). \tag{29}$$

iii) This result is a direct consequence of the result of item ii), because for any $\alpha \ge 1$, we have

$$\frac{dX(\gamma^\alpha)}{d\gamma} = \alpha\gamma^{\alpha-1}\frac{dX(\gamma^\alpha)}{d\gamma^\alpha} < 0, \tag{30}$$

and

$$-\frac{\alpha\lambda_1}{\gamma}X(\gamma^\alpha) \le \frac{dX(\gamma^\alpha)}{d\gamma} \le -\frac{\alpha\lambda_2}{\gamma}X(\gamma^\alpha). \;\square$$

Appendix B. Proof of Theorem 1. Let $\gamma = \theta^{\frac{1}{3}}$. By associating the Lyapunov function: $V(x,\gamma) = x'X(\gamma^2)\,x/\gamma$ to the closed-loop dynamics:

$$\dot{x} = A\,x + \varphi(x,p)\,x - \theta^{\frac{2\beta}{3}}BB'X(\theta^{\frac{2}{3}})x.$$

Then,

$$\dot{V}(x,\gamma) = -\frac{\dot{\gamma}}{\gamma^2}x'X(\gamma^2)x + \frac{2}{\gamma}x'X(\gamma^2)\dot{x} + \frac{\dot{\gamma}}{\gamma}x'\frac{dX(\gamma^2)}{d\,\gamma}x. \tag{31}$$

We have

$$\begin{aligned} \frac{2}{\gamma}x'X(\gamma^2)\dot{x} &= \frac{2}{\gamma}x'X(\gamma^2)\big(A - \gamma^{2\beta}BB'X(\gamma^2)\big)x \\ &+ \frac{2}{\gamma}x'X(\gamma^2)\,\varphi(x,p)\,x. \end{aligned} \tag{32}$$

By using the information of the system dynamics we have the following bound

$$
\begin{aligned}
&\frac{2}{\gamma}x'X(\gamma^2)\dot{x} \\
&\le \frac{1}{\gamma}x'\left[A'X(\gamma^2)+X(\gamma^2)A-2\gamma^{2\beta}X(\gamma^2)BB'X(\gamma^2)\right]x \\
&+\frac{2}{\gamma}x'X(\gamma^2)\,\varphi(x,p)\,x.
\end{aligned}
\tag{33}
$$

Using (4) along with inequality (33) then,

$$
\begin{aligned}
\frac{2}{\gamma}\,x'X(\gamma^2)\dot{x} &\le -\frac{1}{\gamma^3}x'X(\gamma^2)x \\
&+(-2\gamma^{2\beta}+1)\,x'X(\gamma^2)BB'X(\gamma^2)x/\gamma \\
&+\frac{2}{\gamma}x'X(\gamma^2)\,\varphi(x,p)\,x.
\end{aligned}
\tag{34}
$$

Since $X(\gamma^2) = S^{-1}(\gamma^2)X_1S^{-1}(\gamma^2)/\gamma^2$; where $S(q) = \mathrm{diag}(q^{n-i},\ 1\le i\le n)$ then,

$$
\begin{aligned}
&\frac{2}{\gamma}x'X(\gamma^2)\,\varphi(x,p)\,x \\
&\le \frac{2}{\gamma}\left\|X_{\mathrm{ch}}S^{-1}(\gamma^2)\varphi(x,p)\,S(\gamma^2)X_{\mathrm{ch}}^{-1}\right\|x'X(\gamma^2)x \\
&\le \frac{2}{\gamma^5}\sup_{p\in\Omega}\left\|X_{\mathrm{ch}}\varphi(x,p)\,X_{\mathrm{ch}}^{-1}\right\|x'X(\gamma^2)x \\
&\le \frac{\rho(x)}{\gamma^5}\,x'X(\gamma^2)x.
\end{aligned}
\tag{35}
$$

From the dynamics of θ, one can easily extract the dynamics of the scalar γ, that is,

$$
\dot{\gamma} = -\frac{\lambda}{\gamma}+\frac{\rho(x)}{\gamma^3}. \tag{36}
$$

As a consequence, the term: $-2\gamma^{2\beta}+1<0\ \forall\, t$, and hence,

$$
\begin{aligned}
&\frac{2}{\gamma}x'X(\gamma^2)\dot{x} \le -\frac{1}{\gamma^3}x'X(\gamma^2)x+\frac{\rho(x)}{\gamma^5}\,x'X(\gamma^2)x \\
&\quad+(-2\gamma^{2\beta}+1)\,x'X(\gamma^2)BB'X(\gamma^2)x/\gamma.
\end{aligned}
\tag{37}
$$

This immediately implies that

$$
\frac{\dot{\gamma}}{\gamma}\,x'\frac{dX(\gamma^2)}{d\gamma}x \le -\frac{\lambda}{\gamma^2}\,x'\frac{dX(\gamma^2)}{d\gamma}x \le \frac{2\lambda\lambda_1}{\gamma^3}\,x'X(\gamma^2)x \tag{38}
$$

