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Observer-based control of fractional-order linear systems with norm-bounded uncertainties: New convex-optimization conditions ⋆

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Abstract: New sufficient linear-matrix-inequality conditions are provided to ensure the stability of a class of fractional-order systems by means of asymptotic observer-based feedbacks. It is shown that the search of the observer and the controller gains can be obtained by decoupling the necessary matrix inequalities that involve coupled gains. The obtained numerically tractable conditions are formulated as a set of strict linear matrix inequalities and compared to other sufficient conditions with equality constraints. Numerical computations are provided to show the straightforwardness and the efficiency of the proposed control designs.

Keywords: Fractional order systems, Stability, Observer-based control, Linear Matrix Inequalities.

1. INTRODUCTION

Fractional-order calculus has a long history and its serves as a modern powerful tool in analyzing various physical phenomena. The interest in understanding systems governed by fractional-order differential equations has grown up during the last decades and many associated results have been appeared, see e.g., Manabe [1960], Matignon [1996], Farges et al. [2010], Trigeassou et al. [2011]. It was found that diffusion processes, biological systems, and other dynamics of real-world applications can be modeled in terms of fractional-order differential equations Sabatier et al. [2007]. Additionally, the use of fractional-order derivatives and integrals in feedback design has been successful to a large extent in improving the robustness of the closed-loop systems.

Nevertheless, fractional differential equations have not yet received the same attention as ordinary differential equations in the investigation of their stability, simulation, and analysis. Owing to the lack of effective analytic methods for the time-domain analysis and simulation of linear feedback fractional-order systems, a numerical simulation scheme is developed in Hwang et al. [2002]. Exact calculation of fractional-order derivatives of some particular polynomial signals is discussed in Samadi et al. [2004].

Stability of dynamical systems, represented by fractional order derivatives, has been investigated using the Routh-Hurwitz criteria, the pole placement method, and Lyapunov strategies. For linear fractional-order systems, it was found that the stability is equivalent to the repartition of the system poles in a restricted area of the complex plane. Based on this key formulation of stability and the use of convex-optimization algorithms, stated as linear-matrix-inequality conditions, numerous sufficient conditions have been proposed to ensure robust stability of some classes of fractional-order type systems, see e.g., Farges et al. [2010], Ahn and Chen [2008]. A considerable interest has been also devoted to stability and stabilizability of special classes of fractional-order systems, see e.g., Li et al. [2010], Wen et al. [2008]. A new Lyapunov stability analysis of fractional differential equations is discussed in Trigeassou et al. [2011]. The problem of pseudo-state feedback stabilization of fractional-order systems using LMI setting was addressed in Farges et al. [2010]. Other interesting topics in identification, observation, and control of fractional order systems can be traced in the references Wang and Gao [2012], Aoun et al. [2007], Sabatier et al. [2007], Li [2013] and the references therein.

The observer-based control problem is originally stated as a non-convex optimization issue due to the coupling conditions that must satisfy both the observer and the controller gains. For linear fractional-order systems, it has been shown that the stability is assured by placement of the system poles in a defined region of the complex plane. Therefore, the use of an observer-based controller to maintain the system stability, generally leads to a non-convex optimization problem, that must be solved numerically. In the recent paper Lan et al. [2012], the authors have presented a numerical scheme for stabilization of

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uncertain commensurate ($1 < \alpha < 2$) fractional-order systems by means of dynamic output feedback. In Lan et al. [2012] the authors have assumed that the uncertainties are only present in the state and the input matrix. Other recent works on stabilization of triangular fractional-order systems can be traced in Zhang et al. [2013] and the references therein. In this paper, we devote our attention to the control of uncertain fractional-order systems subject to partially-state measurements where the integer-differentiation order is between zero and two. Additionally, we assume that the system uncertainties are randomly distributed in the state matrix, the input matrix, and the output matrices as well. By decoupling the necessary conditions into a set of matrix inequalities, we show that the search of the observer and the controller gains can be transformed into a convex optimization problem. A set of sufficient linear-matrix-inequality conditions are developed to ensure the existence of an observer-based controller assuring the asymptotic stability of the system under consideration. A detailed proof is presented and the efficiency of the proposed design is testified by numerical simulations.

2. PRELIMINARIES

Throughout this paper we note by $\mathbb{R}$, $\mathbb{R}_{>0}$, and $\mathbb{C}$ the set of real number, the set of positive real numbers, and the set of complex numbers, respectively. The notation $A > 0$, with A being an Hermitian matrix (respectively, $A < 0$), means that the matrix A is positive definite (respectively, negative definite). A' is the matrix transpose of A. $X^\star$ stands for the complex conjugate transpose of the matrix X. The notation $\bar{X}$ stands for the matrix conjugate of the complex matrix X. The star element in a given matrix stands for any element that is induced by transposition. The spec(A) denotes the set of eigenvalues of the matrix A. We note by I and $\mathbf{0}$ the identity matrix of appropriate dimension and the null matrix of appropriate dimension, respectively. The Schur Complement Lemma is extensively used in the proof of the main statement, therefore, we would rather recall this important result. Let X be a symmetric real matrix given by:

$$X = \begin{bmatrix} A & B \\ B' & C \end{bmatrix}. \tag{1}$$

Let S be the Schur Complement of A in X, that is: $S = C - B'A^{-1}B$. Then,

- X is positive definite if and only if A and $S = C - B'A^{-1}B$ are both positive definite.
- $X > 0 \Leftrightarrow C > 0,\ A - BC^{-1}B' > 0$.
- If A is positive definite then X is positive semidefinite if and only if S is positive semidefinite; i.e., if $A > 0$ then, $X \geq 0 \Leftrightarrow S = C - B'A^{-1}B \geq 0$.
- If C is positive definite, then X is positive semidefinite if and only if $A - BC^{-1}B'$ is positive semidefinite; i.e., if $C > 0$ then, $X \geq 0 \Leftrightarrow A - BC^{-1}B' \geq 0$.
- In case of complex matrices, the matrix: $\begin{bmatrix} A & B \\ B^\star & C \end{bmatrix} > 0$ if and only if $C - B^\star A^{-1}B > 0$.

Lemma 1. (Boyd et al. [1994]) Given real matrices H and E of appropriate dimensions, the inequality:

$$HF(t)E + E'F'(t)H' < 0 \tag{2}$$

holds for all $F(t)$ satisfying $F'(t)F(t) \leq I$ if and only if there exists an $\varepsilon > 0$ such that

$$\varepsilon HH' + \varepsilon^{-1}E'E < 0. \tag{3}$$

In this paper, Riemann-Liouville fractional differentiation definition is used. Referring to Samko et al. [1987], the fractional integral of a function $f(t)$ is defined by:

$$I^\nu f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-\tau)^{\nu-1} f(\tau) d\tau, \tag{4}$$

where $\nu \in \mathbb{R}_{>0}$ denotes the fractional-integration order, and

$$\Gamma(\nu) = \int_0^{+\infty} e^{-x} x^{\nu-1}\, dx. \tag{5}$$

The order "ν" fractional derivative of a function $f(t)$, $\nu \in \mathbb{R}_{>0}$, is consequently defined by:

$$\begin{aligned} D^\nu f(t) &= \frac{d^m}{dt^m}\left(I^{m-\nu} f(t)\right) \\ &= \frac{1}{\Gamma(m-\nu)} \left(\frac{d}{dt}\right)^m \int_0^t (t-\tau)^{m-\nu-1} f(\tau)\, d\tau. \end{aligned} \tag{6}$$

where m is the smallest integer that exceeds or equal to "ν."

Depending on the value of the fractional-differentiation order "ν" , several stability theorems have been stated. In this study, we provide results for fractional-order systems where ν takes only non-integer values in]0 2[.

Theorem 1. Let $A \in \mathbb{R}^{n \times n}$ be a real matrix. Then, the fractional-order system:

$$D^\alpha x(t) = A\, x(t), \ \ 0 < \alpha < 2, \tag{7}$$

is asymptotically stable, that is, $|\arg(\mathrm{spec}(A))| > \alpha\frac{\pi}{2}$ if and only if there exists a symmetric and positive definite matrix P verifying

$$\begin{bmatrix} (AP + PA')\sin(\theta) & (AP - PA')\cos(\theta) \\ \star & (AP + PA')\sin(\theta) \end{bmatrix} < 0 \tag{8}$$

where $\theta = (1 - \frac{\alpha}{2})\pi$.

Proof. See Farges et al. [2010]. □

The following result concerns the stabilizability of fractional order linear systems by means of pseudo-state feedback where its proof is given in Farges et al. [2010]. As it has been reported in the literature, the stability of fractional-order linear systems is one particular case of domain stability where the eigenvalues of the system should be located in a specific region of the complex plane.