From (31) and taking into account (35), (37) and, (38), it is then possible to have a negative bound of the Lyapunov function $V(x,\gamma)$, that is

$$
\dot{V}(x,\gamma) \le -\frac{1}{2\gamma^2}V(x,\gamma). \tag{39}
$$

From the dynamics of the adaptive parameter "θ", see Eqs. (8), one can conclude that for $\rho(x)>0$ and $\theta=1$, the derivative of θ is always positive, which means that for any initial condition $\theta(0)>1$, the parameter $\theta\ge 1$ for all $t\ge 0$. Since $\dot{V}(x,\gamma)<0$ then the whole state are globally bounded. As matter of fact, the boundedness of the adaptive parameter θ is dependent on the boundedness of $\rho(x)$. This property can be checked by setting the following radially unbounded Lypapunov function: $V_\theta = \theta^{\frac{4}{3}}$, and show that: $\dot{V}_\theta = -4\lambda\,\theta^{\frac{2}{3}}+4\rho(x)$; hence, $\dot{V}_\theta = -4\lambda\sqrt{V_\theta}+4\rho(x)$. This characterizes the Finite-Time-Input-to-State Stability [2] of θ with respect to $\rho(x)$. Since "x" converges to zero then, it is obvious that θ remains bounded by the ISS property. This ends the proof. □

[2] See Reference Hong et al. [2008] for more details on finite-time-input-to-state stability.

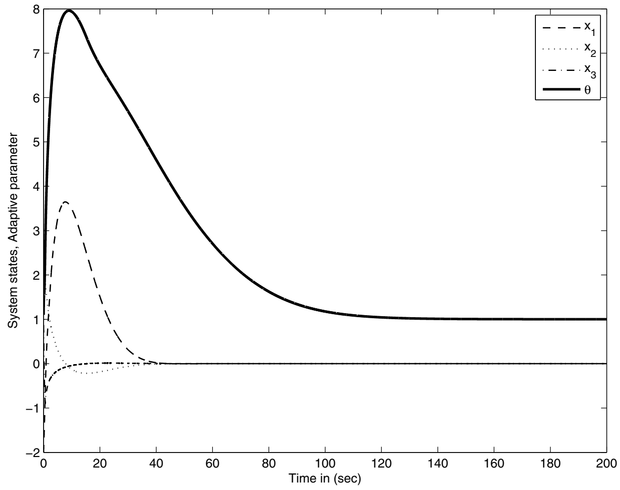

Fig. 1. The system states x_i, $1\le i\le 3$ and θ

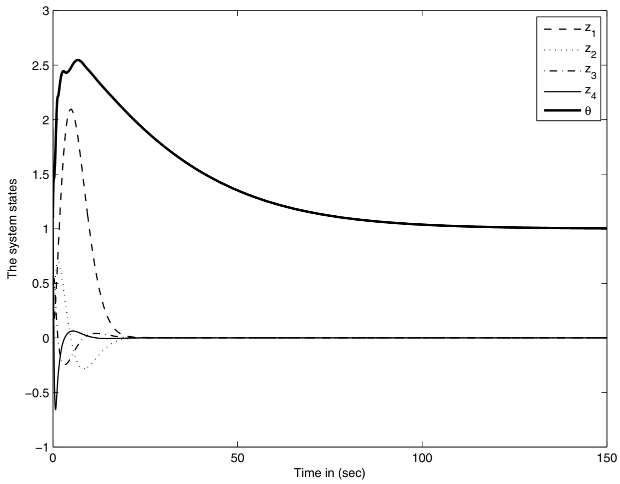

Fig. 2. The system states z_i, $1\le i\le 4$ and θ

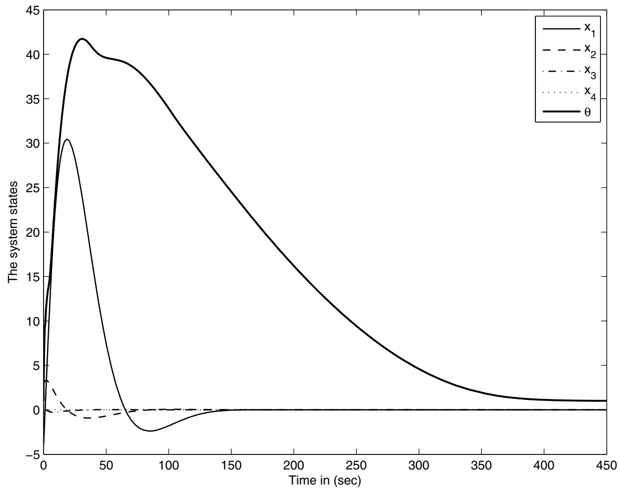

Fig. 3. The system states $(x_i)_{1\le i\le 4}$ and the adaptation θ

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