Theorem 2. (Farges et al. [2010]) The fractional-order system:

$$D^\alpha x(t) = A\, x(t) + B\, u(t) \tag{9}$$

where $0 < \alpha < 1$, $A \in \mathbb{R}^{n \times n}$, and $B \in \mathbb{R}^{n \times m}$, is stabilizable by pseudo-state feedback $u = Y(rX + \bar{r}\bar{X})^{-1}x$ iff $\exists X = X^\star \in \mathbb{C}^{n \times n} > 0$ and $Y \in \mathbb{R}^{m \times n}$ such that

$$(rX + \bar{r}\bar{X})'A' + A(rX + \bar{r}\bar{X}) + BY + Y'B' < 0, \tag{10}$$

where $r = e^{i(1-\alpha)\frac{\pi}{2}}$, $i^2 = -1$.

Theorem 2 will serve as a starting result for further development. The details are given in the following sections.

3. OBSERVER-BASED STABILIZATION

The objective of this section is to develop new LMI conditions for stabilization of fractional-order linear systems subject to state and output uncertainties. The first goal of the proposed development is to decouple the necessary conditions guaranteeing the convergence of the observer-based feedback into two sets of convex sufficient conditions that are numerically tractable. The first set of sufficient conditions assure the existence of a stabilizing feedback without involving the observer gain, and the second set of conditions guarantee the convergence of the observer without incorporating the controller gain. It will be shown later that this decomposition is also possible to derive sufficient stability conditions for uncertain fractional systems.

3.1 Strict sufficient conditions of stability

Consider the fractional-order linear system described by the following dynamics:

$$\begin{aligned} D^\alpha x &= (A+\Delta A)\,x + B\,u, \\ y &= (C+\Delta C)\,x, \end{aligned} \tag{11}$$

where α is the non-integer differentiation order ($0<\alpha<1$), $x=x(t)\in\mathbb{R}^n$ is the state vector, $u=u(t)\in\mathbb{R}^m$ is the control input, and $y=y(t)\in\mathbb{R}^p$ is the system measured output. The real-valued matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, and $C\in\mathbb{R}^{p\times n}$ are assumed to be known and constant. The uncertain parts $\Delta A\in\mathbb{R}^{n\times n}$ and $\Delta C\in\mathbb{R}^{p\times n}$ are assumed to be constant, bounded, and uncertain. We assume that there exist real-valued matrices M_A, M_C, N_A, N_C, F_A and F_C such that:

$$\begin{aligned} \Delta A &= M_A F_A N_A,\ F_A^{'} F_A \le I, \\ \Delta C &= M_C F_C N_C,\ F_C^{'} F_C \le I. \end{aligned} \tag{12}$$

Remark 1. In the general case of linear fractional-order systems of the following form:

$$\begin{aligned} D^\alpha x &= (A+\Delta A)\,x + (B+\Delta B)\,u, \\ y &= (C+\Delta C)\,x + (D+\Delta D)u, \end{aligned} \tag{13}$$

where A, B, C, D, ΔA, ΔB, ΔC, and ΔD are real matrices of appropriate dimensions, it is always possible to convert the dynamics (13) to form (11) by considering the input as an augmented state to the vector x and setting the new input as $v=D^\alpha u$.□

In this subsection, we present sufficient conditions under which an observer-based controller of the following form:

$$\begin{aligned} D^\alpha \hat{x} &= A\,\hat{x} + B\,u + L(C\hat{x}-y), \\ u &= K\,\hat{x}, \end{aligned} \tag{14}$$

could make the closed-loop system:

$$\begin{aligned} D^\alpha x &= (A+\Delta A)\,x + B\,u, \\ u &= K\,\hat{x}, \end{aligned} \tag{15}$$

globally stable, where L and K are some design matrices to be determined later. Before presenting the final result, let us introduce the following results.

Proposition 1. Let $r=e^{i\theta}$ where $i^2=-1$. If there exists an Hermitian matrix $X=X^\star>0$ such that

$$\frac{1}{\cos(\theta)}\Big((rX+\bar{r}\bar{X})+(rX+\bar{r}\bar{X})'\Big) + \frac{i}{\sin(\theta)}\Big((rX+\bar{r}\bar{X})-(rX+\bar{r}\bar{X})'\Big)>0 \tag{16}$$

then there exists an Hermitian matrix $Z=Z^\star>0$ such that

$$(rX+\bar{r}\bar{X})^{-1} = rZ+\bar{r}\bar{Z}. \tag{17}$$

Proof. Setting the matrix Z as

$$\begin{aligned} Z =& \frac{1}{4\cos(\theta)}\Big((rX+\bar{r}\bar{X})^{-1}+(rX'+\bar{r}\bar{X}')^{-1}\Big) \\ &- \frac{i}{4\sin(\theta)}\Big((rX+\bar{r}\bar{X})^{-1}-(rX'+\bar{r}\bar{X}')^{-1}\Big) \end{aligned} \tag{18}$$

then, one can verify that (17) holds. The positive-definiteness property of the matrix Z is verified by pre-multiplying inequality (16) by $(rX+\bar{r}\bar{X})^{-1}$ and post-multiplying the same inequality by $(rX'+\bar{r}\bar{X}')^{-1}$.

Theorem 3. Consider the fractional-order system (11) with $\Delta A=0$ and $\Delta C=0$. Let $\theta=(1-\alpha)\frac{\pi}{2}$ and $r=e^{i\theta}$ where $i^2=-1$. If there exist two complex, Hermitian and positive definite matrices $X_1=X_1^\star\in\mathbb{C}^{n\times n}$, $X_2=X_2^\star\in\mathbb{C}^{n\times n}$, a real, symmetric, and positive definite matrix $W\in\mathbb{R}^{n\times n}$, a positive scalar ε, and two matrices $Y_1\in\mathbb{R}^{m\times n}$, $Y_2\in\mathbb{R}^{n\times p}$ such that the following linear matrix inequalities hold true

$$\frac{1}{\cos(\theta)}(P_2+P_2')+\frac{i}{\sin(\theta)}(P_2-P_2')>0, \tag{19}$$

$$\begin{bmatrix} P_1+P_1'-W & I \\ \star & (2\varepsilon-1)I \end{bmatrix}>0, \tag{20}$$

$$\begin{bmatrix} P_1'A'+AP_1+BY_1+Y_1'B' & BY_1 \\ \star & -I \end{bmatrix}<0 \tag{21}$$

$$\begin{bmatrix} A'P_2+P_2'A+Y_2C+C'Y_2' & \varepsilon I \\ \star & -W \end{bmatrix}<0 \tag{22}$$

where $P_1=(rX_1+\bar{r}\bar{X}_1)$ and $P_2=(rX_2+\bar{r}\bar{X}_2)$ then, the observer-based feedback $u=Y_1P_1^{-1}\hat{x}$ stabilizes system (11) when all the uncertainties are null with $\hat{x}$ being the state trajectory of the fractional-order observer:

$$\begin{aligned} D^\alpha \hat{x} &= A\,\hat{x} + B\,u + (rX_2'+\bar{r}\bar{X}_2')^{-1}Y_2(C\hat{x}-y), \\ u &= Y_1(rX_1+\bar{r}\bar{X}_1)^{-1}\,\hat{x}. \end{aligned} \tag{23}$$

Proof. Let $e=\hat{x}-x$. Then, the observer-error dynamics along with the system dynamics verify the following composite system:

$$D^\alpha\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} A+BY_1P_1^{-1} & BY_1P_1^{-1} \\ \mathbf{0} & A+P_2'^{-1}Y_2C \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix}. \tag{24}$$

According to the result of Theorem 2, system (24) is stable if there exists a positive-definite Hermitian matrix $\mathscr{X}=\mathscr{X}^\star$ of dimensions $2n\times 2n$ such that

$$\begin{aligned} &\begin{bmatrix} A+BY_1P_1^{-1} & BY_1P_1^{-1} \\ \mathbf{0} & A+P_2'^{-1}Y_2C \end{bmatrix}(r\mathscr{X}+\bar{r}\bar{\mathscr{X}}) \\ &+(r\mathscr{X}+\bar{r}\bar{\mathscr{X}})'\begin{bmatrix} A+BY_1P_1^{-1} & BY_1P_1^{-1} \\ \mathbf{0} & A+P_2'^{-1}Y_2C \end{bmatrix}'<0. \end{aligned} \tag{25}$$

Let us take a partition of $\mathscr{X}$ as

$$\mathscr{X}=\begin{bmatrix} X_1 & \mathbf{0} \\ \mathbf{0} & Z_2 \end{bmatrix},\ X_1=X_1^\star>0,\ Z_2=Z_2^\star>0. \tag{26}$$

Under the conditions (19) and (26) there exists $X_2=X_2^\star$ such that $rZ_2+\bar{r}\bar{Z}_2=(rX_2+\bar{r}\bar{X}_2)^{-1}$. Therefore, under conditions (19) and (26), inequality (25) if the following hold true

$$\begin{aligned} &\begin{bmatrix} A+BY_1P_1^{-1} & BY_1P_1^{-1} \\ \mathbf{0} & A+P_2^{'-1}Y_2C \end{bmatrix}\begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2^{-1} \end{bmatrix} \\ &+\begin{bmatrix} P_1 & \mathbf{0} \\ \mathbf{0} & P_2^{-1} \end{bmatrix}'\begin{bmatrix} A+BY_1P_1^{-1} & BY_1P_1^{-1} \\ \mathbf{0} & A+P_2^{'-1}Y_2C \end{bmatrix}'<0, \end{aligned} \tag{27}$$

which is equivalent after simplification to the following matrix inequality:

$$\begin{bmatrix} \mathscr{R}_{11} & \mathscr{R}_{12} \\ \star & \mathscr{R}_{22} \end{bmatrix} < 0, \tag{28}$$

where

$$\begin{aligned} \mathscr{R}_{11} &= AP_1 + P'_1A' + BY_1 + Y_1'B', \\ \mathscr{R}_{12} &= BY_1P_1^{-1}P_2^{-1}, \\ \mathscr{R}_{22} &= AP_2^{-1} + P_2^{'-1}A' + P_2^{'-1}(Y_2C + C'Y_2')P_2^{-1}. \end{aligned} \tag{29}$$

Remark that inequality (28) can be rewritten as

$$\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & P_1^{-1}P_2^{-1} \\ \mathbf{0} & I \end{bmatrix}' \begin{bmatrix} AP_1 + P_1'A' + BY_1 + Y_1'B' & BY_1 & \mathbf{0} \\ \star & -I & \mathbf{0} \\ \star & \star & \mathscr{W}_{33} \end{bmatrix} \times \begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & P_1^{-1}P_2^{-1} \\ \mathbf{0} & I \end{bmatrix} < 0 \tag{30}$$

where

$$\begin{aligned} \mathscr{W}_{33} = &AP_2^{-1} + P_2^{'-1}A' + P_2^{'-1}(Y_2C + C'Y_2')P_2^{-1} \\ &+ P_2^{'-1}P_1^{'-1}P_1^{-1}P_2^{-1}. \end{aligned} \tag{31}$$

Since the matrix:

$$\begin{bmatrix} I & \mathbf{0} \\ \mathbf{0} & P_1^{-1}P_2^{-1} \\ \mathbf{0} & I \end{bmatrix} \tag{32}$$

has a full-column rank than, from the last inequality, it can be deduced that the following matrix inequality is a sufficient condition to fulfill (30), that is

$$\begin{bmatrix} AP_1 + P_1'A' + BY_1 + Y_1'B' & BY_1 & \mathbf{0} \\ \star & -I & \mathbf{0} \\ \star & \star & \mathscr{W}_{33} \end{bmatrix} < 0, \tag{33}$$

Inequality (33) holds true if and only if:

$$\begin{aligned} &\mathscr{W}_{33} < 0, \\ &\begin{bmatrix} AP_1 + P_1'A' + BY_1 + Y_1'B' & BY_1 \\ \star & -I \end{bmatrix} < 0. \end{aligned} \tag{34}$$

Since P_2 is a full-rank matrix then, the condition $\mathscr{W}_{33} < 0$ holds if and only if

$$P_2'\mathscr{W}_{33}P_2 < 0, \tag{35}$$

or

$$P_2'A + A'P_2 + Y_2C + C'Y_2' + P_1^{'-1}P_1^{-1} < 0. \tag{36}$$

Under the conditions: $X_1 = X_1^\star > 0$, the matrix $\bar{X}_1 > 0$. Note that the matrix $P_1 = r X_1 + \bar{r}\bar{X}_1$ is a real matrix which is not necessarily symmetric. This can be proved by showing that

$$rX_1 + \bar{r}\bar{X}_1 = 2\big(\cos(\theta)X_R - \sin(\theta)X_M\big), \tag{37}$$

where the complex matrix X_1 is taken as a sum of real matrix X_R and pure imaginary matrix iX_M, i.e., $X_1 = X_R + iX_M$ with $X_R \in \mathbb{R}^{n\times n}$ and $X_M \in \mathbb{R}^{n\times n}$. In addition, the matrix $P_1 = rX_1 + \bar{r}\bar{X}_1$ has all its eigenvalues in right-hand side of the complex plane due to the fact that for $0 < \alpha < 1$,

$$(rX_1 + \bar{r}\bar{X}_1)' + (rX_1 + \bar{r}\bar{X}_1) = 2\cos(\theta)(X_1 + \bar{X}_1) > 0. \tag{38}$$

Consequently, the matrix $P_1^{'-1}P_1^{-1}$ is a real positive definite, and therefore, we can find a positive $\varepsilon > 0$ and a real matrix W such that

$$P_1^{'-1}P_1^{-1} < \varepsilon^2 W^{-1}, \tag{39}$$

Based upon (36) and (39), a sufficient condition to fulfill (36) is given by:

$$P_2'A + A'P_2 + Y_2C + C'Y_2' + \varepsilon^2 W^{-1} < 0. \tag{40}$$

The last inequality is equivalent by the Schur Complement to (22). The matrix inequality (39) is not linear but, it's equivalent to the following matrix inequality:

$$\varepsilon^2 P_1P_1' - W > 0. \tag{41}$$

Using the result of Lemma 1, we can write

$$P_1 + P_1' \leq \varepsilon^2 P_1P_1' + \varepsilon^{-2} I. \tag{42}$$

Therefore, if the following inequality is satisfied

$$P_1 + P_1' - \varepsilon^{-2}I - W > 0 \tag{43}$$

then (41) is satisfied as well. Using the Schur complement Lemma, a necessary and sufficient condition to fulfill (43) is given by:

$$\begin{bmatrix} P_1 + P_1' - W & I \\ I & \varepsilon^2 I \end{bmatrix} > 0. \tag{44}$$

Using the fact that $\varepsilon^2 > 2\varepsilon - 1$ for $\varepsilon > 0$ then, (20) is a sufficient condition to verify inequality (44). This ends the proof of Theorem 3. □

Remark 2. The results given by Theorem 3 are not conservative because the decoupling of the necessary conditions (27) into conditions (34) is not related to a severe or restricted additional condition. Furthermore, the linearization of the matrix inequalities (34) by imposing (41) is not conservative as well because every positive definite matrix can be easily upper and lower bounded by another positive definite matrix. This has been done by the introduction of ε and W as additional LMI variables which make the design more flexible and straightforward.

3.2 Sufficient conditions with equality constraint

In this part, it is showed that the observer-based conditions for stabilization of fractional-order systems can be reformulated as linear matrix inequalities subject to equality constraint. The obtained result can be applied to fractional order-systems where the degree of differentiation α can take non-integer values inside the interval $]0\ 2[$. The whole design is given by the following statement.

Theorem 4. Consider the fractional-order system

$$\begin{aligned} D^\alpha x &= A\,x + B\,u, \\ y &= C\,x, \end{aligned} \tag{45}$$

where $0 < \alpha < 2$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$. Define the observer-based controller as:

$$\begin{aligned} D^\alpha \hat{x} &= A\,\hat{x} + B\,u + L(C\hat{x} - y), \\ u &= K\,\hat{x}. \end{aligned} \tag{46}$$

If there exist two symmetric matrices $P_1 \in \mathbb{R}^{n\times n} > 0$, $P_2 \in \mathbb{R}^{n\times n} > 0$, a matrix $Y \in \mathbb{R}^{n\times p}$, a matrix $\bar{K} \in \mathbb{R}^{m\times n}$ and a matrix $Z \in \mathbb{R}^{m\times m}$ such that the following set of linear matrix inequality hold simultaneously

$$\begin{aligned} &\begin{bmatrix} \mathscr{R}_1 & \mathscr{R}_2 \\ \star & \mathscr{R}_1 \end{bmatrix} < 0, \\ &P_1B = BZ, \end{aligned} \tag{47}$$

where

$$\mathscr{R}_1 = \begin{bmatrix} \left(A'P_1 + P_1 A + \bar{K}'B' + B\bar{K}\right)\sin(\theta) & B\bar{K}\sin(\theta) \\ \star & \left(A'P_2 + P_2 A + YC + C'Y'\right)\sin(\theta) \end{bmatrix}, \tag{48}$$

$$\mathscr{R}_2 = \begin{bmatrix} \left(P_1 A - A'P_1 - \bar{K}'B' + B\bar{K}\right)\cos(\theta) & \bar{K}'B'\cos(\theta) \\ -\bar{K}'B'\cos(\theta) & \left(P_2 A - A'P_2 + YC - C'Y'\right)\cos(\theta) \end{bmatrix}. \tag{49}$$

Then, system (45) is stable under the action of the observer controller $u = K\hat{x}$ where $\hat{x}$ is the state vector of the fractional-order observer (46) with $K = Z^{-1}\bar{K}$ and $L = P_2^{-1}Y$.

Proof. The system and the observer dynamics are given by the following composite system:

$$\begin{aligned} D^{\alpha}\begin{bmatrix} x \\ e \end{bmatrix} &= \begin{bmatrix} A + BK & BK \\ \mathbf{0} & A + LC \end{bmatrix}\begin{bmatrix} x \\ e \end{bmatrix} \\ &= A_{\text{closed}}\begin{bmatrix} x \\ e \end{bmatrix}. \end{aligned} \tag{50}$$

The design of the observer-controller gains L and K is conditioned by the stability of the fractional-order augmented system (50). Based upon the stability result of fractional-order system, see the statement of Theorem 2, one can conclude that system (50) is stable if there exists a matrix $X = X' > 0$, of appropriate dimension, such that the following holds

$$\begin{bmatrix} A_1 & A_2 \\ \star & A_1 \end{bmatrix} < 0 \tag{51}$$

where $A_1 = (A_{\text{closed}}X + XA'_{\text{closed}})\sin(\theta)$, $A_2 = (A_{\text{closed}}X - XA'_{\text{closed}})\cos(\theta)$, and $\theta = (1 - \frac{\alpha}{2})\pi$. By pre- and post- multiplying the last inequality by the matrix $\text{diag}(X^{-1}, X^{-1})$ and put $X^{-1} = P$ then, inequality (51) is verified if there exits a matrix $P \in \mathbb{R}^{2n\times 2n}$ such that

$$\begin{bmatrix} \Pi_1 & \Pi_2 \\ \star & \Pi_1 \end{bmatrix} < 0, \tag{52}$$

where $\Pi_1 = (PA_{\text{closed}} + A'_{\text{closed}}P)\sin(\theta)$, $\Pi_2 = (PA_{\text{closed}} - A'_{\text{closed}}P)\cos(\theta)$. Let us now take the partition of P as

$$P = \begin{bmatrix} P_1 & \mathbf{0} \\ \star & P_2 \end{bmatrix},\ P_1 \in \mathbb{R}^{n\times n},\ P_2 \in \mathbb{R}^{n\times n}. \tag{53}$$

This gives

$$A'P = \begin{bmatrix} A'P_1 + K'B'P_1 & \mathbf{0} \\ K'B'P_1 & A'P_2 + C'L'P_2 \end{bmatrix}. \tag{54}$$

Let $Z \in \mathbb{R}^{m\times m}$ be any full-rank matrix and let $Y \in \mathbb{R}^{n\times p}$ be any real-valued matrix. Then, by introducing the following constraint:

$$P_1 B = BZ, \tag{55}$$

and set the controller and the observers gains as $K = Z^{-1}\bar{K}$, $L = P_2^{-1}Y$ then,

$$A'P = \begin{bmatrix} A'P_1 + \bar{K}'B' & \mathbf{0} \\ \bar{K}'B' & A'P_2 + C'Y' \end{bmatrix}. \tag{56}$$

This implies that the stability of the fractional-order system (45) is dependent on the solvability of the linear matrix inequalities (47). This ends the proof. □

The statement of Theorem 4 summarizes the design of dynamic-output stabilizing feedbacks for larger class of fractional order linear systems. However, the constraints imposed on the matrix P_1 (*being symmetric, real, and positive definite in addition to equality constraint (55)*) introduces some conservatism. Extensive simulations has also shown that the LMIs of Theorem 4 are not feasible for some single-input fractional-order systems.

4. OBSERVER-BASED CONTROL WITH SYSTEM UNCERTAINTIES

Based on the result of Theorem 3, now we are ready to treat the case where all the system uncertainties are non null. The design is summarized in the following statement.

Theorem 5. Consider system (11) where the uncertain matrices ΔA and ΔC are not null. Define $\theta = (1-\alpha)\frac{\pi}{2}$ and $r = e^{i\theta}$ where $i^2 = -1$. If there exist a set of matrices $X_1 = X_1^{\star} \in \mathbb{C}^{n\times n}$, $X_2 = X_2^{\star} \in \mathbb{C}^{n\times n}$, $W \in \mathbb{R}^{n\times n}$, $Y_1 \in \mathbb{R}^{m\times n}$, $Y_2 \in \mathbb{R}^{n\times p}$ and a set of positive scalars ε_1, ε_2, ε_3, and ε_4 such that the following linear matrix inequalities hold simultaneously

$$\begin{aligned} &X_1 > 0,\ X_2 > 0,\ W = W' > 0, \\ &\frac{1}{\cos(\theta)}(P_2 + P_2') + \frac{i}{\sin(\theta)}(P_2 - P_2') > 0, \end{aligned} \tag{57}$$

$$\begin{bmatrix} P_1 + P_1' - W & I \\ \star & (2\varepsilon_3 - 1)I \end{bmatrix} > 0, \tag{58}$$

$$\begin{bmatrix} \mathscr{L}_{11} & BY_1 & P_1'N_A' & P_1'N_C' & P_1'N_A' \\ \star & -I & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \star & \star & -\varepsilon_1 I & \mathbf{0} & \mathbf{0} \\ \star & \star & \star & -\varepsilon_2 I & \mathbf{0} \\ \star & \star & \star & \star & -\varepsilon_4 I \end{bmatrix} < 0, \tag{59}$$

$$\begin{bmatrix} \mathscr{K}_{11} & P_2'M_A & Y_2 M_C & \varepsilon_3 I \\ \star & (-2+\varepsilon_1)I & \mathbf{0} & \mathbf{0} \\ \star & \star & (-2+\varepsilon_2)I & \mathbf{0} \\ \star & \star & \star & -W \end{bmatrix} < 0, \tag{60}$$

where

$$\begin{aligned} &P_1 = (rX_1 + \bar{r}\bar{X}_1),\ P_2 = (rX_2 + \bar{r}\bar{X}_2), \\ &\mathscr{L}_{11} = P_1'A' + AP_1 + BY_1 + Y_1'B' + BY_1 + \varepsilon_4 M_A M_A', \\ &\mathscr{K}_{11} = A'P_2 + P_2'A + Y_2 C + C'Y_2' \end{aligned} \tag{61}$$

then, the observer-based feedback $u = Y_1 P_1^{-1}\hat{x}$ stabilizes system (11) with $\hat{x} = \hat{x}(t)$ being the state vector of the fractional-order observer:

$$\begin{aligned} &D^{\alpha}\hat{x} = A\,\hat{x} + B\,u + (rX_2' + \bar{r}\bar{X}_2')^{-1}Y_2(C\hat{x} - y), \\ &u = Y_1(rX_1 + \bar{r}\bar{X}_1)^{-1}\,\hat{x}. \end{aligned} \tag{62}$$

Proof. The proof of this result is omitted for space limitation. However, the proof can be easily obtained by composing the resulting state and observer-error dynamical equations and then apply the result of Theorem 2. The rest of the proof is obtained by the use of Lemma 1 and then the decomposition as shown from inequality (33).

5. NUMERICAL SIMULATION

Consider the fractional-order system:

$$\begin{aligned} D^{0.8} &= \left(\begin{bmatrix} -1.5 & 0.5 \\ -2.5 & 2.3 \end{bmatrix} + \Delta A \right) x + \begin{bmatrix} 1 \\ -1 \end{bmatrix} u, \\ y &= \left([\,1\;\;1\,] + \Delta C \right) x, \end{aligned} \tag{63}$$

where $M_A = \begin{bmatrix} 0 & 0.1 \\ 0.1 & 0 \end{bmatrix}$, $N_A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $M_C = [0.2\;\;0]$, $N_C = [1\;\;0.1]$. The LMIs of Theorem 5 were found feasible, where

$$\begin{aligned} X_1 &= \begin{bmatrix} 0.34218 & 0.14063 + 0.0351\,i \\ 0.14063 - 0.0351\,i & 0.1914 \end{bmatrix}, \\ X_2 &= \begin{bmatrix} 30.184 & -0.69362 - 0.1734\,i \\ -0.69362 + 0.1734\,i & 0.068504 \end{bmatrix}, \\ W &= \begin{bmatrix} 0.7622 & 0.3755 \\ 0.3755 & 0.3922 \end{bmatrix}, Y_1 = [-0.2864\;\;0.7106], \\ Y_2 &= \begin{bmatrix} 33.284 \\ -23.879 \end{bmatrix}, \varepsilon_1 = 1.1963, \varepsilon_2 = 0.42387, \\ \varepsilon_3 &= 2.6141, \varepsilon_4 = 10.867. \end{aligned} \tag{64}$$

As a comparison between the LMIs of Theorem 4, that are subject to the equality constraint, and the LMIs of Theorem 3, we found that the LMIs of Theorem 4 are not feasible for the above example where $\Delta A = 0$ and $\Delta C = 0$. Actually, Theorem 4 showed its usefulness in case of multiple-Input-Multiple-Output systems where the equality constraint can be satisfied.

6. CONCLUSION

New Linear-Matrix-Inequality conditions are proposed to solve the problem of observer-based stabilization of a class of fractional-order linear systems. The stability of fractional-order systems by means of fractional-order observers poses more constraints on the choice of the observer and the controller gain. Therefore, the formulation of the gains as a combination of complex variables give more freedom in selecting the appropriate gains assuring the stability of the system and the observer as well. The reduction of the initial non-convex optimization problem into a set of convex optimization problems makes the design simple and straightforward. Extension of the obtained results to other classes of fractional-order systems remains an open problem for further investigation.

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